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ON THE VOLUME OF A LINE BUNDLE

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Using the Calabi–Yau technique to solve Monge-Ampère equations, we translate a result of T. Fujita on approximate Zariski decompositions into an analytic setting and combine this to the holomorphic Morse inequalities in order to express the volume of a line bundle as the maximum of the mean curvatures of all the singular Hermitian metrics on it, with a way to pick an element at which the maximum is reached and satisfying a singular Monge–Ampère equation. This enables us to introduce the volume of any $(1,1)$-class on a compact Kähler manifold, and Fujita's theorem is then extended to this context.

Keywords: Volume; compact Kähler manifold; big line bundle; closed positive current; Monge-Ampère equation; Aubin-Calabi-Yau theorem; pseudoeffective class; Zariski decomposition.

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1. Introduction

For a holomorphic line bundle L on a compact Kähler n-fold X, one defines the *volume* of L as

$$v(L) := \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL) .$$

The line bundle L is then big, i.e. it has maximal Kodaira–Iitaka dimension $\kappa(X, L) = n$, exactly when $v(L) > 0$, and in that case, it is known that the lim sup is a limit (cf. [7]), so that the volume $v(L)$ precisely measures the bigness of L.

As a consequence, we immediately get that $v(kL) = k^n v(L)$ for every $L \in \mathrm{Pic}(X)$ and every integer k, so we can define the volume of a $\mathbf{Q}$-line bundle L by setting $v(L) = k^{-n} v(kL)$ for some k such that kL is an actual line bundle.

It can be shown in general that the volume of L only depends on the first Chern class $c_1(L)$ (this will be a consequence of our results), and we therefore raise the following

Question: Can we find a formula expressing the volume of L in terms of $c_1(L)$?

If L is an ample line bundle, the combination of Serre's vanishing theorem and the asymptotic Riemann–Roch formula shows that $v(L) = L^n$, where L^n is the n-fold intersection number $\int_X c_1(L)^n$. When L is merely a nef (numerically effective) line bundle, one can show using Demailly's Morse inequalities that $h^q(X, kL) = o(k^n)$ for $q > 0$, so that the Riemann–Roch formula again yields $v(L) = L^n$ in that case. We therefore have a positive and simple answer to our question in case L is nef.

Recall now that a line bundle L is said to have an algebraic Zariski decomposition if there exists a nef $\mathbf{Q}$-line bundle P and an effective $\mathbf{Q}$-divisor N such that

(i) $L = P + N$ as $\mathbf{Q}$-line bundles.
(ii) The canonical inclusion of $H^0(X, \mathcal{O}(kP))$ in $H^0(X, \mathcal{O}(kL))$ is surjective for every positive integer k clearing up the denominator of N.

If such a decomposition exists, one of course has $v(L) = v(P) = P^n$, so that the knowledge of the nef part P enables one to compute the volume of L. On a surface, it is known since Zariski that such a decomposition exists, but this is no longer true in general; however the following result, due to T. Fujita ([9], cf. also [7]), states that some kind of approximate Zariski decomposition exists in general, as far as the volume is concerned:

Theorem 1.1 (Approximate Zariski Decomposition). *Let L be a big line bundle on a projective manifold X. Then, for every $\varepsilon > 0$, there exists a modification $\mu : \tilde{X} \to X$, an ample $\mathbf{Q}$-line bundle A and an effective $\mathbf{Q}$-divisor E on $\tilde{X}$ (the data depends on ε) such that*:

(i) $L = A + E$ *as $\mathbf{Q}$-line bundles*,
(ii) $|v(A) - v(L)| < \varepsilon$.

It is thus tempting to think that a Zariski decomposition of L could be obtained by letting ε tend to zero in the above theorem, which cannot be done in an algebraic context. But the theorem of Calabi–Yau affords a way to choose a Hermitian metric on each ample line bundle A such that the product of the curvature eigenvalues is constantly equal to the volume $v(A) = A^n$ of A. Applying this to an approximate Zariski decomposition yields metrics h_ε on L converging to some singular metric h such that the product of the curvature eigenvalues of h is also constantly equal the volume $v(L)$. Combining this with the holomorphic Morse inequalities, we prove the following

Theorem 1.2. *Let L be a pseudoeffective line bundle on a compact Kähler manifold X. Then the volume of L satisfies*

$$v(L) = \max_T \int_X T_{ac}^n$$

for T ranging among the closed positive $(1,1)$-currents in the cohomology class $c_1(L)$.

Furthermore, given a Kähler form ω on X normalized so that $\int_X \omega^n = 1$, there exists a closed positive $(1,1)$-current $T \in c_1(L)$ such that

$$T_{ac}(x)^n = v(L)\omega(x)^n$$

for almost every $x \in X$.

Recall that a class α lying in $H^{1,1}(X, \mathbf{R}) := H^{1,1}(X) \cap H^2(X, \mathbf{R})$ is said to be pseudoeffective if and only if it contains a closed positive current T. The notation T_{ac} stands for the absolutely continuous part of T, cf. Sec. 2.3. In case $\alpha = c_1(L)$ is the first Chern class of a line bundle L, we say that L is pseudoeffective if $c_1(L)$ is. In that case, for every closed positive current T in $c_1(L)$, there exists a singular Hermitian metric h on L such that $T = \Theta_h(L)$ is the curvature current of h. The determinant $T_{ac}(x)^n/\omega(x)^n$ is just the product of the curvature eigenvalues of h, so that Theorem 1.2 really expresses the volume as the maximum mean curvature of metrics on L.

If L is not pseudoeffective, it is *a fortiori* not big, and its volume is zero anyway. Theorem 1.2 thus yields a general answer to our question. The volume is so far defined on the rational Neron–Severi space $NS(X) \otimes \mathbf{Q}$, which is the set of classes α of the form $c_1(L)$ for some $\mathbf{Q}$-line bundle L, but the formula in Theorem 1.2 makes sense for any pseudoeffective class $\alpha \in H^{1,1}(X, \mathbf{R})$, and it therefore seems natural to introduce the following

Definition 1.3. Let X be a compact Kähler n-fold. We define the volume of a cohomology class $\alpha \in H^{1,1}(X, \mathbf{R})$ by

$$v(\alpha) := \sup_T \int_X T_{ac}^n$$

for T ranging over the closed positive $(1,1)$-currents in α, in case α is pseudoeffective. If it is not, we set $v(\alpha) = 0$.

We show that the supremum involved is always finite. The set of pseudoeffective classes in $H^{1,1}(X, \mathbf{R})$ is a closed convex cone called the pseudoeffective cone and denoted by $\mathcal{E} \subset H^{1,1}(X, \mathbf{R})$. A class $\alpha \in H^{1,1}(X, \mathbf{R})$ lies in the interior $\mathcal{E}^0$ if and only if it can be represented by a strictly positive current T (a so-called Kähler current, cf. Sec. 1). The interior $\mathcal{E}^0$ we call the big cone, whose elements are big classes, and it is true that a line bundle L is big if and only if $c_1(L)$ is a big class. In general, we would like to think of the volume $v(\alpha)$ of a class $\alpha \in H^{1,1}(X, \mathbf{R})$ as a quantitative measure of its bigness. It is trivial that the volume $v(\alpha)$ of a big class α is non-zero, but the converse is far from obvious. We prove it by adapting arguments from [8]. We also prove that the volume map $v : H^{1,1}(X, \mathbf{R}) \to \mathbf{R}$ is continuous, and that $\alpha \mapsto v(\alpha)^{1/n}$ is homogeneous and concave on the pseudoeffective cone $\mathcal{E}$. Finally, we give the corresponding version of Fujita's theorem:

Theorem 1.4. *Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbf{R})$ be a big class on X. Then, for every $\varepsilon > 0$, there exists a modification $\mu : \tilde{X} \to X$, a Kähler class ω and an effective real divisor D on $\tilde{X}$ such that*

(i) $\mu^\star a = \omega + \{D\}$ *as cohomology classes,*
(ii) $|v(\alpha) - v(\omega)| < \varepsilon$.

2. Technical Preliminaries

2.1. *Terminology*

Let X be a compact complex n-fold. We will use dd^c to denote the operator $\frac{i}{\pi}\partial\bar{\partial}$. We recall a few more or less standard definitions: a closed real $(1,1)$-current T on X is said to be almost positive if some smooth real $(1,1)$-form γ can be found such that $T \geq \gamma$. A function φ in $L^1_{loc}(X)$ is called almost plurisubharmonic (almost psh for short) if its complex Hessian $dd^c\varphi$ is an almost positive current. This latter property is equivalent to the fact that φ can locally be written as a sum of a plurisubharmonic function and a smooth one. When T is an arbitrary closed $(1,1)$-current, it can locally be written as $dd^c\varphi$ for some current φ of degree 0. The current T is (almost) positive if and only if its local potentials φ are (almost) psh functions.

A closed $(1,1)$-current T is called a Kähler current if one has $T \geq \omega$ for some Hermitian form ω on X (a Hermitian form will always mean a smooth positive definite Hermitian form for us).

We say that a function φ on X has analytic singularities along a subscheme $V(\mathcal{I})$ (corresponding to a coherent ideal sheaf $\mathcal{I}$) if there exists $c > 0$ such that φ is locally congruent to $\frac{c}{2}\log(\sum |f_j|^2)$ modulo smooth functions, where $f_1, \dots, f_r$ are local generators of $\mathcal{I}$. When this holds true, we furthermore say that φ has algebraic singularities if $c > 0$ can be taken to be rational. Note that a function with analytic singularities is automatically almost psh, and that it is smooth away from the support of $V(\mathcal{I})$.

When T is an almost positive $(1,1)$-current, it is always possible to find a smooth form θ such that $T = \theta + dd^c\varphi$ for some almost psh function φ, and we say that T has analytic or algebraic according to φ.

2.2. *Siu decomposition of a current*

Let T be a closed positive current of bidegree (p,p) on a complex n-fold X. We denote by $\nu(T,x)$ its Lelong number at a point $x \in X$. The Lelong super-level sets are defined by $E_c(T) := \{x \in X, \nu(T,x) \geq c\}$ for $c > 0$, and a well known result of Y. T. Siu asserts that $E_c(T)$ is a (closed) analytic subset of X of codimension at least p. As a consequence, for any analytic subset A of X, the generic Lelong number of T along A, defined by

$$\nu(T,A) := \inf\{\nu(T,x), x \in A\},$$

is also equal to $\nu(T,x)$ for a very general $x \in A$. It is also true that, for any irreducible p-codimensional analytic subset A of X, the positive current $\chi_A T$ is a positive multiple of the integration current $[A]$, defined by integrating test forms on the smooth locus of A (χ_A denotes the characteristic function of A), so that $\chi_A T = \nu(T, A)[A]$. Since $E_+(T) := \cup_{c>0} E_c(T)$ is a countable union of analytic subsets of codimension at least p, it contains an at most countable family A_k of irreducible p-codimensional analytic subsets. By what we have said, $T - \nu(T, A_1)[A_1] - \cdots - \nu(T, A_k)[A_k]$ is a positive current for all k, thus the series $\sum_{k\geq 0} \nu(T, A_k)[A_k]$ converges, and we have

$$T = R + \sum_{k\geq 1} \nu(T, A_k)[A_k]$$

for some closed positive (p,p)-current R such that $E_+(R)$ contains no analytic subset of codimension p. The decomposition above is called the Siu decomposition of the closed positive (p,p)-current T. Since $\nu(T, A) = 0$ if A is not contained in $E_+(T)$, it makes sense to write $\sum_k \nu(T, A_k)[A_k] = \sum \nu(T, A)A$, where the sum is implicitely extended over all irreducible analytic subsets of codimension p (we omit the brackets in $[A]$ when no confusion is possible).

When $p = 1$, we call $\sum \nu(T, D)D$ the divisor part of T. It is straightforward using the Lelong–Poincaré formula to check that the divisor part $\sum \nu(T, D)D$ of a closed $(1,1)$-current T with analytic singularities along the subscheme V is just the divisor part of V, times the constant $c > 0$ appearing in the definition of analytic singularities. The residual part R again has analytic singularities, but in codimension at least 2.

2.3. *Lebesgue decomposition of a current*

A current T on an n-fold X can be seen as a form with distribution coefficients. When T is positive, the distributions in question are positive measures which admit a Lebesgue decomposition into an absolutely continuous part (with respect to the Lebesgue measure on X) and a singular part. We therefore get a decomposition of T itself into an absolutely continuous part T_{ac} and a singular part T_{sg}. The decomposition $T = T_{ac} + T_{sg}$ is called the Lebesgue decomposition of T.

The absolutely continuous part T_{ac} is positive, and more generally we have $T_{ac} \geq \gamma$ whenever $T \geq \gamma$ for some smooth real form γ, but the trouble with T_{ac} is that it is in general not closed, even when T is, so that the Lebesgue decomposition doesn't induce a significant decomposition at the cohomological level. For instance, one can check that the absolutely continuous part of $T := i\partial\bar{\partial}\log^+|z|$ on $X = \mathbf{C}^2$ is not closed.

The absolutely continuous current T_{ac} can be seen as (the current associated to) a positive form with L^1_{loc} coefficients, and it therefore makes sense to consider $T_{ac}(x)^k$ for almost every $x \in X$. This yields a positive Borel (k,k)-form, which we denote by T^k_{ac}.

When T is a closed $(1,1)$-current with analytic singularities along a subscheme V, things are much nicer: the absolutely continuous part of T is just $T_{ac} = (1 - \chi_V)T$, where χ_V denotes the characteristic function of (the support of) V. Indeed, $(1 - \chi_V)T$ is absolutely continuous since T is smooth outside V, and $\chi_V T$ is clearly singular since V has zero Lebesgue measure. Since the residual part R of T in its Siu decomposition has analytic singularities along a set of codimension at least 2, and since a closed (almost) positive $(1,1)$-current carries no mass on 2-codimensional analytic subsets, we see that R is absolutely continuous. Now, the divisor part $\sum \nu(T,D)D$ of T is clearly singular, so that the Lebesgue decomposition of T coincides with its Siu decomposition. In particular, T_{ac} is always closed in that case.

The absolutely continuous part T_{ac} of a positive current T does not depend continuously on T, but we have the following semi-continuity property:

Proposition 2.1. *Let T_k be a sequence of positive $(1,1)$-currents converging weakly to T. Then one has $T_{ac}(x)^n \geq \limsup T_{k,ac}(x)^n$ for almost every $x \in X$.*

Proof. Let ω be some fixed Hermitian form. For each positive $(1,1)$-form α, we denote by $\det(\alpha)$ the determinant of α with respect to ω, that is $\det(\alpha(x)) = \alpha(x)^n/\omega(x)^n$. Since the result is local, we may consider a regularizing sequence (ρ_j). Since $T_k \geq T_{k,ac}$, we have

$$\det(T_k \star \rho_j)^{1/n} \geq \det(T_{k,ac} \star \rho_j)^{1/n} .$$

The concavity of the function $A \mapsto \det(A)^{1/n}$ on the convex cone of Hermitian semi-positive matrices of size n then yields

$$\det(T_{k,ac} \star \rho_j)^{1/n} \geq \det(T_{k,ac})^{1/n} \star \rho_j .$$

Since a convolution transforms a weak convergence into a C^∞ one, Fatou's lemma therefore implies:

$$\det(T \star \rho_j)^{1/n} \geq (\liminf_{k\to\infty} \det(T_{k,ac})^{1/n}) \star \rho_j .$$

Now Lebesgue's theorem implies that $T \star \rho_j \to T_{ac}$ a.e. thus we get

$$\det(T_{ac})^{1/n} \geq \liminf_{k\to\infty} \det(T_{k,ac})^{1/n} .$$

We can eventually turn the lim inf into a lim sup by choosing appropriate subsequences pointwise. □

We will also need the following well known facts:

Proposition 2.2. *Let $f : Y \to X$ be a proper surjective holomorphic map. If α is a locally integrable form of bidimension (k,k) on Y, then the pushed-forward current $f_\star\alpha$ is absolutely continuous, hence a locally integrable form of bidimension (k,k). In particular, when T is a positive current on Y, the pushed-forward current $f_\star(T_{ac})$ is absolutely continuous, and we have the formula $f_\star(T_{ac}) = (f_\star T)_{ac}$.*

2.4. *How to improve the singularities of a current*

When $f : Y \to X$ is a *surjective* holomorphic map between compact complex manifolds and T is a closed almost positive $(1,1)$-current on X, it is possible to define its pull back $f^\star T$ by f in the following fashion: write $T = \theta + dd^c\varphi$ for some smooth form θ. φ is then an almost psh and integrable function, which is thus not identically $-\infty$ on $X = f(Y)$. One defines $f^\star T$ to be $f^\star\theta + dd^c f^\star\varphi$, as this is easily seen to be independent of the choices made.

When T has analytic singularities along $V(\mathcal{I})$, its pull-back clearly has analytic singularities along $V(f^{-1}\mathcal{I})$. Therefore, by first blowing-up X along the subscheme $V(\mathcal{I})$ and then resolving the singularities, we get a modification $\mu : \tilde{X} \to X$ such that $\mu^\star T$ has analytic singularities along an effective divisor D only. Its Siu decomposition therefore writes $\mu^\star T = \theta + cD$ for some smooth (semi-)positive form θ and $c > 0$. This operation we call a resolution of the singularities of T. For a general current T, no such procedure is available, but one has two types of regularizations of T inside its cohomology class, both due to J. P. Demailly. In the following results, X denotes a compact complex manifold equipped with a Hermitian form ω.

Theorem 2.3 ([3]). *Let $T = \theta + dd^c\varphi$ be a closed $(1,1)$-current, where θ is a smooth form. Suppose that a smooth $(1,1)$-form γ is given such that $T \geq \gamma$. Then there exists a decreasing sequence of smooth functions φ_k converging to φ such that, if we set $T_k := \theta + dd^c\varphi_k$, we have*

(i) *$T_k \to T$ weakly and $T_{k,ac}(x) \to T_{ac}(x)$ a.e.*
(ii) *$T_k \geq \gamma - C\lambda_k\omega$, where $C > 0$ is a constant depending on (X,ω) only, and λ_k is a decreasing sequence of continuous functions such that $\lambda_k(x) \to \nu(T,x)$ for all $x \in X$.*

The theorem roughly says that it is possible to smooth a current inside its cohomology class, but only with a loss of positivity controled by the Lelong numbers of T. If one is willing to accept analytic singularities, then the loss of positivity can be made as small as desired:

Theorem 2.4. *Let $T = \theta + dd^c\varphi$ be a closed $(1,1)$-current, where θ is a smooth form. Suppose that a smooth $(1,1)$-form γ is given such that $T \geq \gamma$. Then there exists a sequence of functions φ_k with algebraic singularities converging to φ such that, if we set $T_k := \theta + dd^c\varphi_k$, we have*

(i) *$T_k \to T$ weakly and $T_{k,ac}(x) \to T_{ac}(x)$ a.e.*
(ii) *$T_k \geq \gamma - \varepsilon_k\omega$, where $\varepsilon_k > 0$ is a sequence converging to zero.*
(iii) *The Lelong numbers $\nu(T_k,x)$ increase to $\nu(T,x)$ uniformly with respect to $x \in X$.*

Proof. This result is entirely proved in [4], except for the slight refinement about the absolutely continuous parts, which we shall need for our purpose. We therefore choose a sequence $T_k^{(1)} = \theta + dd^c\varphi_k^{(1)}$ of smooth forms as in Theorem 2.3, and also a sequence $T_k^{(2)} = \theta + dd^c\varphi_k^{(2)}$ with algebraic singularities as in the statement of Theorem 2.4, except for the requirement on the absolutely continuous parts. We are going to explain how to glue the two constructions in order to get a sequence $T_k^{(3)} = \theta + dd^c\varphi_k^{(3)}$ satisfying the full statement of Theorem 2.4. □

Denote by A_k the analytic set along which $\varphi_k^{(2)}$ has singularities. Choose an arbitrary sequence $C_k > 0$ increasing to $+\infty$, and a sequence of rationals $\delta_k > 0$ decreasing to 0. Observe that $U_k := \{\varphi_k^{(2)} < -(C_k+1)/\delta_k\}$ is an open neighbourhood of A_k such that we have $\varphi_k^{(2)} < (1-\delta_k)\varphi_k^{(2)} - C_k - 1/2$ on $\bar{U}_k$, so that

$$\varphi \leq \varphi_k^{(2)} < (1-\delta_k)\varphi_k^{(2)} - C_k - 1/2$$

on the compact $\bar{U}_k$. Since $\varphi_k^{(1)}$ is continuous and decreases to φ, we have

$$\varphi_{j_k}^{(1)} < (1-\delta_k)\varphi_k - C_k - 1/2$$

on $\bar{U}_k$ for j_k big enough. We now select a smaller open neighbourhood $W_k \subset\subset U_k$ of A_k, and we set:

$$\varphi_k^{(3)} := \begin{cases} (1-\delta_k)\varphi_k^{(2)} - C_k & \text{on } U_k\,, \\ \max_\eta((1-\delta_k)\varphi_k^{(2)} - C_k, \varphi_{j_k}^{(1)}) & \text{on } X - W_k\,, \end{cases}$$

where $\max_\eta(x,y) := \max \star \rho_\eta$ denotes a regularized maximum function obtained by convolution with a regularizing kernel ρ_η, and η is chosen so small that $\max_\eta(x,y) = x$ when $y < x - 1/2$. The two parts to be glued then coincide on some neighbourhood of ∂U_k, and the gluing property of psh functions thus shows that $\varphi_k^{(3)}$ is almost plurisubharmonic; it has algebraic singularities since it coincides with $\varphi_k^{(2)}$ near each of its poles, and it is clear that $\varphi_k^{(3)}$ converges to φ, since $\varphi_k^{(1)}$ and $\varphi_k^{(2)}$ do. Let us prove that $T_k^{(3)} := \theta + dd^c\varphi_k^{(3)}$ satisfies the statement (i). Since $\varphi_k^{(3)}$ converges to φ, $T_k^{(3)} \to T$ is automatic. As to the second assertion, notice that if $\varphi(x) > -\infty$, then x cannot be in U_k for all k big enough, since otherwise $\varphi_k^{(2)}(x) \leq -(C_k+1)$ for k big enough, which would yield $\varphi(x) = -\infty$ since $\varphi_k^{(2)}(x)$ converges to $\varphi(x)$. Furthermore, for such an x, we have

$$(1-\delta_k)\varphi_k^{(2)}(x) - C_k < \varphi(x) - 1/2$$

for k big enough, since $C_k \to +\infty$. As a consequence, we get that $(1-\delta_k)\varphi_k^{(2)}(x) - C_k < \varphi_{j_k}^{(1)}(x) - 1/2$, and thus $(1-\delta_k)\varphi_k^{(2)} - C_k < \varphi_{j_k}^{(1)} - 1/2$ on some neighbourhood of x (depending on k) contained in $X - W_k$, by continuity. We infer that $\varphi_k^{(3)} = \varphi_{j_k}^{(1)}$ on this neighbourhood. The upshot is: for every x outside the polar set of φ (which has measure 0) we have $T_k^{(3)}(x) = T_{j_k}^{(1)}(x)$ for k big enough, and this certainly implies that $T_k^{(3)}(x) = T_{k,ac}^{(3)}(x) \to T_{ac}(x)$ for almost every x. We now prove (ii): the gluing property of plurisubharmonic functions shows that $T_k^{(3)} - \gamma$ will have a lower bound going to 0 for $k \to +\infty$ if we can show that this is the case for $T_{j_k}^{(1)} - \gamma$ on

$X - W_k$. But we have $\nu(T_k^{(2)}, x) = 0$ for x in this set, thus $\nu(T, x)$ will be uniformly small for x in that set. Since the lack of positivity of $T_{j_k}^{(1)} - \gamma$ is controled by the Lelong numbers of T, we get the result.

2.5. *Boundedness of the mass*

Here we are interested in the control of the mass of T_{ac}^k for a closed positive $(1,1)$-current T. As we have said, T_{ac}^k is *a priori* just a positive Borel (k,k)-form, and it is by no means clear that it should have finite mass; we show that things go well in the Kähler case. In the remainder, X denotes a compact Kähler n-fold, and ω is a fixed Kähler form on it.

Lemma 2.5. *Let T be any closed positive $(1,1)$-current on X. Then the Lelong numbers $\nu(T, x)$ of T can be bounded by a constant depending on (X, ω) and the cohomology class $\{T\}$.*

Proof. One has by definition that $\nu(T, x)$ is (up to a constant depending on ω near x) the limit for $r \to 0_+$ of

$$\nu(T, x, r) := \frac{(n-1)!}{(\pi r^2)^{n-1}} \int_{B(x,r)} T \wedge \omega^{n-1},$$

which is known to be an increasing function of r. Thus if we choose r_0 small enough to ensure that each ball $B(x, r_0)$ is contained in a coordinate chart, we get $\nu(T, x) \leq \nu(T, x, r_0) \leq C \int_X T \wedge \omega^{n-1}$, a quantity depending on the cohomology class $\{T\}$ only since ω is closed. □

Proposition 2.6. *Let T be a closed positive $(1,1)$-current. Then the integrals $\int_X T_{ac}^l \wedge \omega^{n-l}$ are finite for each $l = 0, \ldots, n$ and can be bounded in terms of (X, ω) and the cohomology class of T only.*

Proof. Choose a sequence T_k of smooth forms approximating T as in Theorem 2.3. Since $T_k \geq -C\lambda_k \omega$ for some constant $C > 0$ depending on (X, ω) only and continuous functions $\lambda_k(x)$ decreasing to $\nu(T, x)$, we find using Lemma 2.5 a constant also denoted by C and depending on (X, ω) and the cohomology class $\{T\}$ only such that $T_k + C\omega \geq 0$. But now

$$\int_X (T_k + C\omega)^l \wedge \omega^{n-l} = \{T + C\omega\}^l \{\omega\}^{n-l}$$

does not depend on k, so the result follows by Fatou's lemma, since $T_k + C\omega$ is a smooth form converging to $T_{ac} + C\omega$ a.e. □

3. Volume of a Line Bundle

Unless further notice, X denotes a compact Kähler n-fold.

3.1. *A Morse-type inequality*

In this part, we will prove the following:

Proposition 3.1. *If L is a pseudoeffective line bundle on X and $T \in c_1(L)$ is a closed positive current, one has*:

$$v(L) \geq \int_X T_{ac}^n .$$

In order to prove this, we will appeal to the singular holomorphic Morse inequalities, but to state this result we first need some terminology.

(i) The Nadel multiplier ideal sheaf $\mathcal{I}(T)$ of an almost positive closed $(1,1)$-current T is defined as the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally integrable, for some (hence any) $\varphi \in L^1_{loc}$ such $T = dd^c\varphi$ locally. This sheaf is coherent, as is well known (cf. e.g. [6]).

(ii) The q-index set of an almost positive closed $(1,1)$-current T is the set of $x \in X$ such that the absolutely continuous part T_{ac} of T has exactly q negative eigenvalues at x. This set is denoted by $X(T,q)$, and we also write $X(T, \leq q)$ for the union of the $X(T,j)$'s for $j = 0, \ldots, q$. These sets are only defined up to a null measure set, but since we shall only integrate absolutely continuous forms on them, it is not really annoying here.

Now we can state the following result, due to Bonavero [1]:

Theorem 3.2 (Singular Morse inequalities). *Let L be any holomorphic line bundle on a compact complex n-fold X, and $T \in c_1(L)$ be some closed $(1,1)$-current with algebraic singularities. Then the following holds*:

$$h^0(X, \mathcal{O}(kL) \otimes \mathcal{I}(kT)) - h^1(X, \mathcal{O}(kL) \otimes \mathcal{I}(kT)) \geq \frac{k^n}{n!} \int_{X(T,\leq 1)} T_{ac}^n - o(k^n)$$

for $k \to +\infty$.

Since one has

$$\begin{aligned} h^0(X, \mathcal{O}(kL)) &\geq h^0(X, \mathcal{O}(kL) \otimes \mathcal{I}(kT)) \\ &\geq h^0(X, \mathcal{O}(kL) \otimes \mathcal{I}(kT)) - h^1(X, \mathcal{O}(kL) \otimes \mathcal{I}(kT)) \, , \end{aligned}$$

Theorem 3.2 implies:

Corollary 3.3. *For any line bundle L on X, one has*

$$v(L) \geq \int_{X(T,\leq 1)} T_{ac}^n$$

for every closed $(1,1)$-current $T \in c_1(L)$ with algebraic singularities.

We can now conclude the proof of Proposition 3.1. In fact, we choose a sequence T_k of currents with algebraic singularities as in Theorem 2.4 (here $\gamma = 0$), and we denote by $\lambda_1 \le \cdots \le \lambda_n$ (respectively $\lambda_1^{(k)} \le \cdots \le \lambda_n^{(k)}$) the eigenvalues of T_{ac} (respectively $T_{k,ac}$) with respect to ω. We have by assumption that $\lambda_1 \ge 0$, $\lambda_1^{(k)} \ge -\varepsilon_k$ and $\lambda_j^{(k)}(x) \to \lambda_j(x)$ almost everywhere. We can certainly assume that $\int_X T_{ac}^n > 0$, which means that the set $A := \{\lambda_1 > 0\}$ has positive measure. For each small $\delta > 0$, Egoroff's lemma gives us some $B_\delta \subset A$ such that $\lambda_1^{(k)} \to \lambda_1$ uniformly on B_δ and also $A - B_\delta$ has measure less than δ. Thus we see that $B_\delta \subset X(T_k, 0)$ for k big enough, and consequently $\limsup \int_{X(T_k,0)} T_{k,ac}^n \ge \int_{B_\delta} \liminf T_{k,ac}^n = \int_{B_\delta} T_{ac}^n$, using Fatou's lemma. Letting now δ tend to 0, we get

$$\limsup \int_{X(T_k,0)} T_{k,ac}^n \ge \int_A T_{ac}^n = \int_X T_{ac}^n .$$

Since by Corollary 3.3 above we have $\int_{X(T_k,0)} T_{k,ac}^n \le v(L) - \int_{X(T_k,1)} T_{k,ac}^n$ for every k, the proof of Proposition 3.1 will be over if we can show that $-\int_{X(T_k,1)} T_{k,ac}^n \to 0$. But we observe the following inequalities on $X(T_k, 1)$:

$$0 \le -T_{k,ac}^n \le n\varepsilon_k \omega \wedge (T_{k,ac} + \varepsilon_k \omega)^{n-1} ,$$

from which we get

$$0 \le -\int_{X(T_k,1)} T_{k,ac}^n \le n\varepsilon_k \int_X \omega \wedge (T_{k,ac} + \varepsilon_k \omega)^{n-1} .$$

Now the last integral is bounded uniformly in terms of $\{T\}$ and ω only by Proposition 2.6, which ends the proof.

It is worth noting that the Kähler assumption is needed precisely for this last lemma. Consequently, Proposition 3.1 is also true on a non-Kähler surface.

3.2. *The theorem of Calabi–Yau*

First, let us recall the fundamental result proved in [14]:

Theorem 3.4 (Aubin–Calabi–Yau). *Let (X, ω) be a compact Kähler manifold, and assume that $\int_X \omega^n = 1$. Then, given a Kähler cohomology class α, there exists a Kähler form $\tau \in \alpha$ such that*

$$\tau(x)^n = \left(\int_X \alpha^n \right) \omega(x)^n$$

for every $x \in X$.

We now consider a pseudoeffective line bundle L on a compact Kähler n-fold X, equipped with some fixed Kähler form ω such that $\int_X \omega^n = 1$. We intend to prove Theorem 1.2. Notice that, if L is not big, we will have $v(L) = 0$, and thus $\int_X T_{ac}^n = 0$ for every positive current $T \in c_1(L)$ by Proposition 3.1. Therefore we have $T_{ac}^n(x) = 0 = v(L)\omega(x)^n$ a.e. for every positive current $T \in c_1(L)$, and the proof of Theorem 1.2 is over in that case.

We now assume that L is big, which automatically implies the projectivity of X since the latter will be both a Kähler and a Moishezon manifold. By Theorem 1.1, given $\varepsilon > 0$, we get a modification $\mu : \tilde{X} \to X$ and a decomposition $\mu^\star L = A + E$, where A is an ample **Q**-line bundle, E is an effective **Q**-divisor, and $|v(L) - v(A)| < \varepsilon$. We now choose some Kähler form $\tilde{\omega}$ on $\tilde{X}$, and we apply the Calabi–Yau theorem for every $\delta > 0$ to the Kähler class $c_1(A)$ with respect to the Kähler form $\mu^\star\omega + \delta\tilde{\omega}$, normalized adequately. The ampleness of A yields the equality $v(A) = A^n$, so we get a Kähler form $\tau_\delta \in c_1(A)$ such that

$$\tau_\delta^n = \frac{v(A)}{\int_{\tilde{X}}(\mu^\star\omega + \delta\tilde{\omega})^n}(\mu^\star\omega + \delta\omega)^n \,.$$

Since the set of positive currents in $c_1(A)$ is weakly compact, we can find some weak limit $S = \lim_{\delta\to 0}\tau_\delta$. S is a positive current in $c_1(A)$, and we have $S_{ac}^n \geq v(A)\mu^\star(\omega^n)$ a.e. by semi-continuity (Proposition 2.1). Now consider

$$T_\varepsilon := \mu_\star(S + [E]) \,.$$

It is a closed positive current in $c_1(L)$ and, by Proposition 2.2, we have $T_{\varepsilon,ac} = \mu_\star(S_{ac})$. Since μ is an isomorphism outside sets of measure 0, it is clear that $(\mu_\star(S_{ac}))^n = \mu_\star(S_{ac}^n)$, so that $T_{\varepsilon,ac}^n \geq v(A)\omega^n$ a.e. The data μ, A, E depend on ε; letting $\varepsilon \to 0$, we can one more time select a weak limit $T = \lim_{\varepsilon\to 0} T_\varepsilon$, and apply Proposition 2.2 again to get $T_{ac}^n \geq v(L)\omega^n$ a.e. because $v(A)$ tends to $v(L)$ when ε goes to zero. T is a closed positive current in $c_1(L)$, so we also have $\int_X T_{ac}^n \leq \int_X v(L)\omega^n$ by Proposition 3.1, and this implies that $T_{ac}^n = v(L)\omega^n$ a.e., and in particular $v(L) = \int_X T_{ac}^n$. The proof of Theorem 1.2 is now over.

4. Volume of a Pseudoeffective Class

4.1. *General properties*

In this section, X denotes again a compact Kähler n-fold unless otherwise specified. We propose to extend some results related to the volume of a line bundle, or rather of a rational class $\alpha \in NS(X) \otimes \mathbf{Q} \subset H^{1,1}(X, \mathbf{R})$, to the more general case of an arbitrary class $\alpha \in H^{1,1}(X, \mathbf{R})$. As explained in the introduction, we define the volume of α as follows: if α is not pseudoeffective, we set $v(\alpha) = 0$. Otherwise, we set $v(\alpha) = \sup_T \int_X T_{ac}^n$ with T ranging among the closed positive currents in α. This quantity is finite thanks to Proposition 2.6. We have seen in Sec. 1 that for a nef line bundle L, one has $v(L) = L^n$. This remains true for an arbitrary nef class:

Theorem 4.1. *Let $\alpha \in H^{1,1}(X, \mathbf{R})$ be a nef class. Then one has $v(\alpha) = \alpha^n$.*

The proof is in two steps. We first give the following improvement of Proposition 2.6 for a nef class, which is due to C. Mourougane [13]:

Lemma 4.2. *Given a nef class $\alpha \in H^{1,1}(X, \mathbf{R})$, one has for every positive $T \in \alpha$*

$$\int_X T_{ac}^n \leq \alpha^n \,.$$

Proof. Choose some Kähler form ω, and write as before $T = \theta + dd^c\varphi$ with θ a smooth form. We consider a sequence $T_k^{(1)} = \theta + dd^c\varphi_k^{(1)}$ of smooth forms as given by Theorem 2.3, i.e. such that $T_k^{(1)} \to T$ and $T_k^{(1)}(x) \to T_{ac}(x)$ a.e. with a loss of positivity for $T_k^{(1)}$ controled by the Lelong numbers of T. Since α is nef, there also exists by definition a sequence of smooth functions $\varphi_k^{(2)}$ on X such that $T_k^{(2)} := \theta + dd^c\varphi_k^{(2)}$ has $T_k^{(2)} \geq -\varepsilon_k\omega$ ($T_k^{(2)}$ is of course completely unrelated to T *a priori*). If we set $\varphi_k^{(3)} := \max_\eta(\varphi_k^{(2)} - C_k, \varphi_{j_k}^{(1)})$, then $\varphi_k^{(3)}$ is a smooth function, and the arguments given in the proof of Theorem 2.2 easily show that $T_k^{(3)} := \theta + dd^c\varphi_k^{(3)}$ is a smooth form such that $T_k^{(3)}(x) \to T_{ac}(x)$ a.e. and $T_k^{(3)} \geq -\delta_k\omega$ for some sequence $\delta_k > 0$ converging to zero. Since $T_k^{(3)} + \delta_k\omega$ also converges to T_{ac} a.e. Fatou's lemma yields

$$\int_X T_{ac}^n \leq \liminf_{k\to\infty} \int_X (T_k^{(3)} + \delta_k\omega)^n\,,$$

and the latter integral is just $(\alpha + \delta_k\{\omega\})^n$, thus it converges to α^n. □

As a consequence of Lemma 4.2, we of course get that $v(\alpha) \leq \alpha^n$ for α nef.

To get the converse equality and thus conclude the proof of Theorem 4.1, we again use the Calabi–Yau theorem. If ω is a given Kähler form with $\int_X \omega^n = 1$, for each $\varepsilon > 0$, $\alpha + \varepsilon\{\omega\}$ is a Kähler class since α is nef, hence we can find a Kähler form τ_ε in $\alpha + \varepsilon\{\omega\}$ such that $\tau_\varepsilon^n = (\int(\alpha + \varepsilon\{\omega\})^n)\omega^n$. Since the family τ_ε, $\varepsilon > 0$, represents a bounded set of cohomology classes, it is bounded in mass and we can thus extract some weak limit $T = \lim_{\varepsilon\to 0}\tau_\varepsilon$. By semi-continuity (Proposition 2.1), we get $T_{ac}^n \geq (\int \alpha^n)\omega^n$, and the inequality $v(\alpha) \geq \int \alpha^n$ follows by integrating.

4.2. *A degenerate Calabi–Yau theorem*

In this section, we prove the following singular version of the Calabi–Yau theorem:

Theorem 4.3. *If $\alpha \in H^{1,1}(X, \mathbf{R})$ is a pseudoeffective class and ω is a Kähler form with $\int_X \omega^n = 1$, then there exists a closed positive current T in α such that*

$$T_{ac}(x)^n = v(\alpha)\omega(x)^n$$

almost everywhere.

We first prove Theorem 1.4, which generalizes Fujita's theorem.

Lemma 4.4. *If $\alpha \in H^{1,1}(X, \mathbf{R})$ is a big class, there exists a sequence T_k of Kähler currents with analytic singularities in α such that $\int_X T_{k,ac}^n \to v(\alpha)$.*

Proof. Let $\varepsilon > 0$, and choose some Kähler current T_0 in α. By definition of the volume, there exists a closed positive current S in α such that $\int_X S_{ac}^n > v(\alpha) - \varepsilon$. By Fatou's lemma, we also have

$$\int_X S_{ac}^n \leq \liminf_{\delta\to 0} \int_X ((1-\delta)S + \delta T_0)_{ac}^n\,,$$

thus there exists $\delta > 0$ such that $T_1 := (1-\delta)S + \delta T_0$ is a Kähler current in α with $\int_X T_{1,ac} > v(\alpha) - \varepsilon$. Using Theorem 2.4, we can now choose a sequence T_k of Kähler currents with analytic singularities in α such that $T_{k,ac}(x) \to T_{ac}(x)$ a.e. and the same argument involving Fatou's lemma shows that for k big enough $T := T_k$ is a Kähler current with analytic singularities such that $v(\alpha) - \varepsilon < \int_X T_{ac}^n$, and the lemma follows. ◻

Now we prove Theorem 1.4.

Let α be a big class and let $\varepsilon > 0$. We can choose by Lemma 4.4 a Kähler current with analytic singularities T in α such that $|v(\alpha) - \int_X T_{ac}^n| < \varepsilon$. By Sec. 2.4, there exists a modification $\mu : \tilde{X} \to X$ such that $\mu^\star T = \theta + E$, with θ a smooth form and E an effective **R**-divisor, and we have $\int_X T_{ac}^n = \int_{\tilde{X}} (\mu^\star T)_{ac}^n = \int_{\tilde{X}} \theta^n$. It is true that $\theta \geq \mu^\star \omega$, but θ is not a Kähler form. However, denoting by $E_1, \dots, E_r$ the μ-exceptional prime divisors, it is well known that $\mu^\star\{\omega\} - \{a_1 E_1 + \cdots + a_r E_r\}$ is a Kähler class for some $a_1, \dots, a_r > 0$ (cf. for instance [8, Lemma 3.5]). The class $\{\theta\} - \{a_1 E_1 + \cdots + a_r E_r\}$ is then also Kähler. Since $\{\theta\}$ is nef, $\omega_\delta := \{\theta\} - \frac{\delta}{1+\delta}\{a_1 E_1 + \cdots + a_r E_r\}$ is also a Kähler class for each $\delta > 0$. Since $v(\omega_\delta) = \omega_\delta^n$ tends to $\int_{\tilde{X}} \theta^n = \int_X T_{ac}^n$, we will thus have $|v(\alpha) - v(\omega_\delta)| < \varepsilon$ for $\delta > 0$ small enough, and we obtain the decomposition $\mu^\star \alpha = \omega + \{D\}$ we are after by setting $\omega := \omega_\delta$ and $D := E + \frac{\delta}{1+\delta}\{a_1 E_1 + \cdots + a_r E_r\}$. Theorem 1.4 is proved.

In order to prove Theorem 4.3, we just remark that the arguments given in Sec. 3.2 to prove Theorem 1.2 also prove Theorem 4.3 once the generalization of Fujita's theorem is obtained. As an application, we prove

Proposition 4.5. *The restriction of $\alpha \mapsto v(\alpha)^{1/n}$ to the pseudoeffective cone $\mathcal{E}$ is homogeneous and concave, and is therefore continuous on the big cone $\mathcal{E}^0$. Furthermore, we have $v(\alpha) \geq \limsup_{k\to\infty} v(\alpha_k)$ for every sequence $\alpha_k \in \mathcal{E}$ converging to some α.*

Proof. Let α_1 and α_2 be two pseudoeffective classes, and choose $T_j \in \alpha_j$ a positive current such that $T_{j,ac}(x)^n = v(\alpha_j)\omega(x)^n$ a.e. (we use Theorem 4.3). Since the restriction of $A \mapsto \det(A)^{1/n}$ to the convex cone of Hermitian non-negative matrices is concave and homogeneous, we get $\det(T_{1,ac}(x) + T_{2,ac}(x)) \geq (\det(T_{1,ac}(x))^{1/n} + \det(T_{2,ac}(x))^{1/n})^n = (v(\alpha_1)^{1/n} + v(\alpha_2)^{1/n})^n$ a.e. and thus $v(\alpha_1 + \alpha_2) \geq \int_X \det(T_{1,ac} + T_{2,ac})\omega^n \geq (v(\alpha_1)^{1/n} + v(\alpha_2)^{1/n})^n$. The homogeneity of $\alpha \mapsto v(\alpha)^{1/n}$ being trivial, its concavity ensues. It is well known that a concave map on an open convex subset of some $\mathbf{R}^p$ is continuous, thus the volume is continuous on the interior $\mathcal{E}^0$. To prove the last assertion, select for each k a closed positive current T_k in α_k such that $T_{k,ac}(x)^n = v(\alpha_k)\omega(x)^n$ a.e. By weak compactness, we may assume that T_k converges weakly to some T. T is then a closed positive current in α, and we have $T_{ac}(x)^n \geq \limsup T_{k,ac}(x)^n = \limsup v(\alpha_k)\omega(x)^n$ a.e. by semi-continuity (Proposition 2.1). The result thus follows. ◻

We'd like to thank R.Lazarsfeld for pointing out to us the log-concavity property of the volume, which he previously proved in the context of algebraic geometry.

4.3. *The Grauert–Riemenschneider criterion*

We have seen in Sec. 3.1 that for a pseudoeffective line bundle L on a compact Kähler n-fold X, the existence of a positive current T in $c_1(L)$ with $\int_X T_{ac}^n > 0$ implies that $v(L) > 0$, and thus that L is big. This is a kind of Grauert–Riemenschneider criterion for bigness, which we would like to extend to any (i.e. not necessarily rational) pseudoeffective class. We begin by quoting the following result:

Theorem 4.6 ([8]). *If $\alpha \in H^{1,1}(X, \mathbf{R})$ is a nef class such that $\int \alpha^n > 0$, then α contains a Kähler current.*

The proof of this fact is non-trivial, and is in fact one of the main steps in the proof by J. P. Demailly and M. Paun of their Nakai–Moishezon criterion for Kähler classes. Since $v(\alpha) = \alpha^n$ when α is nef, the following result generalizes Theorem 4.6:

Theorem 4.7. *If X is a compact Kähler manifold, a class $\alpha \in H^{1,1}(X, \mathbf{R})$ is big if and only if $v(\alpha) > 0$.*

Proof. If α is big, it contains a Kähler current T such that $T \geq \omega$ for some Kähler form ω. We then have $T_{ac} \geq \omega$ since $T_{ac} - \omega = (T - \omega)_{ac}$ is the absolutely continuous part of a positive current. This implies that $\int_X T_{ac}^n \geq \int_X \omega^n > 0$, and the volume $v(\alpha)$ is thus positive. To prove the converse direction, we will heavily rely on the proof of Theorem 4.6 given in [8]. If we choose a sequence S_k of closed positive currents in α such that $\int_X S_{k,ac}^n \to v(\alpha)$ and apply Theorem 2.4 combined with Fatou's lemma, we can construct a sequence T_k of closed currents with analytic singularities in α such that $T_k \geq -\varepsilon_k \omega$ and $\int_X (T_{k,ac} + \varepsilon_k \omega)^n \geq c$ for some uniform lower bound $c > 0$. We then choose a modification $\mu_k : X_k \to X$ such that $\mu^\star T_k = \theta_k + E_k$, with $\theta_k \geq -\varepsilon_k \mu^\star \omega$ a smooth form and E_k an real effective divisor. If we apply Theorem 4.6 to the nef class $\alpha_k := \{\theta_k + \varepsilon_k \mu^\star \omega\}$, we get a Kähler current in α_k, and thus a Kähler current $T_k \geq \delta\omega$ in $\alpha + \varepsilon_k\{\omega\}$. But the trouble is that the lower bound δ depends *a priori* on k. What we plan to do is to follow the arguments given in [8] with a special care in the estimates in order to show that the $\delta > 0$ above can chosen uniformly for all k. The element $T_k - \varepsilon_k \omega$ of α will then be a Kähler current for k big enough, as desired. We quote without proof the following result from [8] (cf. Lemma 2.1): □

Lemma 4.8. *Let (X, ω) be a Kähler manifold and $Y \subset X$ be an analytic subset of codimension p. Then there exists a function φ with analytic singularities along Y, a bound $\delta > 0$ and a sequence φ_ε of smooth functions decreasing pointwise to φ such that $\omega_\varepsilon := \omega + dd^c \varphi_\varepsilon$ has $\omega_\varepsilon \geq 2^{-1}\omega$ and*

$$\int_{V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta$$

for every $\varepsilon > 0$, with $V_\varepsilon := \{\varphi < \log \varepsilon\}$.

Let us just give an idea of the proof: let $(f_1, \ldots, f_r)$ be local generators of the ideal $\mathcal{I}_Y$ of Y near some point x_0. The function $\varphi := \frac{1}{2}\log(\sum |f_j|^2)$

has analytic singularities along Y by definition. The wedge product $(dd^c\varphi)^p$ can be defined, and its Siu decomposition writes $(dd^c\varphi)^p = R + \sum[Y_j]$, where the Y_j's are the p-codimensional (local) components of Y through x_0. We then set $\varphi_\varepsilon := \frac{1}{2}\log(\sum|f_j|^2+\varepsilon^2)$ and $\omega_\varepsilon := \omega + dd^c\varphi_\varepsilon$. The volume form $\omega_\varepsilon^p \wedge \omega^{n-p}$ converges weakly to the current $(\omega + dd^c\varphi)^p \wedge \omega^{n-p}$, who has positive mass on Y by what we have said, so the result follows locally. To globalize the construction, one has to glue the data by means of a partition of unity, which procedure entails a loss of positivity.

The next lemma is also extracted from [8], but we will reproduce the proof to show that an explicit estimate can be obtained:

Lemma 4.9. *Let (X,ω) be a compact Kähler manifold and let Y be an analytic subset of X of codimension p. Suppose given the following data: a basis V_ε of neighbourhoods of Y, a family $\omega_\varepsilon \geq 2^{-1}\omega$ of Kähler forms in the class $\{\omega\}$, and a number $\delta > 0$ such that*

$$\int_{V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta$$

for every $\varepsilon > 0$. Then, for every nef class α with $v(\alpha) > 0$, there exists a closed positive current T in α^p such that

$$\int_Y T \wedge \omega^{n-p} \geq \lambda\,,$$

where $\lambda := C_n\delta^2 v(\alpha)/(\int \alpha^{n-p}\wedge\omega^p)v(\omega)$ with $C_n > 0$ depending on n only.

Proof. For each $\varepsilon > 0$, there exists by the Calabi–Yau theorem a smooth Kähler form τ_ε in the Kähler class $\alpha + \varepsilon\{\omega\}$ such that

$$\tau_\varepsilon^n = \frac{v(\alpha+\varepsilon\omega)}{v(\omega)}\omega_\varepsilon^n\,.$$

Denoting by

$$\lambda_1^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon$$

the eigenvalues of τ_ε with respect to ω_ε, we thus find:

(1) $\lambda_1^\varepsilon \cdots \lambda_n^\varepsilon = v(\alpha+\varepsilon\omega)/v(\omega)$,
(2) $\tau_\varepsilon^p \geq (\lambda_1^\varepsilon\cdots\lambda_p^\varepsilon)\omega_\varepsilon^p$,
(3) $\tau_\varepsilon^{n-p}\wedge\omega_\varepsilon^p \geq C_n^{-1}(\lambda_{p+1}^\varepsilon\cdots\lambda_n^\varepsilon)\omega_\varepsilon^n$,

where $C_n = \binom{n}{p}$. Relation (3) implies

$$\int_X (\lambda_{p+1}^\varepsilon\cdots\lambda_n^\varepsilon)\omega_\varepsilon^n \leq C_n\int(\alpha+\varepsilon\omega)^{n-p}\wedge\omega^p =: M_p\,.$$

Thus in particular the set $E_\eta := \{\lambda^\varepsilon_{p+1}\cdots\lambda^\varepsilon_n \ge M_p/\eta\}$ has $\int_{E_\eta}\omega_\varepsilon^n \le \eta$ for every $\eta > 0$. From (1) and (2), we deduce:

$$\int_{V_\varepsilon}\tau_\varepsilon^p\wedge\omega^{n-p} \ge \frac{v(\alpha+\varepsilon\omega)}{v(\omega)}\int_{V_\varepsilon}(\lambda^\varepsilon_{p+1}\cdots\lambda^\varepsilon_n)^{-1}\omega_\varepsilon^p\wedge\omega^{n-p}$$
$$\ge \frac{v(\alpha+\varepsilon\omega)}{v(\omega)}\int_{V_\varepsilon-E_\eta}\frac{\eta}{M_p}\omega_\varepsilon^p\wedge\omega^{n-p}\,.$$

Observe that

$$\int_{V_\varepsilon-E_\eta}\omega_\varepsilon^p\wedge\omega^{n-p} \ge \int_{V_\varepsilon}\omega_\varepsilon^p\wedge\omega^{n-p} - \int_{E_\eta}\omega_\varepsilon^p\wedge\omega^{n-p}\,.$$

The first integral on the right is greater than δ by assumption, and since $\omega_\varepsilon^p\wedge\omega^{n-p}\le 2^{n-p}\omega_\varepsilon^n$ (because $\omega_\varepsilon \ge 2^{-1}\omega$), the second integral on the right will be less than $2^{n-p}\eta$. Combining all this yields in the end

$$\int_{V_\varepsilon}\tau_\varepsilon^p\wedge\omega^{n-p} \ge (v(\alpha+\varepsilon\omega)\eta)(v(\omega)M_p)^{-1}(\delta - 2^{n-p}\eta)\,.$$

We now choose $\eta := \delta/2^{n-p+1}$, so that the last inequality becomes $\int_{V_\varepsilon}\tau_\varepsilon^p\wedge\omega^{n-p}\ge\lambda_\varepsilon$, where $\lambda_\varepsilon > 0$ converges to the λ is defined in the statement of the lemma. Since τ_ε^p belongs to a fixed cohomology class, we can find some weak limit $T = \lim_{\varepsilon\to 0}\tau_\varepsilon^p$, which is then a closed positive (p,p)-current belonging to the cohomology class α^p, and such that $\int_Y T\wedge\omega^{n-p}\ge\lambda$. □

We now come back to the proof of Theorem 4.7. Recall that (X,ω) is a Kähler manifold with $v(\omega) = 1$, $T_k \ge -\varepsilon_k\omega$ is a sequence of closed currents with analytic singularities in the pseudoeffective class $\alpha\in H^{1,1}(X,\mathbf{R})$, and a sequence of modifications $\mu_k : X_k \to X$ is given such that $\mu_k^\star T_k = \theta_k + E_k$ and the nef class $\alpha_k := \{\theta_k + \varepsilon_k\mu_k^\star\omega\}$ has volume $v(\alpha_k)$ uniformly bounded away from zero. We denote by $\tilde\mu_k : \tilde X_k \to \tilde X$ the product map $\mu_k\times\mu_k : X_k\times X_k \to X\times X$, and select on each $\tilde X_k$ a Kähler form $\tilde\omega_k$. On $\tilde X$, we set $\tilde\omega := p^\star\omega + q^\star\omega$ and $\tilde\alpha_k := p_k^\star\alpha_k + q_k^\star\alpha_k$, where p, q (resp. p_k, q_k) denote the two projections from the product $\tilde X$ (resp. $\tilde X_k$). One computes that $v(\tilde\alpha_k) = \binom{2n}{n}v(\alpha_k)^2 \ge c > 0$ for some $c > 0$, since $v(\alpha_k)$ is uniformly bounded away from zero. Finally, we denote by Δ (respectively Δ_k) the diagonal in $\tilde X$ (respectively $\tilde X_k$).

We now state the next

Lemma 4.10. *There exists a uniform constant $\eta > 0$ such that, for each k, the class $\tilde\alpha_k^n$ contains a closed positive (n,n)-current Θ_k with*

$$\Theta_k \ge \eta[\Delta_k]\,.$$

Proof. If we apply Lemma 4.9 with $Y = \Delta$, the diagonal of $\tilde X = X\times X$, we get a basis of neighbourhoods V_ε of Δ, a family of Kähler forms $\omega_\varepsilon \ge 2^{-1}\tilde\omega$ in $\{\tilde\omega\}$ and a uniform bound $\delta > 0$ such that $\int_{V_\varepsilon} \omega_\varepsilon^n \wedge \omega^n \geq \delta$ for every $\varepsilon > 0$. Fix some k, and let $\rho > 0$, which we shall let tend to zero afterwards. Working on $\tilde{X}_k$, we set

$$\omega_k := \tilde{\mu}_k^\star \tilde{\omega} + \rho \tilde{\omega}_k ,$$

and we have a family

$$\omega_{k,\varepsilon} := \tilde{\mu}_k^\star \omega_\varepsilon + \rho \tilde{\omega}_k$$

of Kähler forms in the class $\{\omega_k\}$ such that $\omega_{k,\varepsilon} \geq 2^{-1}\omega_k$. The family $V_{k,\varepsilon} := \tilde{\mu}_k^{-1}(V_\varepsilon)$ defines a basis of neighbourhoods of $Y_k := \tilde{\mu}_k^{-1}(\Delta)$, and we have $\int_{V_{k,\varepsilon}} \omega_{k,\varepsilon}^n \wedge \omega_k^n \geq \delta > 0$ for the same $\delta > 0$ as on $\tilde{X}$. Since $\tilde{\alpha}_k$ is a nef class with positive volume, Lemma 4.9 yields the existence of a closed positive (n,n)-current T_k in $\tilde{\alpha}_k^n$ such that

$$\int_{Y_k} T_k \wedge \omega_k^n \geq \lambda_k ,$$

with $\lambda_k = C_n \delta^2 c / (\int \tilde{\alpha}_k^n \wedge \omega_k^n) v(\omega_k) > 0$ since $v(\tilde{\alpha}_k) \geq c > 0$. As we said, this data was depending on $\rho > 0$. We can find some weak limit $\Theta_k = \lim_{\rho \to 0} T_k$, which is a closed positive (n,n)-current in $\tilde{\alpha}_k^n$ such that

$$\int_{Y_k} \Theta_k \wedge \tilde{\mu}_k^\star \tilde{\omega}^n \geq \eta_k ,$$

with $\eta_k = C_n \delta^2 c / (\int \tilde{\alpha}_k^n \wedge \tilde{\mu}_k^\star \tilde{\omega}^n) v(\tilde{\omega})$ because $\omega_k = \tilde{\mu}_k^\star \tilde{\omega} + \rho \tilde{\omega}_k \to \tilde{\mu}_k^\star \tilde{\omega}$ as $\rho \to 0$, and $v(\tilde{\mu}_k^\star \tilde{\omega}) = v(\tilde{\omega})$.

We deduce from this that

$$\int_{\Delta_k} \Theta_k \wedge \tilde{\mu}_k^\star \tilde{\omega}^n \geq \eta_k ,$$

since any component of $Y_k = \tilde{\mu}_k^{-1}(\Delta)$ distinct from Δ_k is sent by $\tilde{\mu}_k$ to an analytic set of dimension $< n$, on which the bidimension (n,n) closed positive current $\tilde{\mu}_{k,\star}\Theta_k$ carries no mass. By Sec. 2.2, we have

$$\Theta_k \geq \chi_{\Delta_k} \Theta_k = \nu(\Theta_k, \Delta_k)[\Delta_k] ,$$

where $\nu(\Theta_k, \Delta_k)$ is the generic Lelong number of Θ_k along Δ_k, and χ_{Δ_k} is the characteristic function; our goal is now to show that $\nu(\Theta_k, \Delta_k)$ is uniformly bounded away from zero, which will conclude the proof of Lemma 4.10. Since $\int [\Delta_k] \wedge \tilde{\mu}_k^\star \tilde{\omega}^n = \int [\Delta] \wedge \tilde{\omega}^n$, we deduce that

$$\nu(\Theta_k, \Delta_k) \int [\Delta] \wedge \tilde{\omega}^n = \int_{\Delta_k} \Theta_k \wedge \tilde{\mu}_k^\star \tilde{\omega}^n \geq \eta_k ,$$

so it remains to take care of $\eta_k = C_n \delta^2 c / (\int \tilde{\alpha}_k^n \wedge \tilde{\mu}_k^\star \tilde{\omega}^n) v(\tilde{\omega})$. All we have to do is to bound $\int_{X_k \times X_k} \tilde{\alpha}_k^n \wedge \tilde{\mu}_k^\star \omega^n$ from above. But the latter integral can be expressed in terms of $\int_{X_k} \alpha_k^l \wedge \mu_k^\star \omega^{n-1}$, $l = 0, \ldots, n$, which are just $\int_X (T_{k,ac} + \varepsilon\omega)^l \wedge \omega^{n-1}$. These integrals are uniformly bounded by Proposition 2.6, so the proof of Lemma 4.10 is over.

We can now conclude the proof of Theorem 4.7, along the path of [8]. Let p, q denote the projections $\tilde{X}_k = X_k \times X_k \to X_k$. Since $\tau_k := \mu_k^\star \omega$ is a smooth semi-positive form on X_k, the current $S_k := q_\star(T_k \wedge p^\star \tau_k)$ is a closed current in the $(1,1)$-class $(\tilde{\alpha}_k^n)_\star(\tau_k)$ such that $S_k \geq \eta q_\star([\Delta_k] \wedge p^\star \tau_k) = \eta \tau_k$. One easily computes that the class $(\tilde{\alpha}_k^n)_\star(\tau_k)$ is just $n(\int \alpha_k^{n-1} \wedge \tau_k)\alpha_k$, so that $U_k := (n(\int \alpha_k^{n-1} \wedge \tau_k)^{-1} S_k$ is a closed positive current in α_k with $U_k \geq c_k \mu_k^\star \omega$ for $c_k := (n(\int \alpha_k^{n-1} \wedge \tau_k)^{-1}\eta$. Finally, the push-forward $\mu_{k,\star} U_k$ is a Kähler current on X, bounded from below by $c_k \omega$ and who belongs to the class $\mu_{k,\star}\alpha_k = \alpha + \varepsilon_k\{\omega\}$. Since $\varepsilon_k \to 0$, it remains therefore to ensure that $(\int \alpha_k^{n-1} \wedge \tau_k)^{-1}$ is uniformly bounded away from zero. But we have $\int \alpha_k^{n-1} \wedge \tau_k = \int (\theta_k + \varepsilon_k \mu_k^\star \omega)^{n-1} \wedge \mu_k^\star \omega = \int_X (T_{k,ac} + \varepsilon_k \omega)^{n-1} \wedge \omega$, and the latter quantity is uniformly bounded by Proposition 2.6. The proof of Theorem 4.7 is over. □

Corollary 4.11. *The volume $v : H^{1,1}(X, \mathbf{R}) \to \mathbf{R}$ is a continuous function.*

Proof. In view of Proposition 4.5 and the fact that $v(\alpha) = 0$ for a class $\alpha \in H^{1,1}(X, \mathbf{R})$ lying outside $\mathcal{E}$, the continuity of the volume on the whole of $H^{1,1}(X, \mathbf{R})$ is equivalent to the fact that $v(\alpha) = 0$ for $\alpha \in \partial\mathcal{E}$, which is exactly the assertion of Theorem 4.7. □

The following result is well known for line bundles:

Proposition 4.12. *Let $f : Y \to X$ be a surjective holomorphic map between compact Kähler n-folds, and α, β be real $(1,1)$-cohomology classes on X and Y respectively. Then:*

(i) *α big if and only if $f^\star\alpha$ is big,*
(ii) *$f_\star\beta$ is big if β is, but not conversely.*

Proof. The second point stems from the fact from the easy fact that pushing forward a Kähler current yields a Kähler current; since $f_\star f^\star \alpha = (\deg f)\alpha$, this also proves sufficiency in (i). If α is big, we can choose a Kähler current T in it. Since f is generically finite, we have $\int_Y (f^\star T)_{ac}^n = (\deg f) \int_X T_{ac}^n > 0$, thus $f^\star\alpha$ is big. □

4.4. *Miscellaneous*

4.4.1. *The non-Kähler case*

Thanks to Proposition 2.2, the property of uniform boundedness of the mass (Proposition 2.6) is invariant under modifications, thus still holds true on any compact complex manifold in the Fujiki class $\mathcal{C}$ (a Fujiki manifold for short). Since a compact complex manifold is Fujiki if and only if it carries a big class (cf. [8]), all the results which are not an equality of Calabi–Yau type remain true on a Fujiki manifold. It is tempting to think that the Grauert–Riemenschneider criterion (Theorem 4.7) holds true in general, which comes down to the following conjecture (cf. also [8]):

Conjecture: *If a compact complex manifold X carries a closed positive $(1,1)$-current T with $\int_X T_{ac}^n > 0$, then X is Fujiki.*

This conjecture is true for $\dim X = 2$, as we shall see.

4.4.2. *The case of a surface*

In this section, we assume that X is a compact complex surface. Consider a closed positive $(1,1)$-current T on X, and let $T = R + \sum \nu(T,D)D$ be its Siu decomposition. It is well known that the $\partial\bar{\partial}$-cohomology class $\{R\}$ is nef, and is even Kähler when T is a Kähler current. This is for instance a straightforward consequence of a result of M. Paun given in [8], which says that a pseudoeffective class (respectively a big class) $\{T\}$ is nef (respectively Kähler) if and only if its restriction $\{T\}_{|Y}$ is nef (respectively Kähler) for every irreducible analytic subset contained in some Lelong super-level set $E_c(T)$.

In particular, X is a Fujiki surface if and only if it is a Kähler surface. Now, if we assume that $\int_X T_{ac}^2 > 0$, then since $\{R\}$ is nef we get $\{R\}^2 \geq \int_X R_{ac}^2 = \int_X T_{ac}^2 > 0$, using Lemma 4.2 (which holds on a surface without the Kähler assumption as one immediately sees). The intersection form on $H^{1,1}(X, \mathbf{R})$ can therefore not be negative definite, and this compels $b_1(X)$ to be even by classical results due to Kodaira (cf. e.g. [12]). This in turn forces X to be Kähler by the main result of [12] or [2]. Therefore the above conjecture is true on a surface.

4.4.3. *Behaviour of the volume in deformations*

In this last part we prove the following

Proposition 4.13. *The volume is upper-semicontinuous in Kähler deformations in the following sense: if $\mathcal{X} \to S$ is a deformation of Kähler n-folds and $\alpha \in H^{1,1}(X, \mathbf{R})$ is a pseudoeffective class on the central fibre $X_0 =: X$, then one has*

$$v(\alpha) \geq \limsup_{k \to +\infty} v(\alpha_k)$$

for every sequence $\alpha_k \in H^{1,1}(X_{t_k}, \mathbf{R})$ converging to α (with $t_k \to 0$).

Proof. Upon shrinking the base S, we may assume that the deformation is topologically trivial, and that there exist Kähler metrics ω_t on X_t depending smoothly on t; we normalize them so that $\int_{X_t} \omega_t^n = 1$. Let then $\alpha_k \in H^{1,1}(X_{t_k}, \mathbf{R}) \subset H^2(X, \mathbf{R})$ be a sequence of classes converging to α. We may assume that $v(\alpha_k) > 0$ for k big enough, otherwise the result is trivial. α_k is then pseudoeffective (by definition of the volume), and we can choose a closed positive $(1,1)$-current $T_k \in \alpha_k$ on X_{t_k} such that $T_{k,ac}^n = v(\alpha_k)\omega_{t_k}^n$ almost everywhere. Since the sequence α_k converges, it is bounded, thus T_k is bounded in mass, and we may assume that T_k converges to a closed positive current T in α. By semi-continuity (Proposition 2.1), we get $T_{ac}^n \geq \limsup v(\alpha_k)\omega^n$, which yields the result. ∎

As a consequence, we get the following quantitative version of a result of Huybrechts [10]:

Corollary 4.14. *If X is a compact hyper-Kähler manifold, then $v(\alpha) \geq \int \alpha^{\dim X}$ for each class $\alpha \in H^{1,1}(X, \mathbf{R})$ lying in the positive quadratic cone of X.*

Proof. We can choose a sequence of very general points t_k converging to 0 in the universal deformation space of X such that α is a limit of classes $\alpha_k \in H^{1,1}(X_{t_k}, \mathbf{R})$ lying in the positive quadratic cone of X_{t_k}. By [10], the positive quadratic cone of a very general hyper-Kähler manifold coincides with its Kähler cone, therefore α_k is a Kähler class and we have $v(\alpha_k) = \int \alpha_k^{\dim X}$ for each k by Theorem 4.1. The result then follows from the preceding proposition. □

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