Chapter 7
Monge–Ampère Equations on Complex Manifolds with Boundary

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Abstract We survey the proofs of two fundamental results on the resolution of Monge–Ampère equations on complex manifolds with boundary. The first result guarantees the existence of smooth solutions to non-degenerate complex Monge–Ampère equations admitting subsolutions, it is a continuation of results due to Caffarelli–Kohn–Nirenberg–Spruck, Guan and Błocki. The second result shows the existence of almost C^2 solutions to degenerate complex Monge–Ampère equations admitting subsolutions, and yields as a special case X.X.Chen's result on the existence of almost C^2 geodesics in the space of Kähler metrics.

7.1 Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension $\dim X =: n$. Let ω_0, ω_1 be two Kähler forms in the cohomology class of ω, which can be written as $\omega_j = \omega + dd^c u_j$ for some $u_j \in C^\infty(X)$, $j = 0, 1$ by the dd^c-lemma. If we let $A \subset \mathbb{C}$ be a closed annulus then the functions u_0, u_1 induce a radially symmetric smooth function u on the boundary of $M := X \times A$ such that $\omega + dd^c u > 0$ on each X-slice. As we saw in Kolev's lectures (cf. [Kol, Proposition 6]) it has been observed by Semmes and Donaldson that there exists a (smooth) geodesic

$$\omega_t = \omega + dd^c u_t,\ 0 \le t \le 1$$

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joining ω_0 to ω_1 in the space of Kähler metrics cohomologous to ω iff the equations

$$(\omega + dd^c v)^{n+1} = 0,\ v|_{\partial M} = u, \tag{7.1}$$

i.e. the Dirichlet problem for a degenerate complex Monge–Ampère equation on the complex manifold with boundary M, admits a solution $v \in C^\infty(M)$ such that $\omega + dd^c v$ is furthermore *positive on X-slices* in the sense that

$$(\omega + dd^c v)|_{X\times z} > 0 \text{ for each } z \in A. \tag{7.2}$$

Here ω is identified with its pull back to M, a semipositive $(1,1)$-form on which is however *not* a Kähler form (since it vanishes in the A-directions).

The existence of $v \in C^\infty(M)$ satisfying both (7.1) and (7.2) is a very difficult analytic problem. Indeed Donaldson showed in [Don02, Theorem 2] that even for the simpler case where $M = X \times D$ with $D \subset \mathbb{C}$ a closed disc there always exist boundary data $u \in C^\infty(\partial M)$ with $\omega + dd^c u > 0$ on X-slices such that no smooth solution v to both (7.1) and (7.2) exists. Note however that the boundary data $u \in C^\infty(\partial M)$ is a priori not radially symmetric in Donaldson's construction.

It is easily seen that (7.1) and (7.2) imply that $\omega + dd^c v \geq 0$ on M, i.e. v is ω-plurisubharmonic (ω-psh for short) on the product space $M = X \times A$. The maximum principle for the Monge–Ampère operator shows that there exists at most one continuous ω-psh function v on M satisfying (7.1) (in the weak sense of Bedford–Taylor, cf. Corollary 7.7). The problem at hand therefore splits in two parts:

1. Show that (7.1) admits a smooth ω-psh solution $v \in C^\infty(M)$.
2. Show that this solution must satisfy $\omega + dd^c v > 0$ on X-slices.

More generally let $M = X \times S$ with S a compact Riemann surface with boundary. Then M is a complex manifold with boundary such that ∂M is Levi flat, and M has the property that any smooth function φ on ∂M such that $\omega + dd^c\varphi > 0$ on X-slices admits a smooth extension $\widetilde{\varphi}$ to M such that $\eta := \omega + dd^c\widetilde{\varphi} > 0$ on M (cf. Proposition 7.10 below). Since an ω-psh function v satisfies (7.1) iff $\psi := v - \widetilde{\varphi}$ satisfies

$$(\eta + dd^c\psi)^m = 0 \text{ and } \psi|_{\partial M} = 0 \tag{7.3}$$

we can trade the semipositive form ω for an actual Kähler form η.

We are going to present essentially self-contained proofs of the following general results, which contain as special cases [CKNS85] and [Che00]:

Theorem A. *Let (M, η) be an m-dimensional compact Kähler manifold with boundary. Given $\varphi \in C^\infty(\partial M)$ and a smooth positive volume form μ there exists a smooth η-psh solution ψ to the Dirichlet problem. In that case, the solution ψ is furthermore unique.*

$$\begin{cases} (\eta + dd^c\psi)^m = \mu \\ \psi|_{\partial M} = \varphi \end{cases}$$

iff there exists a subsolution, i.e. a smooth η-psh function $\widetilde{\varphi}$ such that

$$\begin{cases} (\eta + dd^c\widetilde{\varphi})^m \geq \mu \\ \widetilde{\varphi}|_{\partial M} = \varphi \end{cases}$$

Theorem B. *Let (M, η) be an m-dimensional compact Kähler manifold with boundary. Let $\varphi \in C^\infty(\partial M)$ and assume that φ admits a smooth η-psh extension $\widetilde{\varphi} \in C^\infty(M)$. Then we have:*

(i) There exists a unique Lipschitz continuous η-psh function ψ such that

$$\begin{cases} (\eta + dd^c\psi)^m = 0 \\ \psi|_{\partial M} = \varphi \end{cases}$$

(ii) If we assume furthermore that ∂M is weakly pseudoconcave *then $dd^c\psi$ has L^∞_{loc} coefficients.*

See Definition 7.4 for the definition of weakly pseudoconcave in this setting. Note that $dd^c\psi \in L^\infty_{\mathrm{loc}}$ is equivalent to ψ having bounded Laplacian on M (since ψ is quasi-psh). It implies that $\psi \in C^{1,\alpha}(M)$ for each $\alpha < 1$ by usual elliptic regularity but it is however a priori weaker than $\psi \in C^{1,1}(M)$, which means that the full *real* Hessian of ψ is bounded (cf. Sect. 7.2.3).

History of the results. We claim no originality in the proofs of Theorems A and B, which as we shall see are a combination of techniques and ideas from [Yau78, CKNS85, Gua98, Che00, Bl09a, Bl09b, PS09]. But since the history of these two results happens to be somewhat complicated we find it worthwhile to discuss it here in some detail.

Let us first consider the case where $M \subset \mathbb{C}^m$ is a smooth bounded domain (and $\eta = dd^c|z|^2$ for instance).

If ∂M is furthermore assumed to be strictly pseudoconvex then a subsolution always exists (cf. Proposition 7.10) and Theorem A was proved in that case in [CKNS85, Theorem 1.1]. A special case of [BT76] Theorem D proves (i) of Theorem B. The solution ψ was furthermore shown in [BT76, Theorem D] to be locally $C^{1,1}$ in the interior when M is a ball (cf. [GZ09] in this volume), and [Kry89] shows that $\psi \in C^{1,1}$ up to the boundary in the general case. We refer to Delarue's lecture in this volume [Del09] for a presentation of Krylov's results, which rely on completely different probabilistic tools in the setting of optimal control. It is interesting to note that $\psi \in C^{1,1}(M)$ up to boundary is not even known for a ball in $\mathbb{C}^2$ using barrier arguments.

For a general smooth bounded domain Theorem A was obtained in [Gua98] by improving the barrier arguments of [CKNS85], and Theorem B might follow as well from [Kry89].

Let us now consider the case where (M, η) is an arbitrary compact Kähler manifold with boundary. The first general results were obtained in [Che00] by combining techniques from [Yau78] (which settled the analogue of Theorem A when M has no boundary) and [Gua98] with a blow-up argument.

In fact [Che00] proved (ii) of Theorem B when $M = X \times S$ is a product of a compact Kähler manifold X with a compact Riemann surface with boundary S, but the product structure only turns out to matter near the boundary, so that [Che00] basically contains the proof of Theorem B when ∂M is Levi flat as was observed in [PS07, Lemma 1]. However there appears to be a small difficulty in X.X. Chen's argument which will be discussed in Sect. 7.3.2. This difficulty was subsequently settled in [Bl09b], which also provided a proof of Theorem A in the general case. We add here the minor observation that the proof goes through in the weakly pseudoconcave case as well.

Finally part (i) of Theorem B is a direct consequence of Theorem A combined with Błocki's gradient estimate [Bl09a]. It is a special case of [PS09, Theorem 2].

Nota Bene. What follows is an expanded set of notes written by Sébastien Boucksom, after the lecture he delivered in Marseille, March 2009. A first draft of these notes had originally been written by Benoît Claudon and Philippe Eyssidieux.

7.2 Preliminaries

7.2.1 Complex Manifolds with Boundary

We recall the following definitions.

Definition 7.1 *A* complex manifold with boundary M *is a C^∞-manifold with boundary endowed with a system of coordinate patches*

$$\Psi_j : U_j \simeq \{z \in B,\, r_j(z) \leq 0\}$$

where B denotes the open unit ball in $\mathbb{C}^m$, r_j is a local defining function*, i.e. a smooth function on $\overline{B}$ with $dr_j \neq 0$ along $\{r_j = 0\}$ and $\Psi_j \circ \Psi_i^{-1}$ is holomorphic on $\Psi_i(U_i \cap U_j) \cap \{r_i < 0\}$.*

The *holomorphic tangent bundle* of ∂M is the largest complex subbundle of T_M contained in $T_{\partial M}$, i.e.

$$T^h_{\partial M} = T_{\partial M} \cap J T_{\partial M}$$

where $J : T_M \to T_M$ denotes the complex structure of M.

We will use the following elementary calculus lemma.

Lemma 7.2 *Let r be a smooth function defined near 0 in $\mathbb{R}^d$ with coordinates $t_1, ..., t_d$. Assume that $r_{t_d}(0) = -1$ and $r_{t_i}(0) = 0$ for all $i < d$. Then the restrictions of $t_1, ..., t_{d-1}$ to $N := \{r = 0\}$ yield local coordinates on N and for each smooth function v near 0 in $\mathbb{R}^d$ we have*

$$(v|_N)_{t_i|_N}(0) = v_{t_i}(0) + v_{t_p}(0) r_{t_i}(0)$$

and

$$(v|N)_{t_i|_N t_j|_N}(0) = v_{t_i t_j}(0) + v_{t_p}(0) r_{t_i t_j}(0)$$

for all $i, j < d$

Lemma 7.3 *Let M be a complex manifold with boundary and let r be a local defining function of ∂M. If v is a smooth function on M such that $v|_{\partial M} \equiv 0$ and ν is a local vector field that is normal to ∂M then we have*

$$dd^c v|_{T^h_{\partial M}} = \frac{\nu \cdot v}{\nu \cdot r} dd^c r|_{T^h_{\partial M}}.$$

Proof. We can choose the local coordinates $z_1, ..., z_m$ at a given point $0 \in \partial M$ such that

$$T^h_{\partial M,0} = \mathrm{Vect}\left(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_{m-1}}\right)$$

and $\nu = \partial/\partial x_m$ at 0. Then the result directly follows from Lemma 7.2. □

Given an outward pointing normal ν to ∂M (i.e. $\nu \cdot r > 0$ along ∂M) the Hermitian form $L_{\partial M,\nu}$ on $T^h_{\partial M}$ defined by $(\nu \cdot r)^{-1} dd^c r|_{T^h_{\partial M}}$ is thus independent of the choice of r and is called the *Levi form* of ∂M (with respect to ν).

Definition 7.4 *If M is a complex manifold with boundary then ∂M is said to be weakly (resp. strictly) pseudoconcave (resp. pseudoconvex) if the Levi form $L_{\partial M,\nu}$ of ∂M (with respect to an outward pointing normal) satisfies $L_{\partial M,\nu} \leq 0$ (resp. < 0, ≥ 0, > 0).*

7.2.2 Maximum Principles

We first state a simple version of the maximum principle for complex Monge–Ampère equations.

Proposition 7.5 *Let (M, η) be a compact Kähler manifold with boundary and let $\psi_1, \psi_2 \in C^\infty(M)$ be two strictly η-psh functions such that*

(i) $\psi_1 \le \psi_2$ *on* ∂M.
(ii) $(\eta + dd^c\psi_1)^m \ge (\eta + dd^c\psi_2)^m$.

Then we have $\psi_1 \le \psi_2$ *on* M.

Proof. The following argument is basically due to Calabi [Cal55] (compare also [CKNS85] Lemma 1.1, [Bl09b] Proposition 2.1). We write

$$0 \le (\eta + dd^c\psi_1)^m - (\eta + dd^c\psi_2)^m = dd^c(\psi_1 - \psi_2) \wedge T \tag{7.4}$$

with

$$T := \sum_{j=0}^{m-1} (\eta + dd^c\psi_1)^j \wedge (\eta + dd^c\psi_2)^{m-1-j},$$

which is a (strictly) positive form of bidegree $(1,1)$. We thus see that $u := \psi_2 - \psi_1$ satisfies $Lu \ge 0$ where $Lu := dd^c u \wedge T$ is a second order elliptic operator, and the result follows from the usual (linear) maximum principle.

In order to get uniqueness in Theorem B we prove the following version of the comparison principle.

Proposition 7.6 *Let* $\varphi, \psi \in C^0(M)$ *be two* η*-psh functions such that* $\varphi \le \psi$ *on* ∂M. *Then we have*

$$\int_{\{\psi<\varphi\}} (\eta + dd^c\varphi)^m \le \int_{\{\psi<\varphi\}} (\eta + dd^c\psi)^m.$$

Here the Monge–Ampère measures are defined in the sense of Bedford–Taylor.

Proof. Let $\delta > 0$ and set $\Omega := \{\psi < \varphi - \delta\}$. For each $\varepsilon > 0$ set

$$\varphi_\varepsilon := \max(\varphi - \delta, \psi + \varepsilon).$$

We then have $\varphi_\varepsilon = \psi + \varepsilon$ in a neighbourhood of $\partial\Omega$, thus

$$\int_\Omega (\eta + dd^c\varphi_\varepsilon)^m = \int_\Omega (\eta + dd^c\psi)^m.$$

Indeed we have $\partial\Omega \cap \partial M = \emptyset$ since $\varphi \le \psi$ on ∂M so the result follows from Stokes' theorem since

$$(\eta + dd^c\varphi_\varepsilon)^m - (\eta + dd^c\psi)^m$$

is exact by (7.4).

On the other hand φ_ε decreases to $\varphi - \delta$ as $\varepsilon \to 0$ thus Bedford–Taylor's monotone continuity theorem for the Monge–Ampère operator implies that

$$(\eta + dd^c\varphi_\varepsilon)^m \to (\eta + dd^c\varphi)^m$$

in the weak topology and we get

$$\int_\Omega (\eta + dd^c\varphi)^m \le \liminf_{\varepsilon\to 0} \int_\Omega (\eta + dd^c\varphi_\varepsilon)^m.$$

We have thus proved

$$\int_{\{\psi<\varphi-\delta\}} (\eta + dd^c\varphi)^m \le \int_{\{\psi<\varphi-\delta\}} (\eta + dd^c\psi)^m$$

for each $\delta > 0$ and the result follows by monotone convergence. □

Corollary 7.7 *Let $\psi \in C^0(M)$ be an η-psh function such that $(\eta+dd^c\psi)^m = 0$. Then for every continuous η-psh function φ on M we have*

$$\sup_M(\varphi - \psi) = \sup_{\partial M}(\varphi - \psi).$$

Proof. Upon adding a constant we may assume that $\sup_{\partial M}(\varphi - \psi) = 0$ and we have to show that $\varphi \le \psi$. For each $0 < \delta \ll 1$ we have

$$\begin{aligned}\int_{\{\psi<(1-\delta)\varphi\}} (\delta\eta)^m &\le \int_{\{\psi<(1-\delta)\varphi\}} (\eta + (1-\delta)dd^c\varphi)^m \\ &\le \int_{\{\psi<(1-\delta)\varphi\}} (\eta + dd^c\psi)^m = 0\end{aligned}$$

by the comparison principle, thus $\psi \ge (1-\delta)\varphi$ holds for each $\delta > 0$ and the result follows.

7.2.3 Elliptic Regularity

In this section we quickly recall some facts about second order linear elliptic PDE's. Let Δ be the Laplacian (with respect to a Riemannian metric g) locally near $0 \in \mathbb{R}^d$ and let u be a distribution near 0. By [GT83] we have

$$\Delta u \in L^p \Longrightarrow u \in L^p_2 \tag{7.5}$$

for each $1 \le p < +\infty$. Here L^p_k denotes the Sobolev space of functions whose derivatives of order at most k belongs to L^p (locally). We thus have

$$L^\infty_2 = C^{1,1}.$$

Note however that (7.5) fails in general when $p = \infty$. Indeed the following example can be found in [GT83]:

$$u(x, y) := |x||y| \log(|x| + |y|)$$

has bounded Laplacian near $0 \in \mathbb{R}^2$ but $\frac{\partial^2 u}{\partial x \partial y}$ is not locally bounded near 0. On the other hand we have

Lemma 7.8 *Suppose that $\Delta u \in L^\infty$ locally. Then we have $u \in C^{1,\alpha}$ for each $\alpha < 1$.*

Proof. By (7.5) we have $u \in L^p_2$ for each finite $p \geq 1$. But

$$L^p_2 \subset C^{1,\alpha}$$

for each $\alpha < 1 - d/p$ by Sobolev's embedding theorem and the result follows.

7.2.4 Miscellanea

Recall that the *trace* $\mathrm{tr}_\beta(\alpha)$ of a $(1,1)$-form α with respect to a positive $(1,1)$-form β is defined as the sum of the eigenvalues of α with respect to β. It satisfies

$$\mathrm{tr}_\beta(\alpha) = m \frac{\alpha \wedge \beta^{m-1}}{\beta^m}$$

We will use the following reformulation of the arithmetico-geometric inequality.

Proposition 7.9 *Let α, β be two positive $(1,1)$-forms on M. Then we have*

$$\frac{1}{m} \mathrm{tr}_\beta(\alpha) \geq \left(\frac{\alpha^m}{\beta^m} \right)^{1/m}.$$

Finally we state a general extension result.

Proposition 7.10 *Let (X, ω) be a compact Kähler manifold (without boundary) and let Y be a compact complex manifold with boundary such that there exists a smooth strictly psh function χ on Y with $\chi|_{\partial Y} = 0$.*

Then every $\varphi \in C^\infty(X \times \partial Y)$ which is a strictly ω-psh on X-slices admits a strictly ω-psh extension in $\widetilde{\varphi} \in C^\infty(X \times Y)$.

Note that the condition on Y is also necessary (apply the extension property to $\varphi = 0$ and X a point) and implies that ∂Y is strictly pseudoconvex.

Proof. Let U be an open neighbourhood of ∂Y in Y with a smooth retraction $\rho : U \to \partial Y$ and let $0 \leq \vartheta \leq 1$ be a smooth function with compact support in

U such that $\vartheta \equiv 1$ on a neighbourhood of ∂Y. Then $\psi(x, y) := \vartheta(y)\varphi(x, \rho(y))$ is smooth and strictly ω-psh on X-slices. Thus setting $\widetilde{\varphi} := \psi + C\chi$ with $C \gg 1$ yields the desired extension of φ.

7.3 A Priori Estimates

Let (M, η) be a compact Kähler manifold with boundary of complex dimension m and denote by Δ the analyst's Laplacian with respect to η so that

$$\Delta u = \mathrm{tr}_\eta(dd^c u)$$

for every function u.

The following a priori estimate is the key to the proof of Theorem A.

Theorem 7.11 *Let (M, η) be a compact Kähler manifold with boundary of complex dimension m. Then for every $A > 0$ there exists $C, \alpha > 0$ such that the following holds: given a non-positive function $F \in C^\infty(M)$ with*

$$\|F\|_{C^2(M)} \leq A$$

and a smooth η-psh function ψ on M with

$$(\eta + dd^c\psi)^m = e^F \eta^m,\ \psi|_{\partial M} \equiv 0$$

the a priori estimate $\|\psi\|_{C^{2+\alpha}(M)} \leq C$ holds.

7.3.1 A Series of Lemma

Let (M, η) be a compact Kähler manifold with boundary. In this whole section we let ψ be a smooth η-psh function such that

$$(\eta + dd^c\psi)^m = e^F \eta^m \text{ and } \psi|_{\partial M} = 0$$

with $F \in C^\infty(M)$ such that

(i) $-A_0 \leq F \leq 0$
(ii) $\sup_M |\nabla F| \leq A_1$
(iii) $\Delta F \geq -A_2$

for some given $A_0, A_1, A_2 > 0$.

In this section we are going to show in a series of lemma following [CKNS85, Gua98, Che00, Bl09b] that there exists $C > 0$ only depending on A_0, A_1, A_2 (and even only on A_1, A_2 when ∂M is pseudoconcave) such that

$$\sup_M (|\psi| + |\Delta\psi|) \leq C. \tag{7.6}$$

We will explain in the next section how to deduce Theorem 7.11 as a consequence of a general result on fully non-linear second order elliptic PDE's ([CKNS85, Theorem 1]) by adding a result of [Bl09b].

Remark 7.12 *Let us fix some notation and terminology. We fix once and for all a vector field ν on M which is outward pointing, of unit length and orthogonal to $T_{\partial M}$ (with respect to η) at every point of ∂M. We also fix a finite cover of ∂M by coordinate half-balls $B^{(\alpha)}$ with complex coordinates $z^{(\alpha)}$ and defining function r_α for $\partial M \cap B_\alpha$ such that the half-balls B'_α of radius half that of B_α still cover ∂M. We will write as usual*

$$\eta_\psi := \eta + dd^c\psi$$

and denote by tr_ψ and Δ_ψ the trace and Laplacian with respect to η_ψ, so that $\Delta_\psi u = tr_\psi(dd^c u)$.

When we say for instance that a constant depends only on A_0 we mean that it only depends on A_0 together with the background data η, ν and $(B^{(\alpha)}, z^{(\alpha)}, r_\alpha)$.

Remark 7.13 *The following situation occurs several times below. Given point $0 \in \partial M$ we will want to choose an adapted data B, r, z where B is a coordinate half-ball B, r is a defining function for $\partial M \cap B$ and the coordinates z on B are centered at 0 and satisfy*

$$r = -x_m + \Re\left(\sum_{1\leq j,k\leq m} a_{jk} z_j \overline{z}_k\right) + O\left|z|^3\right) \tag{7.7}$$

near 0. We will then estimate the value at 0 of certain partial derivatives of ψ in the z-coordinates in a way that only depends on, say, A_0. In order to ensure the uniformity with respect to the choice of the data B, r, z the latter will implicitely have been constructed as follows. We choose α such that 0 belongs to B'_α, we let B be the translate of B'_α centered at 0 and let w be the translate of $z^{(\alpha)}$. The Taylor series expansion of r in the w-coordinates writes

$$r = \Re\left(\sum_{1\leq j\leq m} c_j w_j + \sum_{1\leq j,k\leq m} b_{jk} w_j w_k + \sum_{1\leq j,k\leq m} a_{jk} w_j \overline{w}_k\right) + O\left(|w|^3\right)$$

and we first perform a linear change of coordinates $w \mapsto w'$ to arrange that $c_m = -1$ and $c_j = 0$ for $j < m$ in the w'-coordinates, and next set

$$z_m := w'_m - \sum_{1\le j,k\le m} b_{jk} w'_j w'_k$$

in order to kill the b_{jk}'s. It is now clear that any quantity which is uniform with respect to certain derivatives in the z-coordinates of a given background function independent of ψ will also be uniform with respect to certain derivatives of the same function in the original coordinates $z^{(\alpha)}$.

Lemma 7.14 *There exists $C > 0$ independent of ψ such that*

$$\sup_M |\psi| + \sup_{\partial M} |\nabla\psi| \le C.$$

Proof. The inequality $\psi \ge 0$ follows from the maximum principle for complex Monge–Ampère equations (Proposition 7.5) since we have $(\eta + dd^c\psi)^m \le \eta^m$ and $\psi = 0$ on ∂M (recall that we have assumed $F \le 0$).

On the other hand let $h \in C^\infty(M)$ be the unique function on M such that

$$\Delta h = -m \text{ and } h|_{\partial M} = 0.$$

Then

$$\Delta(\psi - h) = \Delta\psi + m = \mathrm{tr}_\eta(\eta + dd^c\psi)$$

is non-negative and $\psi - h = 0$ on ∂M thus $\psi \le h$ as a consequence of the maximum principle, this time for subharmonic functions. Now $0 \le \psi \le h$ and $h = 0$ on ∂M shows that

$$\sup_{\partial M} |\nabla\psi| \le \sup_{\partial M} |\nabla h|$$

and the result follows. □

Lemma 7.15 *There exists a constant $C > 0$ only depending on A_2 such that*

$$\sup_M |\Delta\psi| \le C(1 + \sup_{\partial M} |\Delta\psi|).$$

This result is a rather direct consequence of Yau's estimates (compare [Che00, Corollary 1]).

Proof. Yau's famous pointwise inequality ([Yau78] p. 350 (2.18) and (2.20)) states that $(\eta + dd^c\psi)^m = e^F\eta^m$ implies

$$\begin{aligned} &e^{B\psi}\Delta_\psi\left(e^{-B\psi}(m + \Delta\psi)\right) \\ &\quad \ge -Bm(m + \Delta\psi) + Be^{-F/m-1}(m + \Delta\psi)^{1+1/m-1} + \Delta F - B \end{aligned}$$

where $B > 0$ is a lower bound for the holomorphic bisectional curvature of η and Δ_ψ denotes the Laplacian with respect to the Kähler form $\eta + dd^c\psi$. Let $x_0 \in M$ be a point where $e^{-B\psi}(m + \Delta\psi)$ achieves its maximum. If x_0 lies on ∂M then we get

$$\sup_M (m + \Delta\psi) \le e^{B(\sup_M \psi - \inf_M \psi)} \sup_{\partial M}(m + \Delta\psi),$$

where the oscillation $\sup_M \psi - \inf_M \psi$ is bounded in terms of η by Lemma 7.14. If x_0 lies in the interior of M then Yau's inequality implies that

$$Bm(m + \Delta\psi) + B(m + \Delta\psi)^{1+1/m-1} - A_2 - B \le 0$$

at x_0, using that $F \le 0$. It follows that $0 < (m + \Delta\psi)(x_0)$ is bounded above in terms of B and A_2 and the result follows using Lemma 7.14 again to bound the oscillation of ψ.

Lemma 7.16 *There exists $\varepsilon > 0$ only depending on A_0 such that*

$$(\eta + dd^c\psi)|_{T^h_{\partial M}} \ge \varepsilon\eta|_{T^h_{\partial M}}.$$

When ∂M is weakly pseudoconcave we can even take $\varepsilon = 1$.

The first assertion was proved in [CKNS85, pp. 221–223] in the strictly pseudoconvex case, and their argument was extended to the general case in [Gua98, pp.694–696]. In the product case of [Che00] the trivial independence of ε on A_0 was implicit and was made explicit in the Levi flat case in [Bl09b, Theorem 3.2']. Here we add the easy observation that the result holds in the weakly pseudoconcave case as well.

Proof. Let $0 \in \partial M$ and choose an adapted data B, r, z as in Remark 7.13.

We have $\psi \ge 0$ by Lemma 7.14 and $\psi|_{\partial M} = 0$ thus $dd^c\psi$ is equal to a non-positive multiple of $dd^c r$ by Lemma 7.3 and we see that

$$\eta_\psi|_{T^h_{\partial M}} \ge \eta|_{T^h_{\partial M}}$$

if $dd^c r|_{T^h_{\partial M}} \le 0$, which settles the second assertion.

We now consider the first assertion and we first claim that it is enough to show the existence of $\varepsilon > 0$ only depending on A_0 such that

$$\eta_\psi|_{T^h_{\partial M}} \ge \varepsilon dd^c r|_{T^h_{\partial M}}. \tag{7.8}$$

Indeed, Lemma 7.3 implies on the one hand that

$$(\eta_\psi - \eta)|_{T^h_{\partial M}} = dd^c\psi|_{T^h_{\partial M}} = \frac{\nu \cdot \psi}{\nu \cdot r} dd^c r|_{T^h_{\partial M}}.$$

On the other hand, there exists $C > 0$ independent of A, A_0 such that

$$-C \leq \frac{\nu \cdot \psi}{\nu \cdot r} \leq 0 \text{ on } \partial M$$

by Lemma 7.14. We thus see that (7.8) implies

$$\eta_\psi|_{T^h_{\partial M}} \geq -C^{-1}\varepsilon(\eta_\psi - \eta)|_{T^h_{\partial M}}$$

and the desired result follows easily.

In order to show (7.8) it is enough to concentrate on vectors

$$v = \sum_{j<m} v_j \frac{\partial}{\partial z_j} \in T^h_{\partial M,0}$$

such that $\sum |v_j|^2 = 1$ and $dd^c r(v) > 0$. After possibly performing a unitary change of coordinates we may thus assume that $v = \partial/\partial z_1$, since the unitary change of coordinates will preserve uniformity with respect to the background data (cf. Remark 7.13).

We now follow the local barrier argument of [CKNS85, pp. 221–223] and [Gua98, pp. 694–696].

Step 1: Choice of a Kähler potential. There exists a locally defined smooth function τ such $dd^c\tau = \eta$ and $\tau(0) = 0$. As a first step we claim that τ may be chosen so as to satisfy

$$\tau|_{\partial M} = \Re\left(\sum_{j=2}^{m} c_j z_1 \overline{z}_j\right) + O\left(|z_2|^2 + ... + |z_m|^2\right) \tag{7.9}$$

for some $c_j \in \mathbb{C}$. Indeed note that we can use the restriction of $(z_1, ..., z_{m-1}, y_m)$ to ∂M as local coordinates for the boundary. By Lemma 7.2 we have

$$(v|_{\partial M})_{z_1|_{\partial M}\overline{z}_j|_{\partial M}}(0) = v_{z_1\overline{z}_j}(0) + \delta_{1j} a_{11} v_{x_m}(0) \tag{7.10}$$

for every $j < m$ and every smooth function v on M, with a_{11} as in (7.7). Applying (7.10) to $v := x_m$ shows that there exists $b \in \mathbb{C}$ such that

$$x_m|_{\partial M} = a_{11}|z_1|^2 + y_m\Re(bz_1) + O\left(y_m|z_1|^2 + |z_2|^2 + ... + |z_{m-1}|^2 + y_m^2\right).$$

Since we assume $a_{11} = (dd^c r)(v) > 0$ it follows that the following holds near 0 on ∂M:

(i) $|z_1|^2$ writes as the real part of a complex linear combination of $z_m, z_1 z_m$ and $z_1\overline{z}_m$ modulo $O\left(|z_2|^2 + ... + |z_m|^2\right)$

(ii) $z_1|z_1|^2$ writes as the real part of a complex linear combination of $z_1 z_m$ and $z_1\overline{z}_m$ modulo $O\left(|z_2|^2+...+|z_m|^2\right)$
(iii) $|z_1|^4$, $y_m|z_1|^2$ and $z_j|z_1|^2$ are $O\left(|z_2|^2+...+|z_m|^2\right)$ for $j=2,...,m-1$.

As a consequence, Taylor series considerations show that any smooth real-valued function v on ∂M near 0 writes as the real part of a complex linear combination of

$$z_1, z_2, ..., z_m, z_1^2, z_1 z_2, ..., z_1 z_m, z_1\overline{z}_2, ..., z_1\overline{z}_m \text{ and } z_1^3$$

modulo $O\left(|z_2|^2+...+|z_m|^2\right)$. Since we can add to τ the real part of a holomorphic polynomial in $z_1,...,z_m$ without changing the condition $dd^c\tau=\eta$, the above fact applied to $v:=\tau|_{\partial M}$ shows that τ may indeed be chosen so as to satisfy (7.9).

Step 2: Choice of a barrier function. Let us now consider the barrier function

$$b(z_1,...,z_m) := -\varepsilon_1 x_m + \varepsilon_2|z|^2 + \frac{1}{2\mu}\sum_{j=2}^m |c_j z_1 + \mu z_j|^2 \tag{7.11}$$

with $\varepsilon_1, \varepsilon_2, \mu > 0$ and let $B \subset M$ be a coordinate half-ball centered at 0. We are going to show that we may choose the radius α of B, $\varepsilon_1, \varepsilon_2$ and μ in terms of A_0 only such that

$$b \geq \tau + \psi \text{ on } B. \tag{7.12}$$

By (7.7) there exists $C>0$ such that

$$|z_m| \geq x_m \geq \Re\left(\sum_{1\leq j,k\leq m} a_{jk} z_j \overline{z}_k\right) - C|z|^3 \text{ on } B.$$

Since we have $|z|=\alpha$ on $\partial B - B\cap\partial M$ and $a_{11}>0$ we may thus shrink the radius α of B so that there exists $\beta>0$ with

$$\sum_{j=2}^m |z_j|^2 \geq \beta \text{ on } \partial B - B\cap\partial M. \tag{7.13}$$

On the other hand $r=0$ on ∂M so there exists $C>0$ such that

$$-\varepsilon_1 x_m + \varepsilon_2|z|^2 \geq 0 \text{ on } \partial M \cap B \tag{7.14}$$

as soon as $\varepsilon_2 \geq C\varepsilon_1$.

Having fixed such a choice of B we next claim that there exists $\mu > 0$ independent of ψ such that for any $\varepsilon_1, \varepsilon_2$ with $\varepsilon_2 \geq C\varepsilon_1$ we have

$$\tau + \psi \leq b \text{ on } \partial B. \tag{7.15}$$

Indeed this holds on $B \cap \partial M$ by (7.9) and (7.11) since $\psi|_{\partial M} = 0$ and $-\varepsilon_1 x_m + \varepsilon_2 |z|^2 \geq 0$. On the other hand since $\sup \psi$ is under control by Lemma 7.14 the claim also holds on $\partial B - B \cap \partial M$ by (7.13).

Next we pick $\varepsilon_2 > 0$ only depending on μ and A_0 such that

$$(dd^c b)^m \leq e^{-A_0} \eta^m \leq e^F \eta^m = (dd^c(\tau + \psi))^m \text{ on } B, \tag{7.16}$$

which is possible since

$$\left(dd^c \sum_{j=2}^{m} |c_j z_1 + \mu z_j|^2 \right)^m = 0$$

thus $(dd^c b)^m = O(\varepsilon_2)$. Finally we choose ε_1 such that $\varepsilon_2 \geq C\varepsilon_1$. By (7.15) and (7.16) we finally get $\tau + \psi \leq b$ on B as desired by the maximum principle (Proposition 7.5).

Step 3: Conclusion. Since we have $\tau + \psi \leq b$ on B and $b(0) = (\tau + \psi)(0) = 0$ it follows that

$$(\tau + \psi)_{x_m}(0) \leq b_{x_m}(0) = -\varepsilon_1.$$

On the other hand (7.9) and (7.10) yield

$$\tau_{z_1 \bar{z}_1}(0) + \tau_{x_m}(0) a_1 = 0,$$

and $\psi|_{\partial M} = 0$ and (7.10) similarly imply

$$\psi_{z_1 \bar{z}_1}(0) + \psi_{x_m}(0) a_1 = 0.$$

Putting all this together we thus obtain

$$\eta_\psi(v) = (\tau + \psi)_{z_1 \bar{z}_1}(0) \geq \varepsilon_1 a_1 = \varepsilon_1 (dd^c r)(v)$$

which shows that (7.8) holds and concludes the proof. □

Lemma 7.17 *There exists $C = C(A_0, A_1)$ (resp. $C = C(A_1)$ when ∂M is weakly pseudoconcave) such that*

$$\sup_{\partial M} |\nabla^2 \psi| \leq C(1 + \sup_M |\nabla \psi|^2).$$

This result was proved in [CKNS85, Gua98, Che00] (see also [PS09] Lemma 1).

Proof. Let $0 \in \partial M$ and choose adapted data B, r, z as in Remark 7.13. We set for convenience

$$t_1 = y_1, t_2 = x_2, ..., t_{2m-1} = y_m, t_{2m} = x_m.$$

Let $D_1, ..., D_{2m}$ be the dual basis of

$$dt_1, ..., dt_{2m-1}, -dr,$$

so that

$$D_j = \frac{\partial}{\partial t_j} - \frac{r_{t_j}}{r_{x_m}} \frac{\partial}{\partial x_m} \tag{7.17}$$

for $j < 2m$ and

$$D_{2m} = -\frac{1}{r_{x_m}} \frac{\partial}{\partial x_m}. \tag{7.18}$$

Note that the D_j's commute and are tangent to ∂M for $j < 2m$, so that we have a trivial control on the tangent-tangent derivatives

$$D_i D_j \psi(0) = 0 \text{ for } i, j < 2m.$$

We set

$$K := \sup_M |\nabla \psi|$$

and note that there exists $C_0 > 0$ only depending on r such that

$$|D_j \psi| \leq C_0 K, \ j = 1, ..., 2m. \tag{7.19}$$

In the next two steps we are going to show the existence of $C = C(A_1)$ such that the normal-tangent derivatives of ψ satisfy

$$|D_j D_{2m} \psi(0)| \leq C(1 + K)$$

for $j < 2m$.

In the third step we will show how this combines with Lemma 7.16 to get $C = C(A_1, A_0)$ (resp. $C = C(A_1)$ in the weakly pseudoconcave case) such that

$$|D_{2m}^2 \psi(0)| \leq C(1 + K^2).$$

Step 1: Construction of a barrier function. As in Lemma 7.14 we let h be the unique function on M such that $h|_{\partial M} = 0$ and $\Delta h = -m$ and we introduce the barrier function

$$b := \psi + \varepsilon h - \mu r^2$$

on a coordinate half-ball $B \subset M$ centered at 0. We claim that we may choose B, ε and μ independently of ψ such that

$$b \geq 0 \text{ on } B \tag{7.20}$$

and

$$\Delta_\psi b \leq -\frac{1}{2}\,\mathrm{tr}_\psi(\eta) \text{ on } B. \tag{7.21}$$

Since

$$dd^c(r^2) = 2r dd^c r + 2dr \wedge d^c r$$

there exists C_1 only depending on h and r such that

$$\Delta_\psi b \leq m - \mathrm{tr}_\psi(\eta) + C_1(\varepsilon - 2\mu r)\,\mathrm{tr}_\psi(\eta) - 2\mu\,\mathrm{tr}_\psi(dr \wedge d^c r). \tag{7.22}$$

Using $dr \neq 0$ at 0 and $\eta > 0$ we may choose B small enough so that

$$dr \wedge d^c r \wedge \eta^{m-1} \geq \gamma \eta^m$$

on B for some $\gamma > 0$, hence

$$dr \wedge d^c r \wedge \eta^{m-1} \geq \gamma \eta_\psi^m$$

since

$$\eta_\psi^m = f\eta^m \leq \eta^m$$

by assumption. If we choose $\mu > 0$ such that $m\gamma\mu \geq 2$ we then get

$$(\frac{1}{4}\eta + 2\mu dr \wedge d^c r)^m \geq \gamma^{-1} dr \wedge d^c r \wedge \eta^{m-1} \geq \eta_\psi^m$$

and it follows from the arithmetico-geometric inequality (Proposition 7.9) that

$$\mathrm{tr}_\psi(\frac{1}{4}\eta + 2\mu dr \wedge d^c r) \geq m.$$

By (7.22) we obtain

$$\Delta_\psi b \leq (C_1(\varepsilon - 2\mu r) - 3/4)\,\mathrm{tr}_\psi(\eta).$$

Since $r(0) = 0$ we may assume upon shrinking B and choosing $\varepsilon > 0$ small enough with respect to μ and C_1 that $C_1(\varepsilon - 2\mu r) \leq 1/4$ on B and we get (7.21).

Let us now show how to obtain (7.20). Since $\psi \geq 0$ it is enough to guarantee

$$\varepsilon h \geq \mu r^2. \tag{7.23}$$

But we have $\Delta h = -m$ thus we get $\Delta(h+cr) \le 0$ on the whole of M if $c > 0$ is small enough and the maximum principle implies $h + cr \ge 0$ on M since $(h+cr)|_{\partial M} = 0$. We thus see that (7.23) holds if $-c\varepsilon r \ge \mu r^2$, i.e. $r \ge -c\varepsilon\mu^{-1}$, which will hold on B upon possibly shrinking it further in terms of ε, μ only.

Step 2: Bounding the normal-tangent derivatives. Let $j < 2m$ and consider the tangential vector field D_j. We claim that there exist μ_1, μ_2 only depending on A_1 such that

$$v := K(\mu_1 b + \mu_2 |z|^2) \pm D_j\psi \tag{7.24}$$

satisfies

$$v \ge 0 \text{ on } \partial B \tag{7.25}$$

and

$$\Delta_\psi v \le 0 \text{ on } B. \tag{7.26}$$

We first take care of (7.25). On the one hand we have $D_j\psi = b = 0$ on $B \cap \partial M$ thus $v \ge 0$ on $B \cap \partial M$. On the other hand on $\partial B - B \cap \partial M$ we have $|z|^2 = \alpha^2$ where $\alpha > 0$ denotes the radius of B. Since $b \ge 0$ on B it follows that $v \ge 0$ on ∂B as soon as $K\mu_2\alpha^2 \ge |\psi_{u_j}|$, which holds as soon as $\mu_2 \ge C_0\alpha^{-2}$ by (7.19).

Having fixed such a choice of μ_2 we now show how to choose μ_1 so that (7.26) holds. Applying D_j to

$$\log \frac{(\eta + dd^c\psi)^m}{\eta^m} = F$$

yields

$$\operatorname{tr}_\psi(D_j\eta + D_j dd^c\psi) = D_j F + \operatorname{tr}_\eta(D_j\eta) \tag{7.27}$$

where D_j acts on $(1,1)$-forms componentwise (in the $dz_k \wedge d\bar{z}_l$ basis). Since we have $\operatorname{tr}_\psi(\eta) \ge m$ by the arithmetico-geometric inequality (Proposition 7.9) we thus see that there exists $C = C(A_1)$ such that

$$|\operatorname{tr}_\psi(D_j dd^c\psi)| \le C \operatorname{tr}_\psi(\eta). \tag{7.28}$$

By (7.17) we have

$$D_j = \frac{\partial}{\partial t_j} + a\frac{\partial}{\partial x_m}$$

with $a := -r_{t_j}/r_{x_m}$ and an easy computation yields

$$dd^c(D_j\psi) = D_j dd^c\psi + \psi_{x_m} dd^c a + 2da \wedge d^c\psi_{x_m}$$

thus

$$\Delta_\psi(D_j\psi) = \operatorname{tr}_\psi(D_j dd^c\psi) + \psi_{x_m}\Delta_\psi a + 2\operatorname{tr}_\psi(da \wedge d^c\psi_{x_m}).$$

Now on the one hand one easily checks that

$$d^c\psi_{x_m} = d^c(i_{\partial/\partial x_m} d\psi) = -i_{\partial/\partial y_m} d^c d\psi = i_{\partial/\partial y_m} dd^c\psi$$

where i denotes the contraction operator. On the other hand applying $i_{\partial/\partial y_m}$ to the trivial relation

$$da \wedge \eta_\psi^m = 0$$

yields

$$\mathrm{tr}_\psi \left(da \wedge (i_{\partial/\partial y_m}\eta_\psi)\right) = a_{y_m}$$

so we get

$$\mathrm{tr}_\psi(da \wedge d^c\psi_{x_m}) + \mathrm{tr}_\psi(da \wedge i_{\partial/\partial y_m}\eta) = a_{y_m}.$$

Putting all this together we obtain

$$\Delta_\psi(D_j\psi) = \mathrm{tr}_\psi(D_j dd^c\psi) + \psi_{x_m}\Delta_\psi a + 2a_{y_m} - 2\,\mathrm{tr}_\psi(da \wedge i_{\partial/\partial y_m}\eta)$$

which combines with (7.28) to yield

$$|\Delta_\psi(D_j\psi)| \le C(1+K)\,\mathrm{tr}_\psi(\eta) \tag{7.29}$$

for some $C = C(A_1)$.

By (7.21) we thus get

$$\Delta_\psi v \le \left(-\frac{K}{2}\mu_1 + C(1+K)\right)\eta \wedge \eta_\psi^{m-1} \text{on } B$$

for some $C = C(A_1)$ and we may thus choose $\mu_1 = \mu_1(A_1)$ such that (7.26) holds.

Now (7.25) and (7.26) imply $v \ge 0$ on B by the maximum principle hence $D_{2m}v(0) \ge 0$ since $v(0) = 0$. In other words we have proved that

$$|D_{2m}D_j(0)| \le CK\left(1 + D_{2m}b(0)\right)$$

for some $C = C(A_1)$. But the gradient of $b = \psi + \varepsilon h - \mu r^2$ at 0 is bounded by a constant C independent of ψ by Lemma 7.14, and we thus get

$$|D_{2m}D_j\psi(0)| \le C(1+K)$$

for some $C = C(A_1)$ as desired.

Step 3: Bounding the normal-normal derivatives. In this last step we are going to show that there exists a constant $C = C(A_1, A_0)$ (resp. $C = C(A_1)$ in the weakly pseudoconcave case) such that

$$\left|D_{2m}^2\psi(0)\right| \le C(1+K^2).$$

By the bound on $D_iD_j(0)$ and $D_{2m}D_j\psi(0)$ for $i, j < 2m$ it is equivalent to show that

$$|\psi_{z_m\overline{z}_m}(0)| \leq C(1 + K^2) \tag{7.30}$$

and we already know that

$$\left|\psi_{z_j\overline{z}_m}(0)\right| \leq C(1 + K) \tag{7.31}$$

for all $j < m$ and $C = C(A_1)$. If we write

$$\eta = i\sum_{j,k}\eta_{jk}dz_j \wedge d\overline{z}_k$$

then expanding out the determinant thus yields

$$\left|\det(\eta_{jk} + \psi_{z_j\overline{z}_k})_{1\leq j,k\leq m} - (\eta_{mm} + \psi_{z_m\overline{z}_m})\det(\eta_{jk} + \psi_{z_j\overline{z}_k})_{1\leq j,k<m}\right| \leq C(1 + K^2)$$

at 0.

Now on the one hand the equation $(\eta + dd^c\psi)^m = f\eta^m$ shows that

$$0 \leq \det(\eta_{jk} + \psi_{z_j\overline{z}_k})_{1\leq j,k\leq m} \leq \det(\eta_{jk}) \leq C.$$

On the other hand since $T^h_{\partial M}$ is spanned by $\frac{\partial}{\partial z_j}$, $j = 1, ..., m - 1$ at 0 Lemma 7.16 yields $\varepsilon > 0$ only depending on A_0 (resp. $\varepsilon = 1$ in the weakly pseudoconcave case) such that

$$\det(\eta_{jk} + \psi_{z_j\overline{z}_k})_{1\leq j,k<m} \geq \varepsilon$$

and Lemma 7.17 follows. □

Following [Che00] Sect. 3.2 we now use a blow-up argument to show:

Lemma 7.18 *There exists $C > 0$ only depending on A_0, A_1, A_2 (resp.on A_1, A_2 in the weakly pseudoconcave case) such that* $\sup_M |\nabla\psi| \leq C$.

Proof. By Lemma 7.15 and Lemma 7.17 we have

$$\sup_M |\Delta\psi| \leq C(1 + \sup_M |\nabla\psi|^2) \tag{7.32}$$

for some $C > 0$ only depending on A_0, A_1, A_2.

Assume by contradiction that the result fails. Then there exists sequences $x_j \in M$ and ψ_j such that

$$|\nabla\psi_j(x_j)| = \sup_M |\nabla\psi_j| =: C_j \to +\infty.$$

We may assume that $x_j \to x_\infty \in M$. Pick a coordinate half-ball B centered at x_∞ and set

$$\widetilde{\psi_j}(z) := \psi_j(x_j + C_j^{-1} z) \tag{7.33}$$

which satisfies

$$|\nabla \widetilde{\psi_j}(0)| = 1 \tag{7.34}$$

and

$$\sup_B |\Delta \widetilde{\psi_j}| \le C. \tag{7.35}$$

by (7.32). By (7.35) $\widetilde{\psi_j}$ stays in a compact subset of $C^1(B)$, so we can assume that $\widetilde{\psi_j} \to \rho$ in $C^1(B)$, and (7.34) implies $|\nabla\rho(0)| = 1$, so that ρ is non-constant. Now there are two cases.

If $x_\infty \in \partial M$ then Lemma 7.14 implies $\rho \equiv 0$ and contradicts the fact that ρ is non-constant.

If $x_\infty \notin \partial M$ then (7.33) makes sense on a ball of size C_j, which shows that ρ is actually defined on the whole of $\mathbb{C}^m$. Since

$$dd^c \widetilde{\psi_j} \ge C_j^{-2} dd^c \psi_j \ge -C_j^{-2}\eta$$

we also see that ρ is psh on $\mathbb{C}^m$. On the other hand ρ is uniformly bounded above on $\mathbb{C}^m$ by Lemma 7.14, and these two properties imply that ρ is constant (for instance because ρ extends as a psh function on the complex projective space $\mathbb{P}^m$), which contradicts again $|\nabla\rho(0)| = 1$. □

7.3.2 Proof of Theorem 7.11

By Lemma 7.15, Lemma 7.17 and Lemma 7.18 there exists $C > 0$ only depending on A such that

$$\eta + dd^c\psi \le C\eta.$$

Since we also have $(\eta + dd^c\psi)^m \ge e^{-A}\eta^m$ it follows upon possibly enlarging C that

$$C^{-1}\eta \le \eta + dd^c\psi \le C\eta.$$

This means that the Monge–Ampère equation

$$(\eta + dd^c\psi)^m = e^F\eta^m$$

is elliptic with ellipticity constants that are uniform with respect A, and one would like to conclude that the a priori bound on the complex Hessian $dd^c\psi$ should yield an a priori bound on the $C^{2+\alpha}$ norm of ψ for some $\alpha > 0$ by an Evans-Krylov-type result.

Indeed [Ev82] yields *inner* $C^{2+\alpha}$ estimates for solutions of uniformly elliptic fully non-linear second order PDE's as soon as a C^2 bound is available. Similarly [CKNS85, Theorem 1] yields $C^{2+\alpha}$ estimates *up to the boundary* in a similar situation.

However the difficulty in our case (which seems to have been overlooked in [Che00] Remark 1) is that we only have an a priori bound on $\Delta\psi$, which doesn't provide a bound on the C^2-norm in general.

A similar situation occurs in the proof of the Aubin–Yau theorem, i.e. for the analogue of Theorem A when M has no boundary. The solution adopted in [Siu87] was to reprove the Evans–Krylov theorem in the complex case, replacing the C^2-norm by a control on the complex Hessian. In principle it should be possible to follow the same path in our situation, i.e. to try and adapt the proof of [CKNS85, Theorem 1] to the complex case. However another difficulty occurs since it is assumed in [CKNS85, p. 231] that M is locally a half-plane near a given point of ∂M, which is of course trivial in the real case but cannot hold in the complex case unless ∂M is Levi flat. We thus see that a different strategy has to be proposed in the general case where no assumption is made on ∂M.

The way out of this difficulty is provided by the following result ([Bl09b, Theorem 3.4]).

Lemma 7.19 *Let (M, η) be a compact Kähler manifold with boundary. Given $A > 0$ there exists $C > 0$ such that the following holds. Let ψ be a smooth η-psh function such that*

$$(\eta + dd^c\psi)^m = e^F\eta^m \text{ and } \psi|_{\partial M} = 0$$

with $F \in C^\infty(M)$. If

(i) $A^{-1}\eta \le \eta_\psi \le A\eta$
(ii) $\|F\|_{C^2(M)} \le A$
(iii) $\|\psi\|_{C^1(M)} \le A$

then

$$\sup_M |\nabla^2\psi| \le C(1 + \sup_{\partial M} |\nabla\psi|).$$

Proof. We only sketch the argument, referring to [Bl09b, Theorem 3.4] for computational details.

Let first D be a local vector field on M with constant coefficients with respect to one of the given coordinate patches and of norm (with respect to η) at most 1. Applying D to

$$\log \frac{(\eta + dd^c\psi)^m}{\eta^m} = F \tag{7.36}$$

yields

$$\Delta(D\psi) \ge -C$$

with $C = C(A)$ by (i) and (ii). Similarly, applying D^2 to (7.36) implies

$$\Delta(D^2\psi) \geq -C$$

with $C = C(A)$ by (i) and (ii).

Now there exists $C > 0$ only depending on η such that

$$\sup_M |\nabla^2\psi| \leq C \sup_M D^2\psi \tag{7.37}$$

for some globally defined vector field D of length at most 1.

Somewhat tedious computations using (i) and (iii) then show that

$$\Delta\left(|\nabla\psi|^2\right) \geq C^{-1}|\nabla^2\psi|^2 - C$$

and

$$\Delta\left(D^2\psi\right) \geq -C(1 + |\nabla^2\psi|)$$

with $C = C(A)$, so that

$$\Delta\left(|\nabla\psi|^2 + D^2\psi\right) \geq C^{-1}|\nabla^2\psi|^2 - C(1 + |\nabla\psi|^2) \text{ on } M. \tag{7.38}$$

Now the result follows as in Lemma 7.15: pick $x_0 \in M$ at which $|\nabla\psi|^2 + D^2\psi$ achieves its maximum. If x_0 belongs to ∂M then we're done. Otherwise (7.38) yields

$$|\nabla^2\psi|(x_0) \leq C$$

for some $C = C(A)$. But we have

$$\sup_M D^2\psi \leq D^2\psi(x_0) + C \leq C(|\nabla^2\psi(x_0)| + 1)$$

and we infer $\sup_M |\nabla^2\psi| \leq C$ by (7.37). □

7.4 Proof of Theorem A and Theorem B

In this section we explain how to deduce Theorems A and B from the a priori estimates obtained in the previous section. The proof relies on the continuity method and we won't give much details for the standard parts of the procedure, referring for example to [Bl05b, Sect. 2] in this volume.

7.4.1 Proof of Theorem A

Uniqueness follows from Proposition 7.5.

Upon replacing η with $\eta + dd^c\widetilde{\varphi}$ we may assume that $\widetilde{\varphi} = 0$ (and in particular $\varphi = 0$). If we write $\mu = e^F\eta^m$ with $F \in C^\infty(M)$ we thus have $F \le 0$.

We follow the continuity method: consider the set I of all $t \in [0,1]$ such that there exists a smooth (strictly) η-psh function ψ_t on M with

1. $\psi_t|_{\partial M} = 0$.
2. $(\eta + dd^c\psi_t)^m = ((1-t)e^F + t)\eta^m$.

Note that ψ_t is unique by the maximum principle (Proposition 7.5). The set I is non-empty since it contains 1 (with $\psi_1 \equiv 0$).

Since the linearization of the operator

$$\psi \mapsto \log \frac{(\eta + dd^c\psi)^m}{\eta^m}$$

at a given smooth strictly η-psh function ψ is equal to Δ_ψ, it follows from standard elliptic regularity and the inverse function theorem applied in appropriate Sobolev spaces that I is open.

On the other hand, the C^2 norm of

$$\log\left((1-t)e^F + t\right)$$

is clearly bounded independently of t thus, Theorem 7.11 yields $C > 0$ and $\alpha > 0$ such that $\|\psi_t\|_{C^{2+\alpha}(M)} \le C$ for all $t \in I$. The usual compactness and elliptic bootstrapping argument using Schauder's estimates therefore shows that I is closed and we conclude that $I = [0,1]$, so that $0 \in I$ as desired.

7.4.2 Proof of Theorem B

We use the same strategy as in [PS09] Sect. 4.2.

Uniqueness follows from Corollary 7.7. Now let $\widetilde{\varphi}$ be the given η-psh extension of $\varphi \in C^\infty(\partial M)$ and set $\vartheta := \eta + dd^c\widetilde{\varphi}$. Here ϑ is merely semipositive so we cannot directly replace η with ϑ.

However we have $(1-t)\vartheta + t\eta > 0$ for each $t > 0$ and

$$((1-t)\vartheta + t\eta)^m \ge t^m\eta^m,$$

so by Theorem A there exists a unique smooth $((1-t)\vartheta + t\eta)$-psh function ψ_t on M such that

$$((1-t)\vartheta + t\eta + dd^c\psi_t)^m = t^m\eta^m \text{ and } \psi_t|_{\partial M} = 0. \qquad (7.39)$$

In what follows $C > 0$ denotes a constant independent of t. Since $(1-t)\vartheta + t\eta \leq C\eta$ we have in particular $dd^c\psi_t \geq -C\eta$ and we get

$$\sup_M |\psi_t| + \sup_{\partial M} |\nabla\psi_t| \leq C \qquad (7.40)$$

by replacing h with the solution of $\Delta h = -C$, $h|_{\partial M} = 0$ in the proof of Lemma 7.14. Now observe that (7.39) rewrites in terms of

$$\rho_t := (1-t)\widetilde{\varphi} + \psi_t$$

as

$$(\eta + dd^c\rho_t)^m = e^{F_t}\eta^m \text{ and } \rho_t|_{\partial M} = (1-t)\varphi. \qquad (7.41)$$

with $F_t :\equiv m \log t$ (and thus $\nabla F_t = \Delta F_t \equiv 0$).

We claim that we have

$$\sup_M |\nabla\rho_t| \leq C,$$

and

$$\sup_M \Delta\rho_t \leq C$$

when ∂M is furthermore weakly pseudoconcave.

The bound on $\nabla\rho_t$ follows directly from (the proof of) Blocki's gradient estimate ([Bl09a] Theorem 1) since $\nabla\rho_t$ is bounded on ∂M by (7.40).

Assume that ∂M is weakly pseudoconcave. Then (the proof of) Lemma 7.16 and Lemma 7.17 shows that

$$\sup_{\partial M} \Delta\rho_t \leq C$$

and we infer $\sup_M \Delta\rho_t \leq C$ by Yau's inequality just as in Lemma 7.15.

Let us now conclude the proof of Theorem B. Since

$$\sup_M(|\psi_t| + |\nabla\psi_t|) \leq C$$

the ψ_t's stay in a compact subset of $C^0(M)$. If $\psi = \lim_{t_j \to 0_+} \psi_{t_j}$ is any limit point in $C^0(M)$ then ψ is η-psh and satisfies

$$(\eta + dd^c\psi)^m = 0 \text{ and } \psi|_{\partial M} = 0$$

by continuity of the Monge–Ampère operator in the topology of uniform convergence ([BT76]). Since $\sup_M |\nabla\psi_t| \leq C$ we also get that ψ is Lipschitz continuous which proves (i) of Theorem B.

If we assume furthermore that ∂M is weakly pseudoconcave then we have

$$\sup_M |\Delta\psi_t| \leq C$$

thus $\Delta\psi \in L^\infty$.

Since ψ is η-psh, it follows that $dd^c\psi$ has L^∞_{loc} coefficients. $\square$