Chapter 4
Regularizing Properties of the Kähler–Ricci Flow

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Abstract These notes present a general existence result for degenerate parabolic complex Monge–Ampère equations with continuous initial data, slightly generalizing the work of Song and Tian on this topic. This result is applied to construct a Kähler–Ricci flow on varieties with log terminal singularities, in connection with the Minimal Model Program. The same circle of ideas is also used to prove a regularity result for elliptic complex Monge–Ampère equations, following Székelyhidi–Tosatti.

Introduction

As we saw in Chap. 3, each initial Kähler form ω_0 on a compact Kähler manifold X uniquely determines a solution $(\omega_t)_{t\in[0,T_0)}$ to the (unnormalized) Kähler–Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t).$$

Along the flow, the cohomology class $[\omega_t] = [\omega_0] + t[K_X]$ must remain in the Kähler cone, and this is in fact the only obstruction to the existence of the flow. In other words, the maximal existence time T_0 is either infinite, in which case K_X is nef and X is thus a *minimal model* by definition, or T_0 is finite and $[\omega_0] + T_0[K_X]$ lies on the boundary of the Kähler cone.

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In [ST09], J. Song and G. Tian proposed to use the Minimal Model Program (MMP for short) to continue the flow beyond time T_0. At least when $[\omega_0]$ is a rational cohomology class (and hence X is projective), the MMP allows to find a mildly singular projective variety X' birational to X such that $[\omega_0] + t[K_X]$ induces a Kähler class on X' for $t > T_0$ sufficiently close to T_0. It is therefore natural to try and continue the flow on X', but new difficulties arise due to the singularities of X'. After blowing-up X' to resolve these singularities, the problem boils down to showing the existence of a unique solution to a certain degenerate parabolic complex Monge–Ampère equation, whose initial data is furthermore singular.

The primary purpose of this chapter is to present a detailed account of Song and Tian's solution to this problem. Along the way, a regularizing property of parabolic complex Monge–Ampère equations is exhibited, which can in turn be applied to prove the regularity of weak solutions to certain elliptic Monge–Ampère equations, following [SzTo11].

The chapter is organized as follows. In Sect. 4.1 we gather the main analytic tools to be used in the proof: a Laplacian inequality, the maximum principle, and Evans–Krylov type estimates for parabolic complex Monge–Ampère equations. In Sect. 4.2, we first consider the simpler case of non-degenerate parabolic complex Monge–Ampère equations involving a time-independent Kähler form. We show that such equations smooth out continuous initial data, and give a proof of the main result of [SzTo11]. Sections 4.3–4.5 contain the main result of the chapter, dealing with the general case of degenerate parabolic complex Monge–Ampère equations, basically following [ST09] (and independently of Sect. 4.2). In the final Sect. 4.6, we apply the previous results to study the Kähler–Ricci flow on varieties with log terminal singularities.

Nota Bene. This text is an expanded version of a series of lectures delivered by the two authors during the second ANR-MACK meeting (8–10 June 2011, Toulouse, France). As the audience mostly consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

4.1 An Analytic Toolbox

4.1.1 A Laplacian Inequality

If θ and ω are $(1,1)$-forms on a complex manifold X with $\omega > 0$, θ can be diagonalized with respect to ω at each point of X, with real eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$, and the *trace* of θ with respect to ω is defined as $\operatorname{tr}_\omega(\theta) = \sum_i \lambda_i$. More invariantly, we have

$$\operatorname{tr}_\omega(\theta) = n\frac{\theta \wedge \omega^{n-1}}{\omega^n}.$$

The Laplacian of a function φ with respect to ω is given by

$$\Delta_\omega \varphi = \operatorname{tr}_\omega(dd^c\varphi).$$

For later use, we record an elementary eigenvalue estimate:

Lemma 4.1.1. *If ω and ω' are two positive $(1,1)$-forms on a complex manifold X, then*

$$\left(\frac{\omega'^n}{\omega^n}\right)^{\frac{1}{n}} \le \tfrac{1}{n}\operatorname{tr}_\omega(\omega') \le \left(\frac{\omega'^n}{\omega^n}\right)(\operatorname{tr}_{\omega'}(\omega))^{n-1}.$$

Proof. In terms of the eigenvalues $0 < \lambda_1 \le \ldots \le \lambda_n$ of ω' with respect to ω (at a given point of X), the assertion writes

$$\left(\prod_i \lambda_i\right)^{1/n} \le \frac{1}{n}\sum_i \lambda_i \le \left(\prod_i \lambda_i\right)\left(\sum_i \lambda_i^{-1}\right)^{n-1}.$$

The left-hand inequality is nothing but the arithmetico-geometric inequality. By homogeneity, we may assume that $\prod_i \lambda_i = 1$ in proving the right-hand inequality. We then have

$$\left(\sum_i \lambda_i^{-1}\right)^{n-1} \ge \lambda_1^{-1}\ldots\lambda_{n-1}^{-1} = \lambda_n \ge \tfrac{1}{n}\sum_i \lambda_i. \qquad \square$$

The next result is a Laplacian inequality, which basically goes back to [Aub78, Yau78] and is the basic tool for establishing second order a priori estimates for elliptic and parabolic complex Monge–Ampère equations. In its present form, the result is found in [Siu87, pp. 97–99]; we include a proof for the reader's convenience.

Proposition 4.1.2. *Let ω, ω' be two Kähler forms on a complex manifold X. If the holomorphic bisectional curvature of ω is bounded below by a constant $B \in \mathbb{R}$ on X, then*

$$\Delta_{\omega'} \log \operatorname{tr}_\omega(\omega') \ge -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega')}{\operatorname{tr}_\omega(\omega')} + B\operatorname{tr}_{\omega'}(\omega).$$

Proof. Since this is a pointwise inequality, we can choose normal holomorphic coordinates (z_j) at a given point $p \in X$ so that $\omega = i\sum_{k,l}\omega_{kl}dz_k \wedge d\bar{z}_l$ and $\omega' = i\sum_{k,l}\omega'_{kl}dz_k \wedge d\bar{z}_l$ satisfy

$$\omega_{kl} = \delta_{kl} - \sum_{i,j} R_{ijkl} z_i \bar{z}_j + O(|z|^3)$$

and

$$\omega'_{kl} = \lambda_k \delta_{kl} + O(|z|)$$

near p. Here R_{ijkl} denotes the curvature tensor of ω, δ_{kl} stands for the Kronecker symbol, and $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of ω' with respect to ω at p.

Observe that the inverse matrix $(\omega^{kl}) = (\omega_{kl})^{-1}$ satisfies

$$\omega^{kl} = \delta_{kl} + \sum_{i,j} R_{ijkl} z_i \bar{z}_j + O(|z|^3). \tag{4.1}$$

Recall also that the curvature tensor of ω' is given in the local coordinates (z_i) by

$$R'_{ijkl} = -\partial_i \overline{\partial}_j \omega'_{kl} + \sum_{p,q} \omega'_{pq} \partial_i \omega'_{kq} \overline{\partial}_j \omega'_{pl},$$

hence

$$R'_{ijkl} = -\partial_i \overline{\partial}_j \omega'_{kl} + \sum_{p} \lambda_p^{-1} \partial_i \omega_{kp} \overline{\partial}_j \omega'_{pl} \tag{4.2}$$

at p. Set $u := \operatorname{tr}_\omega(\omega')$, and note that

$$\Delta_{\omega'} \log u = u^{-1} \Delta_{\omega'} u - u^{-2} \operatorname{tr}_{\omega'}(du \wedge d^c u).$$

At the point p we have

$$\Delta_{\omega'} u = \sum_{ik} \lambda_i^{-1} \partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk})$$

and

$$\operatorname{tr}_{\omega'}(du \wedge d^c u) = \sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll},$$

with

$$\partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk}) = \lambda_k R_{iikk} + \partial_i \overline{\partial}_i \omega'_{kk}$$

thanks to (4.1). It follows that

$$\Delta_{\omega'} \log u = u^{-1} \left(\sum_{ik} \lambda_i^{-1} \lambda_k R_{iikk} + \sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk} \right) - u^{-2} \left(\sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll} \right) \tag{4.3}$$

holds at p. On the one hand, the assumption on the holomorphic bisectional curvature of ω reads $R_{iikk} \geq B$ for all i, k, hence

$$\sum_{ik} \lambda_i^{-1} \lambda_k R_{iikk} \geq B \left(\sum_i \lambda_i^{-1} \right) \left(\sum_k \lambda_k \right) = B \operatorname{tr}_{\omega'}(\omega) u. \tag{4.4}$$

On the other hand, (4.2) yields

$$\sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk} = - \sum_{i,k} \lambda_i^{-1} R'_{iikk} + \sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2.$$

Note that $\sum_{i,k} \lambda_i^{-1} R'_{iikk} = \operatorname{tr}_{\omega} \operatorname{Ric}(\omega')$, while

$$\sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2 \geq \sum_{i,k} \lambda_i^{-1} \lambda_k^{-1} |\partial_i \omega'_{kk}|^2 \geq u^{-1} \sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll}$$

by the Cauchy–Schwarz inequality. Combining this with (4.3) and (4.4) yields the desired inequality. □

4.1.2 The Maximum Principle

The following simple maximum principle (or at least its proof) will be systematically used in what follows to establish a priori estimates.

Proposition 4.1.3. *Let U be a complex manifold and $0 < T \leq +\infty$. Let $(\omega_t)_{t \in [0,T)}$ be a smooth path of Kähler metrics on U, and denote by $\Delta_t = \operatorname{tr}_{\omega_t} dd^c$ the Laplacian with respect to ω_t. Assume that*

$$H \in C^0 \left(U \times [0,T) \right) \cap C^\infty \left(U \times (0,T) \right)$$

satisfies either

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) H \leq 0,$$

or

$$\frac{\partial H}{\partial t} \leq \log \left[\frac{(\omega_t + dd^c H)^n}{\omega_t^n} \right]$$

on $U \times (0,T)$. When U is non compact, assume further that $H \to -\infty$ at infinity on $U \times [0,T']$, for each $T' < T$. Then we have

$$\sup_{U \times [0,T)} H = \sup_{U \times \{0\}} H.$$

If we replace $\leq$ with $\geq$ in the above differential inequalities and assume that $H \to +\infty$ in the non compact case, then the conclusion is that

$$\inf_{U\times[0,T)} H = \inf_{U\times\{0\}} H.$$

Proof. Upon replacing H with $H - \varepsilon t$ (resp. $H + \varepsilon t$ for the reverse inequality) with $\varepsilon > 0$ and then letting $\varepsilon \to 0$, we may assume in each case that the differential inequality is strict. It is enough to show that $\sup_{U\times[0,T']} H = \sup_{U\times\{0\}} H$ for each $T' < T$. The properness assumption guarantees that H achieves its maximum (resp. minimum) on $U \times [0, T']$, at some point $(x_0, t_0) \in U \times [0, T']$, and the strict differential inequality forces $t_0 = 0$. Indeed, we would otherwise have $dd^c H \leq 0$ at (x_0, t_0), $\frac{\partial H}{\partial t} = 0$ if $t_0 < T'$, and at least $\frac{\partial H}{\partial t} \geq 0$ if $t_0 = T'$, which would at any rate contradict the strict differential inequality. □

4.1.3 Evans–Krylov Type Estimates for Parabolic Complex Monge–Ampère Equations

Since it will play a crucial in what follows, we want to give at least a brief idea of the proof of the next result, which says in essence that it is enough to control the time derivative and the Laplacian to get smooth solutions to parabolic complex Monge–Ampère equations.

Theorem 4.1.4. *Let $U \Subset \mathbb{C}^n$ be an open subset and $T \in (0, +\infty)$. Suppose that $u, f \in C^\infty\left(\bar{U} \times [0, T]\right)$ satisfy*

$$\frac{\partial u}{\partial t} = \log\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) + f, \tag{4.5}$$

and assume also given a constant $C > 0$ such that

$$\sup_{U\times(0,T)} \left(\left|\frac{\partial u}{\partial t}\right| + |\Delta u|\right) \leq C.$$

For each compact $K \Subset U$, each $\varepsilon > 0$ and each $p \in \mathbb{N}$, the C^p norm of u on $K \times [\varepsilon, T]$ can then be bounded in terms of the constant C and of the C^q norm of f on $\bar{U} \times [0, T]$ for some $q \geq p$.

The first ingredient in the proof are the Schauder estimates for linear parabolic equations. If f is a function on the cylinder $Q = U \times (0, T)$, recall from Chap. 2 that for $0 < \alpha < 1$ the *parabolic α-Hölder norm* of f on Q is defined as

$$\|f\|_{C^\alpha_P(Q)} := \|f\|_{C^0(Q)} + [f]_{\alpha;Q}\,,$$

where $[f]_{\alpha;Q}$ denotes the α-Hölder seminorm with respect to the parabolic distance

$$d_P\left((z,t),(z',t')\right) = \max\left\{|z-z'|, |t-t'|^{1/2}\right\}.$$

For each $k \in \mathbb{N}$, the *parabolic $C^{k,\alpha}$-norm* is then defined as

$$\|f\|_{C_P^{k,\alpha}(Q)} := \sum_{|\beta|+2j\le k} \|D_x^\beta D_t^j f\|_{C_P^\alpha(Q)}.$$

If $(\theta_t)_{t\in(0,T)}$ is a path of differential forms on U, we can similarly consider $[\theta_t]_{\alpha,Q}$ and $\|\theta_t\|_{C_P^{k,\alpha}(Q)}$, with respect to the flat metric ω_U on U. In our context, the parabolic Schauder estimates can then be stated as:

Lemma 4.1.5. *Let $(\omega_t)_{t\in(0,T)}$ be a smooth path of Kähler metrics on U, and assume that $u, f \in C^\infty(Q)$ satisfy*

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) u = f,$$

with Δ_t the Laplacian with respect to ω_t, and setting as above $Q = U \times (0,T)$. Suppose also given $C > 0$ and $0 < \alpha < 1$ such that on Q we have

$$C^{-1}\omega_U \le \omega_t \le C\omega_U \text{ and } [\omega_t]_{\alpha,Q} \le C.$$

For each $Q' = U' \times (\varepsilon, T)$ with $U' \Subset U$ and $\varepsilon \in (0,T)$, we can then find a constant $A > 0$ only depending on U', ε and C such that

$$\|u\|_{C_P^{2,\alpha}(Q')} \le A\left(\|u\|_{C^0(Q)} + \|f\|_{C_P^\alpha(Q)}\right).$$

This result follows for instance from [Lieb96, Theorem 4.9] (see also Chap. 2 in the present volume). Note that these estimates are interior only with respect to the parabolic boundary, i.e. the upper limit of the time interval is the same on both sides of the estimates.

The second ingredient in the proof of Theorem 4.1.4 is the following version of the Evans–Krylov estimates for parabolic complex Monge–Ampère equations. We refer to [Gill11, Theorem 4.9] for the proof, which relies on a Harnack estimate for linear parabolic equations.

Lemma 4.1.6. *Suppose that $u, f \in C^\infty(Q)$ satisfy*

$$\frac{\partial u}{\partial t} = \log\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) + f,$$

and assume also given a constant $C > 0$ such that

$$C^{-1} \le \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \le C \text{ and } \left|\frac{\partial f}{\partial t}\right| + |dd^c f| \le C.$$

For each $Q' = U' \times (\varepsilon, T)$ with $U' \Subset U$ an open subset and $\varepsilon \in (0, T)$, we can then find $A > 0$ and $0 < \alpha < 1$ only depending on U', ε and C such that

$$[dd^c u]_{\alpha, Q'} \le A.$$

Proof of Theorem 4.1.4. The proof consists in a standard boot-strapping argument. Consider the path $\omega_t := dd^c u_t$ of Kähler forms on U. By assumption, we have $\omega_t \le C_1 \omega_U$ with $C_1 > 0$ under control. Since

$$\omega_t^n = \exp\left(\frac{\partial u}{\partial t} - f\right) \omega_U^n$$

where $\frac{\partial u}{\partial t} - f$ is bounded below by a constant under control thanks to the assumptions, simple eigenvalue considerations show that $\omega_t \ge c\omega_U$ with $c > 0$ under control. We can thus apply the Evans–Krylov estimates of Lemma 4.1.6 and assume, after perhaps slightly shrinking Q, that $[\omega_t]_{\alpha, Q}$ is under control for some $0 < \alpha < 1$.

Now let D be any first order differential operator with constant coefficients. Differentiating (4.5), we get

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) Du = Df. \tag{4.6}$$

Since $\left|\frac{\partial u}{\partial t}\right| + |\Delta u|$ is under control, the elliptic Schauder estimates (for the flat Laplacian Δ) show in particular, after perhaps shrinking U (but not the time interval), that the C^0 norm of Du is under control. By the parabolic Schauder estimates of Lemma 4.1.5, the parabolic $C^{2,\alpha}$ norm of Du is thus under control as well. Applying D to (4.6) we find

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) D^2 u = D^2 f + \sum_{j,k} \left(D\omega_t^{jk}\right) \frac{\partial^2 Du}{\partial z_j \partial \bar{z}_k},$$

where the parabolic C^α norm of the right-hand side is under control. By the parabolic Schauder estimates, the parabolic $C^{2,\alpha}$ norm of $D^2 u$ is in turn under control, and iterating this procedure concludes the proof of Theorem 4.1.4. □

4.2 Smoothing Properties of the Kähler–Ricci Flow

By analogy with the regularizing properties of the heat equation, it is natural to expect that the Kähler–Ricci flow can be started from a singular initial data (say a positive current, rather than a Kähler form), instantaneously smoothing out the latter.

The goal of this section is to illustrate positively this expectation by explaining the proof of the following result of Szekelyhidi–Tosatti [SzTo11]:

Theorem 4.2.1. *Let* (X, ω) *be a* n*-dimensional compact Kähler manifold. Let* $F : \mathbb{R} \times X \to \mathbb{R}$ *be a smooth function and assume* $\psi_0 \in \mathrm{PSH}(X, \omega)$ *is continuous*[1] *and satisfies*

$$(\omega + dd^c \psi_0)^n = e^{-F(\psi_0, x)} \omega^n.$$

Then $\psi_0 \in C^\infty(X)$ *is smooth.*

As the reader will realize later on, the proof is a good warm up, as the arguments are similar to the ones we are going to use when proving Theorem 4.3.3.

Let us recall that such equations contain as a particular case the Kähler–Einstein equation. Namely when the cohomology class $\{\omega\}$ is proportional to the first Chern class of X,[2] $\lambda\{\omega\} = c_1(X)$ for some $\lambda \in \mathbb{R}$, then the above equation is equivalent to

$$\mathrm{Ric}(\omega + dd^c \psi_0) = \lambda(\omega + dd^c \psi_0),$$

when taking

$$F(\varphi, x) = \lambda\varphi + h(x)$$

with $h \in C^\infty(X)$ such that $\mathrm{Ric}(\omega) = \lambda\omega + dd^c h$. Szekelyhidi and Tosatti's result is thus particularly striking since the solutions to such equations, if any, are in general not unique.[3]

The interest in such regularity results stems for example from the recent works [BBGZ13, EGZ11] which provide new tools to construct weak solutions to such complex Monge–Ampère equations.

The idea of the proof is both simple and elegant, and goes as follows: assume we can run a complex Monge–Ampère flow

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\omega + dd^c \varphi)^n}{\omega^n}\right] + F(\varphi, x)$$

with an initial data $\varphi_0 \in \mathrm{PSH}(X, \omega) \cap C^0(X)$ in such a way that

$$\varphi \in C^0\left(X \times [0, T]\right) \cap C^\infty\left(X \times (0, T]\right).$$

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[1] The authors state their result assuming that ψ_0 is merely bounded, but they use in an essential way the continuity of ψ_0, which is nevertheless known in this context by Kołodziej [Kol98].

[2] This of course assumes that $c_1(X)$ has a definite sign.

[3] In the Kähler–Einstein Fano case, a celebrated result of Bando and Mabuchi [BM87] asserts that any two solutions are connected by the flow of a holomorphic vector field.

Then ψ_0 will be a fixed point of such a flow hence if ψ_t denotes the flow originating from ψ_0, $\psi_0 \equiv \psi_t$ has to be smooth!

To simplify our task, we will actually give full details only in case

$$F(s,x) = -G(s) + h(x) \text{ with } s \mapsto G(s) \text{ being } \textit{convex}$$

and merely briefly indicate what extra work has to be done to further establish the most general result. Note that this particular case nevertheless covers the Kähler–Einstein setting.

In the sequel we consider the above flow starting from a smooth initial potential φ_0 and establish various a priori estimates that eventually will allow us to start from a poorly regular initial data. We fix once and for all a finite time $T > 0$ (independent of φ_0) such that all flows to be considered are well defined on $X \times [0,T]$: it is standard that the maximal interval of time on which such a flow is well defined can be computed in cohomology, hence depends on the cohomology class of the initial data rather than on the (regularity properties of the) chosen representative.

4.2.1 A Priori Estimate on φ_t

We consider in this section on $X \times [0,T]$ the complex Monge–Ampère flow (*CMAF*)

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right] + F(\varphi, x)$$

with initial data $\varphi_0 \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$. Our aim is to bound $\|\varphi\|_{L^\infty(X\times[0,T])}$ in terms of $\|\varphi_0\|_{L^\infty(X)}$ and T.

4.2.1.1 Heuristic Control

Set $M(t) = \sup_X \varphi_t$. It suffices to bound $M(t)$ from above, the bound from below for $m(t) := \inf_X \varphi_t$ will follow by symmetry.

Assume that we can find $t \in [0,T] \mapsto x(t) \in X$ a differentiable map such that $M(t) = \varphi_t(x(t))$. Then M is differentiable and satisfies

$$M'(t) = \frac{\partial\varphi_t}{\partial t}(x(t)) \leq F(\varphi_t(x(t)), x(t)) \leq \overline{F}(M(t)),$$

where

$$\overline{F}(s) := \sup_{x\in X} F(s,x)$$

is a Lipschitz map.

It follows therefore from the Cauchy–Lipschitz theory of ODE's that $M(t)$ is bounded from above on $[0, T]$ in terms of T, $M(0) = \sup_X \varphi_0$ and $\overline{F}$ (hence F).

4.2.1.2 A Precise Bound

We now would like to establish a more precise control under a simplifying assumption:

Lemma 4.2.2. *Assume that* $\varphi, \psi \in C^\infty(X \times [0, T])$ *define* ω*-psh functions* φ_t, ψ_t *for all* t *and satisfy*

$$\frac{\partial \varphi}{\partial t} \le \log\left[\frac{(\omega + dd^c \varphi)^n}{\omega^n}\right] + F(\varphi, x)$$

and

$$\frac{\partial \psi}{\partial t} \ge \log\left[\frac{(\omega + dd^c \psi)^n}{\omega^n}\right] + F(\psi, x)$$

on $X \times [0, T]$*, where*

$$F(s, x) = \lambda s - G(s, x) \text{ with } s \mapsto G(s, \cdot) \text{ non-decreasing.}$$

Then we have

$$\sup_{X \times [0,T]} (\varphi - \psi) \le e^{\lambda T} \max\{\sup_X(\varphi_0 - \psi_0), 0\}.$$

Proof. Set $u(x, t) := e^{-\lambda t}(\varphi_t - \psi_t)(x) - \varepsilon t \in C^0(X \times [0, T])$, where $\varepsilon > 0$ is fixed (arbitrary small). Let $(x_0, t_0) \in X \times [0, T]$ be a point at which u is maximal.

If $t_0 = 0$, then $u(x, t) \le (\varphi_0 - \psi_0)(x_0) \le \sup_X(\varphi_0 - \psi_0)$ and we obtain the desired upper bound by letting $\varepsilon > 0$ decrease to zero.

Assume now that $t_0 > 0$. Then $\dot{u} \ge 0$ at this point, hence

$$0 \le -\varepsilon - \lambda e^{-\lambda t}(\varphi_t - \psi_t) + e^{-\lambda t}(\dot{\varphi}_t - \dot{\psi}_t).$$

On the other hand $dd^c_x u \le 0$, hence $dd^c_x \varphi_t \le dd^c_x \psi_t$ and

$$\begin{aligned}\dot{\varphi}_t - \dot{\psi}_t &\le F(\varphi_t, x) - F(\psi_t, x) + \log\left[\frac{(\omega + dd^c \varphi_t)^n}{(\omega + dd^c \psi_t)^n}\right] \\ &\le F(\varphi_t, x) - F(\psi_t, x).\end{aligned}$$

Recall now that $F(s, x) = \lambda s - G(s, x)$. Previous inequalities therefore yield

$$G(\varphi_t, x) < G(\psi_t, x) \text{ at point } (x, t) = (x_0, t_0).$$

Since $s \mapsto G(s, \cdot)$ is assumed to be non-decreasing, we infer $\varphi_{t_0}(x_0) \le \psi_{t_0}(x_0)$, so that for all $(x,t) \in X \times [0,T]$,

$$u(x,t) \le u(x_0,t_0) \le 0.$$

Letting ε decrease to zero yields the second possibility for the upper bound. □

By reversing the roles of φ_t, ψ_t, we obtain the following useful:

Corollary 4.2.3. *Assume φ, ψ are solutions of (CMAF) with F as above. Then*

$$\|\varphi - \psi\|_{L^\infty(X\times[0,T])} \le e^{\lambda T}\|\varphi_0 - \psi_0\|_{L^\infty(X)}.$$

As a consequence, if $\varphi_{0,j}$ is a sequence of smooth ω-psh functions decreasing to $\varphi_0 \in \mathrm{PSH}(X,\omega) \cap C^0(X)$, and φ_j are the corresponding solutions to (*CMAF*) on $X \times [0,T]$, then the sequence φ_j converges uniformly on $X \times [0,T]$ to some $\varphi \in C^0(X \times [0,T])$ as $j \to +\infty$.

4.2.2 A Priori Estimate on $\frac{\partial\varphi}{\partial t}$

We assume here again that on $X \times [0,T]$

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right] + F(\varphi, x)$$

with initial data $\varphi_0 \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$.

Lemma 4.2.4. *There exists $C > 0$ which only depends on $\|\varphi_0\|_{L^\infty(X)}$ such that for all $t \in [0,T]$,*

$$\|\dot{\varphi}_t\|_{L^\infty(X)} \le e^{CT}\|\dot{\varphi}_0\|_{L^\infty(X)}.$$

Let us stress that such a bound requires both that the initial potential φ_0 is uniformly bounded and that the initial density

$$f_0 = \frac{(\omega + dd^c\varphi_0)^n}{\omega^n} = \log\dot{\varphi}_0 - F(\varphi_0, x)$$

is uniformly bounded away from zero and infinity. We shall consider in the sequel more general situations with no a priori control on the initial density f_0.

Proof. Observe that

$$\frac{\partial\dot{\varphi}}{\partial t} = \Delta_t\dot{\varphi} + \frac{\partial F}{\partial s}(\varphi, x)\dot{\varphi},$$

where Δ_t denotes the Laplace operator associated to $\omega_t = \omega + dd^c\varphi_t$.

Since F is C^1, we can find a constant $C > 0$ which only depends on (F and) $\|\varphi\|_{L^\infty(X\times[0,T])}$ such that

$$-C < \frac{\partial F}{\partial s}(\varphi, x) < +C.$$

Consider $H_+(x,t) := e^{-Ct}\dot{\varphi}_t(x)$ and let (x_0, t_0) be a point at which H_+ realizes its maximum on $X \times [0, T]$. If $t_0 = 0$, then $\dot{\varphi}_t(x) \le e^{CT}\sup_X \varphi_0$ for all $(x,t) \in X \times [0, T]$. If $t_0 > 0$, then

$$0 \le \left(\frac{\partial}{\partial t} - \Delta_t\right)(H_+) = e^{-Ct}\left[\frac{\partial F}{\partial s}(\varphi_t, x) - C\right]\dot{\varphi}$$

hence $\dot{\varphi}_{t_0}(x_0) \le 0$, since $\frac{\partial F}{\partial s}(\varphi_t, x) - C < 0$. Thus $\dot{\varphi}_t(x) \le 0$ in this case. All in all, this shows that

$$\dot{\varphi}_t \le e^{CT}\max\left\{\sup_X \dot{\varphi}_0, 0\right\}.$$

Considering the minimum of $H_-(x,t) := e^{+Ct}\dot{\varphi}_t(x,t)$ yields a similar bound from below and finishes the proof since $\max\{\sup_X \dot{\varphi}_0, -\inf_X \dot{\varphi}_0\} \ge 0$. □

4.2.3 A Priori Estimate on $\Delta\varphi_t$

Recall that we are considering on $X \times [0, T]$

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right] + F(\varphi_t, x)$$

with initial data $\varphi_0 \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$. Our aim in this section is to establish an upper bound on $\Delta_\omega\varphi_t$, which is uniform as long as $t > 0$ and is allowed to blow up when t decreases to zero.

4.2.3.1 A Convexity Assumption

To simplify our task, we shall assume that

$$F(s,x) = -G(s) + h(x), \text{ with } s \mapsto G(s) \text{ being convex.}$$

This assumption allows us to bound from above $\Delta_\omega F(\varphi, x)$ as follows:

Lemma 4.2.5. *There exists $C > 0$ which only depends on $\|\varphi_0\|_{L^\infty(X)}$ such that*

$$\Delta_\omega (F(\varphi_t, x)) \leq C [1 + \mathrm{tr}_\omega(\omega_t)],$$

where $\omega_t = \omega + dd^c\varphi_t$.

Recall here that for any smooth function h and $(1,1)$-form β,

$$\Delta_\omega h := n\frac{dd^c h \wedge \omega^{n-1}}{\omega^n} \quad \text{while} \quad \mathrm{tr}_\omega\beta := n\frac{\beta \wedge \omega^{n-1}}{\omega^n}.$$

Proof. Observe that

$$dd^c (F(\varphi, x)) = -G''(\varphi)d\varphi \wedge d^c\varphi - G'(\varphi)dd^c\varphi \leq -G'(\varphi)dd^c\varphi$$

since G is convex. Now $dd^c\varphi = (\omega + dd^c\varphi) - \omega = \omega_\varphi - \omega = \omega_t - \omega$ is a difference of positive forms and $-C \leq -G'(\varphi) \leq +C$, therefore

$$dd^c (F(\varphi, x)) \leq C (\omega_t + \omega),$$

which yields the desired upper bound. □

Our simplifying assumption thus yields a bound from above on $\Delta_\omega (F(\varphi, x))$ which depends on $\mathrm{tr}_\omega(\omega_\varphi)$ (and $\|\varphi_0\|_{L^\infty(X)}$) but not on $\|\nabla\varphi_t\|_{L^\infty(X\times[\varepsilon,T])}$. A slightly more involved bound from above is available in full generality, which relies on Blocki's gradient estimate [Bło09]. We refer the reader to the proofs of [SzTo11, Lemmata 2.2 and 2.3] for more details.

4.2.3.2 The Estimate

Proposition 4.2.6. *Assume that $F(s, x) = -G(s) + h(x)$, with $s \mapsto G(s)$ convex. Then*

$$0 \leq \mathrm{tr}_\omega(\omega_t) \leq C \exp(C/t)$$

where $C > 0$ depends on $\|\varphi_0\|_{L^\infty(X)}$ and $\|\dot{\varphi}_0\|_{L^\infty(X)}$.

Proof. We set $u(x,t) := \mathrm{tr}_\omega(\omega_t)$ and

$$\alpha(x,t) := t \log u(x,t) - A\varphi_t(x),$$

where $A > 0$ will be specified later. The desired inequality will follow if we can uniformly bound α from above. Our plan is to show that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) \leq C_1 + (Bt + C_2 - A)\mathrm{tr}_{\omega_t}(\omega)$$

for uniform constants $C_1, C_2 > 0$ which only depend on $\|\varphi_0\|_{L^\infty(X)}$, $\|\dot{\varphi}_0\|_{L^\infty(X)}$.

Observe that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) = \log u + \frac{t}{u}\frac{\partial u}{\partial t} - A\dot{\varphi}_t - t\Delta_t \log u + A\Delta_t\varphi_t.$$

The last term yields $A\Delta_t\varphi_t = An - A\mathrm{tr}_{\omega_t}(\omega)$. The for to last one is estimated thanks to Proposition 4.1.2,

$$-t\Delta_t \log u \leq Bt\,\mathrm{tr}_{\omega_t}(\omega) + t\frac{\mathrm{tr}_\omega(\mathrm{Ric}(\omega_t))}{\mathrm{tr}_\omega(\omega_t)}.$$

It follows from Lemma 4.2.5 that

$$\begin{aligned}
\frac{t}{u}\frac{\partial u}{\partial t} &= \frac{t}{u}\Delta_t\left(\log\frac{\omega_t^n}{\omega^n}\right) + \frac{t}{u}\Delta_\omega F(\varphi_t, x)\\
&= \frac{t}{u}\{-\mathrm{tr}_\omega(\mathrm{Ric}\,\omega_t) + \mathrm{tr}_\omega(\mathrm{Ric}\,\omega)\} + \frac{t}{u}\Delta_\omega F(\varphi_t, x)\\
&\leq -t\frac{\mathrm{tr}_\omega(\mathrm{Ric}\,\omega_t)}{\mathrm{tr}_\omega(\omega_t)} + C\frac{(1+u)}{u}.
\end{aligned}$$

We infer

$$-t\Delta_t \log u + \frac{t}{u}\frac{\partial u}{\partial t} \leq Bt\,\mathrm{tr}_{\omega_t}(\omega) + C_1,$$

using that u is uniformly bounded below as follows from Proposition 4.1.2 again.

To handle the remaining (first and third) terms, we simply note that $\dot{\varphi}_t$ is uniformly bounded below, while

$$\log u \leq \log\left[C\,\mathrm{tr}_{\omega_t}(\omega)^{n-1}\right] \leq C_2 + C_3\mathrm{tr}_{\omega_t}(\omega)$$

by Proposition 4.1.2 and the elementary inequality $\log x < x$. Altogether this yields

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) \leq C_4 + (Bt + C_3 - A)\,\mathrm{tr}_{\omega_t}(\omega) \leq C_4,$$

if we choose $A > 0$ so large that $Bt + C_3 - A < 0$. The desired inequality now follows from the maximum principle. □

4.2.4 Proof of Theorem 4.2.1

4.2.4.1 Higher Order Estimates

By Theorem 4.1.4, it follows from our previous estimates that higher order a priori estimates hold as well:

Proposition 4.2.7. *For each fixed $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $C_k(\varepsilon) > 0$ which only further depends on $\|\varphi_0\|_{L^\infty(X)}$ and $\|\dot{\varphi}_0\|_{L^\infty(X)}$ such that*

$$\|\varphi_t\|_{C^k(X\times[\varepsilon,T])} \leq C_k(\varepsilon).$$

4.2.4.2 A Stability Estimate

Let $0 \leq f, g \in L^2(\omega^n)$ be densities such that

$$\int_X f\omega^n = \int_X g\omega^n = \int_X \omega^n.$$

It follows from the celebrated work of Kolodziej [Kol98] that there exists unique continuous ω-psh functions φ, ψ such that

$$(\omega + dd^c\varphi)^n = f\omega^n, (\omega + dd^c\psi)^n = g\omega^n \quad \text{and} \quad \int_X (\varphi - \psi)\omega^n = 0.$$

We shall need the following stability estimates:

Theorem 4.2.8. *There exists $C > 0$ which only depends on $\|f\|_{L^2}, \|g\|_{L^2}$ such that*

$$\|\varphi - \psi\|_{L^\infty(X)} \leq C\|f - g\|^\gamma_{L^2(X)},$$

for some uniform exponent $\gamma > 0$.

Such stability estimates go back to the work of Kolodziej [Kol03] and Blocki [Blo03]. Much finer stability results are available by now (see [DZ10, GZ12]). We sketch a proof of this version for the convenience of the reader.

Proof. The proof decomposes in two main steps. We first claim that

$$\|\varphi - \psi\|_{L^2(X)} \leq C\|f - g\|^{\frac{1}{2^n-1}}_{L^2(X)}, \tag{4.7}$$

for some appropriate $C > 0$. Indeed we are going to show that

$$\int_X d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge \omega^{n-1} \leq C_1 I(\varphi, \psi)^{2^{-(n-1)}}, \tag{4.8}$$

where

$$I(\varphi,\psi) := \int_X (\varphi-\psi)\{(\omega+dd^c\psi)^n - (\omega+dd^c\varphi)^n\} \geq 0$$

is non-negative, as the reader can check that an alternative writing is

$$I(\varphi,\psi) = \sum_{j=0}^{n-1}\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi^j\wedge\omega_\psi^{n-1-j}.$$

In our case the Cauchy–Schwarz inequality yields

$$I(\varphi,\psi) = \int_X (\varphi-\psi)(g-f)\omega^n \leq \|\varphi-\psi\|_{L^2}\|f-g\|_{L^2},$$

therefore (4.7) is a consequence of (4.8) and Poincaré's inequality.

To prove (4.8), we write $\omega = \omega_\varphi - dd^c\varphi$ and integrate by parts to obtain,

$$\begin{aligned}
&\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega^{n-1}\\
&\quad= \int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi\wedge\omega^{n-2}\\
&\qquad -\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge dd^c\varphi\wedge\omega^{n-2}\\
&\quad= \int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_{\varphi_1}\wedge\omega^{n-2}\\
&\qquad +\int d(\varphi-\psi)\wedge d^c\varphi\wedge(\omega_\varphi-\omega_\psi)\wedge\omega^{n-2}
\end{aligned}$$

We take care of the last term by using Cauchy–Schwarz inequality, which yields

$$\int d(\varphi-\psi)\wedge d^c\varphi\wedge\omega_\varphi\wedge\omega^{n-2} \leq A\left(\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi\wedge\omega^{n-2}\right)^{1/2},$$

where

$$A^2 = \int d\varphi\wedge d^c\varphi\wedge\omega_\varphi\wedge\omega^{n-2}$$

is uniformly bounded from above, since φ is uniformly bounded in terms of $\|f\|_{L^2(X)}$ by the work of Kolodziej [Kol98]. Similarly

$$-\int d(\varphi-\psi)\wedge d^c\varphi\wedge\omega_\psi\wedge\omega^{n-2} \leq B\left(\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\psi\wedge\omega^{n-2}\right)^{1/2},$$

where

$$B^2 = \int d\varphi \wedge d^c\varphi \wedge \omega_\psi \wedge \omega^{n-2}$$

is uniformly bounded from above. Note that both terms can be further bounded from above by the same quantity by bounding from above ω_φ (resp. ω_ψ) by $\omega_\varphi + \omega_\psi$.

Going on this way by induction, replacing at each step ω by $\omega_\varphi + \omega_\psi$, we end up with a control from above of $\int d(\varphi-\psi) \wedge d^c(\varphi-\psi) \wedge \omega^{n-1}$ by a quantity that is bounded from above by $CI(\varphi,\psi)^{2^{-(n-1)}}$ (there are $(n-1)$-induction steps), for some uniform constant $C > 0$. This finishes the proof of the first step.

The second step consists in showing that

$$\|\varphi - \psi\|_{L^\infty(X)} \leq C_2 \|\varphi - \psi\|^{\gamma}_{L^2(X)}$$

for some constants $C_2, \gamma > 0$. We are not going to dwell on this second step here, as it would take us too far. It relies on the comparison techniques between the volume and the Monge–Ampère capacity, as used in [Kol98]. □

4.2.4.3 Conclusion

We are now in position to conclude the proof of Theorem 4.2.1 [at least in case $F(s,x) = -G(s) + h(x)$, with G convex]. Let $\psi_0 \in \mathrm{PSH}(X,\omega)$ be a *continuous* solution to

$$(\omega + dd^c\psi_0)^n = e^{-F(\psi_0,x)}\omega^n.$$

Fix $u_j \in C^\infty(X)$ arbitrary smooth functions which uniformly converge to ψ_0 and let $\psi_j \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$ be the unique smooth solutions of

$$(\omega + dd^c\psi_j)^n = c_j e^{-F(u_j,x)}\omega^n,$$

normalized by $\int_X (\psi_j - \psi_0)\omega^n = 0$. Here $c_j \in \mathbb{R}$ are normalizing constants wich converge to 1 as $j \to +\infty$, such that

$$c_j \int_X e^{-F(u_j,x)}\omega^n = \int_X \omega^n,$$

and the existence (and uniqueness) of the ψ_j's is provided by Yau's celebrated result [Yau78]. It follows from the stability estimate (Theorem 4.2.8) that

$$\|\psi_j - \psi_0\|_{L^\infty(X)} \longrightarrow 0 \text{ as } j \to +\infty,$$

hence

$$\|\psi_j - u_j\|_{L^\infty(X)} \longrightarrow 0 \text{ as } j \to +\infty.$$

Consider the complex Monge–Ampère flows

$$\frac{\partial \varphi_{t,j}}{\partial t} = \log\left[\frac{(\omega + dd^c \varphi_{t,j})^n}{\omega^n}\right] + F(\varphi_{t,j}, x) - \log c_j,$$

with initial data $\varphi_{0,j} := \psi_j$. It follows from Lemma 4.2.2 that

$$\|\varphi_{t,j} - \varphi_{t,k}\|_{L^\infty(X\times[0,T])} \le e^{\lambda T}\|\psi_j - \psi_k\|_{L^\infty(X)} + \left|\log c_j - \log c_k\right|,$$

thus $(\varphi_{t,j})_j$ is a Cauchy sequence in the Banach space $C^0(X \times [0,T])$. We set

$$\varphi_t := \lim_{j\to+\infty} \varphi_{t,j} \in C^0(X \times [0,T]).$$

Note that $\varphi_t \in \mathrm{PSH}(X,\omega)$ for each $t \in [0,T]$ fixed and $\varphi_0 = \psi_0 = \lim \varphi_{0,j}$ by continuity. Proposition 4.2.7 shows moreover that $(\varphi_{t,j})_j$ is a Cauchy sequence in the Fréchet space $C^\infty(X \times (0,T])$, hence $(x,t) \mapsto \varphi_t(x) \in C^\infty(X \times (0,T])$. Observe that

$$\|\dot\varphi_{0,j}\|_{L^\infty(X)} = \|F(\psi_j, x) - F(u_j, x)\|_{L^\infty(X)} \le C\|\psi_j - u_j\|_{L^\infty(X)} \to 0.$$

Lemma 4.2.4 therefore yields for all $t > 0$,

$$\|\dot\varphi_t\|_{L^\infty(X)} = \lim_{j\to+\infty}\|\dot\varphi_{t,j}\|_{L^\infty(X)} \le C\lim_{j\to+\infty}\|\dot\varphi_{0,j}\|_{L^\infty(X)} = 0.$$

This shows that $t \mapsto \varphi_t$ is constant on $(0,T]$, hence constant on $[0,T]$ by continuity. Therefore $\psi_0 \equiv \varphi_t$ is smooth, as claimed.

4.3 Degenerate Parabolic Complex Monge–Ampère Equations

Until further notice, (X, ω_X) denotes a compact Kähler manifold of dimension n endowed with a reference Kähler form.

4.3.1 The Ample Locus

Recall that the *pseudoeffective cone* in $H^{1,1}(X,\mathbb{R})$ is the closed convex cone of classes of closed positive $(1,1)$-currents in X. A $(1,1)$-class α in the interior of the pseudoeffective cone is said to be *big*. Equivalently, α is big iff it can be represented by a *Kähler current*, i.e. a closed $(1,1)$-current T which is strictly positive in the sense that $T \geq c\omega_X$ for some $c > 0$. In the special case where the $(1,1)$-form θ is semipositive, it follows from [DemPaun04] that its class is big iff $\int_X \theta^n > 0$, i.e. iff θ is a Kähler form on at least an open subset of X.

The following result is a consequence of Demailly's regularization theorem [Dem92] (cf. [DemPaun04, Theorem 3.4]).

Lemma 4.3.1. *Let θ be a closed real $(1,1)$-form on X, and assume that its class in $H^{1,1}(X,\mathbb{R})$ is big. Then there exists a θ-psh function $\psi_\theta \leq 0$ such that:*

(i) ψ_θ is of class C^∞ on a Zariski open set $\Omega \subset X$,
(ii) $\psi_\theta \to -\infty$ near $\partial\Omega$,
(iii) $\omega_\Omega := (\theta + dd^c\psi_\theta)|_\Omega$ is the restriction to Ω of a Kähler form on a compactification $\tilde{X}$ of Ω dominating X.

More precisely, condition (iii) means that there exists a compact Kähler manifold $(\tilde{X}, \omega_{\tilde{X}})$ and a modification $\pi : \tilde{X} \to X$ such that π is an isomorphism over Ω and $\pi^*\omega_\Omega = \omega_{\tilde{X}}$ on $\pi^{-1}(\Omega)$.

By the Noetherian property of closed analytic subsets, it is easy to see that the set of all Zariski open subsets Ω so obtained admits a largest element, called the *ample locus* of θ and denoted by $\mathrm{Amp}\,(\theta)$ (see [Bou04, Theorem 3.17]). Note that $\mathrm{Amp}\,(\theta)$ only depends on the cohomology class of θ.

For later use, we also note:

Lemma 4.3.2. *Let θ be a closed real $(1,1)$-form with big cohomology class, and let $U \subset \mathrm{Amp}\,(\theta)$ be an arbitrary Zariski open subset. We can then find a θ-psh function τ_U such that τ_U is smooth on U and $\tau_U \to -\infty$ near ∂U.*

Proof. Let ψ_θ be a function as in Lemma 4.3.1, with $\Omega = \mathrm{Amp}\,(\theta)$. Since $A := X \setminus U$ is a closed analytic subset, it is easy to construct an ω_X-psh function ρ with logarithmic poles along A (see for instance [DemPaun04]). We then set $\tau_U := \psi_\theta + c\rho$ with $c > 0$ small enough to have $\theta + dd^c\psi_\theta \geq \delta\omega_X$ for some $\delta > 0$. □

4.3.2 The Main Result

In the next sections, we will provide a detailed proof of the following result, which is a mild generalization of the technical heart of [ST09]. The assumptions on the measure μ will become more transparent in the context of the Kähler–Ricci flow on varieties with log-terminal singularities, cf. Sect. 4.6.

Theorem 4.3.3. *Let X be a compact Kähler manifold, $T \in (0,+\infty)$, and let $(\theta_t)_{t\in[0,T]}$ be a smooth path of closed semipositive $(1,1)$-forms such that $\theta_t \geq \theta$ for a fixed semipositive $(1,1)$-form θ with big cohomology class. Let also μ be positive measure on X of the form*

$$\mu = e^{\psi^+ - \psi^-} \omega_X^n$$

where

- $\psi^\pm$ *are quasi-psh functions on* X *(i.e. there exists* $C > 0$ *such that* $\psi^\pm$ *are both* $C\omega_X$*-psh);*
- $e^{-\psi^-} \in L^p$ *for some* $p > 1$*;*
- $\psi^\pm$ *are smooth on a given Zariski open subset* $U \subset \mathrm{Amp}(\theta)$.

For each continuous θ_0*-psh function* $\varphi_0 \in C^0(X) \cap \mathrm{PSH}(X, \theta_0)$*, there exists a unique bounded continuous function* $\varphi \in C_b^0(U \times [0, T))$ *with* $\varphi|_{U \times \{0\}} = \varphi_0$ *and such that on* $U \times (0, T)$ φ *is smooth and satisfies*

$$\frac{\partial \varphi}{\partial t} = \log \left[\frac{(\theta_t + dd^c \varphi)^n}{\mu} \right]. \tag{4.9}$$

Furthermore, φ *is in fact smooth up to time* T*, i.e.* $\varphi \in C^\infty(U \times (0, T])$.

Remark 4.3.4. Since $\varphi|_{X \times \{t\}}$ is bounded and θ_t-psh on a Zariski open set of X, it uniquely extends to a bounded θ_t-psh function on X by standard properties of psh functions. We get in this way a natural quasi-psh extension of φ to a bounded function on $X \times [0, T]$, but note that no continuity property is claimed on $X \times [0, T]$ (see however Theorem 4.3.5 below).

As we shall see, uniqueness in Theorem 4.3.3 holds in a strong sense: we have

$$\sup_{U \times [0,T]} |\varphi - \varphi'| = \sup_{U \times \{0\}} |\varphi - \varphi'|$$

for any two

$$\varphi, \varphi' \in C_b^0(U \times [0, T)) \cap C^\infty(U \times (0, T))$$

satisfying (4.9) and such that the restriction to $U \times \{0\}$ of either φ or φ' extends continuously to $X \times \{0\}$.

In the geometric applications to the (unnormalized) Kähler–Ricci flow, the path (θ_t) will be affine as a function of t. In that case, we have a global control on the time derivative:

Theorem 4.3.5. *With the notation of Theorem 4.3.3, assume further that* $(\theta_t)_{t \in [0,T]}$ *is an* affine *path. For each* $\varepsilon > 0$, $\frac{\partial \varphi}{\partial t}$ *is then bounded above on* $U \times [\varepsilon, T]$*, and bounded below on* $U \times [\varepsilon, T - \varepsilon]$*. In particular, the quasi-psh extension of* φ *to* $X \times [0, T]$ *is continuous on* $X \times (0, T)$*, and on* $X \times \{T\}$ *as well.*

4.4 A Priori Estimates for Parabolic Complex Monge–Ampère Equations

4.4.1 Setup

Recall that (X, ω_X) is a compact Kähler manifold endowed with a reference Kähler form. In this section, $(\theta_t)_{t\in[0,T]}$ denotes a smooth path of *Kähler forms* on X, and we assume given a *semipositive* $(1,1)$-form θ with big cohomology class such that

$$\theta_t \geq \theta \text{ for } t \in [0,T].$$

Let also μ be a smooth positive volume form on X, and suppose that $\varphi \in C^\infty(X \times [0,T])$ satisfies

$$\frac{\partial \varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right]. \tag{4.10}$$

Our goal is to provide a priori estimates on φ that only depend on θ, the sup norm of $\varphi_0 := \varphi|_{X\times\{0\}}$, and the L^p-norm and certain Hessian bounds for the density f of μ. More precisely, we will prove the following result:

Theorem 4.4.1. *With the above notation, suppose that μ is written as*

$$\mu = e^{\psi^+ - \psi^-}\omega_X^n$$

with $\psi^\pm \in C^\infty(X)$, and assume given $C > 0$ and $p > 1$ such that

(i) $-C \leq \sup_X \psi^\pm \leq C$ *and* $dd^c\psi^\pm \geq -C\omega_X$.
(ii) $\|e^{-\psi^-}\|_{L^p} \leq C$.
(iii) $\|\varphi_0\|_{C^0} \leq C$.

The C^0 norm of φ on $X \times [0,T]$ is then bounded in terms of θ, C, T, p and a bound on the volume $\int_X \theta_t^n$ for $t \in [0,T]$.

Further, φ is bounded in C^∞ topology on $\operatorname{Amp}(\theta) \times (0,T]$, *uniformly in terms of θ, C, T, p and C^∞ bounds for (θ_t) on $X \times [0,T]$ and for $\psi^\pm$ on* $\operatorname{Amp}(\theta)$. *More explicitly, for each compact set $K \Subset \operatorname{Amp}(\theta)$, each $\varepsilon > 0$ and each $k \in \mathbb{N}$, the C^k-norm of φ on $K \times [\varepsilon, T]$ is bounded in terms of θ, C, T, p and C^∞ bounds for (θ_t) on $X \times [0,T]$ and for $\psi^\pm$ in any given neighborhood of K.*

During the proof, we shall use the following notation. We introduce the smooth path of Kähler forms

$$\omega_t := \theta_t + dd^c\varphi_t,$$

and denote by $\Delta_t = \operatorname{tr}_{\omega_t} dd^c$ the corresponding time-dependent Laplacian operator on functions. We trivially have

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)\varphi = \dot{\varphi} + \operatorname{tr}_{\theta_t}(\theta_t) - n, \tag{4.11}$$

where $\dot{\varphi}$ is a short-hand for $\frac{\partial \varphi}{\partial t}$. Writing $\dot{\theta}_t$ for the time-derivative of θ_t, it is also immediate to see that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)\dot{\varphi} = \operatorname{tr}_{\theta_t}(\dot{\theta}_t) \tag{4.12}$$

To simplify the notation, we set $\Omega := \operatorname{Amp}(\theta)$, and choose a θ-psh function ψ_θ as in Lemma 4.3.1, so that $\psi_\theta \to -\infty$ near $\partial\Omega$ and

$$\omega_\Omega := (\theta + dd^c\psi_\theta)|_\Omega$$

is the restriction to Ω of a Kähler form on a compactification of Ω dominating X. Since $\theta_t \geq \theta$ for all t, (4.11) shows that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\varphi - \psi_\theta) \geq \dot{\varphi} + \operatorname{tr}_{\theta_t}(\omega_\Omega) - n \tag{4.13}$$

on $\Omega \times [0, T]$.

As a matter of terminology, we shall say that a quantity is *under control* if it can be bounded by a constant only depending on the desired quantities within the proof of a given lemma.

4.4.2 A Global C^0-Estimate

Lemma 4.4.2. *Suppose that $\varphi \in C^\infty(X \times [0, T])$ satisfies (4.10). Assume given $C > 0$, $p > 1$ such that*

(i) $\int_X \theta_t^n \leq C$ for $t \in [0, T]$;
(ii) $\int_X \mu \geq C^{-1}$ and $\|f\|_{L^p} \leq C$ for the density $f := \mu/\omega_X^n$.

Then there exists a constant $A > 0$ only depending on θ, p, T and C such that

$$\sup_{X\times[0,T]} |\varphi| \leq \sup_{X\times\{0\}} |\varphi| + A.$$

Proof. **Step 0: an auxiliary construction.** We introduce an auxiliary function, which will also be used in the proof of Lemma 4.4.4 below. For $\varepsilon \in (0, 1/2]$ introduce the Kähler form

$$\eta_\varepsilon := (1-\varepsilon)\theta + \varepsilon^2\omega_X,$$

and set

$$c_\varepsilon := \log\left(\frac{\int \eta_\varepsilon^n}{\int \mu}\right).$$

Since θ_t is a continuous family of Kähler forms, we can fix $\varepsilon > 0$ small enough such that $\theta_t \geq \varepsilon\omega_X$ for all $t \in [0, T]$. Since $\int_X \theta^n$ is positive, c_ε is under control, even though ε itself is not! Observe also that $\theta_t \geq (1-\varepsilon)\theta + \varepsilon\theta_t$, and hence

$$\theta_t \geq \eta_\varepsilon \text{ for } t \in [0, T]. \tag{4.14}$$

By [Yau78] there exists a unique smooth η_ε-psh function ρ_ε such that $\sup_X \rho_\varepsilon = 0$ and

$$(\eta_\varepsilon + dd^c\rho_\varepsilon)^n = e^{c_\varepsilon}\mu. \tag{4.15}$$

Since the L^p-norm of the density of $e^{c_\varepsilon}\mu$ is under control and since

$$\tfrac{1}{2}\theta \leq \eta_\varepsilon \leq \theta + \omega_X \leq C_1\omega_X$$

with $C_1 > 0$ only depending on θ, the uniform version of Kolodziej's L^∞-estimates [EGZ09] shows that the C^0 norm of ρ_ε is under control.
Step 1: lower bound. Consider η_ε and ρ_ε as in Step 0, and set

$$H := \varphi - \rho_\varepsilon - c_\varepsilon t.$$

By (4.15) and (4.14) we get

$$\frac{\partial H}{\partial t} = \log\frac{(\theta_t + dd^c\rho_\varepsilon + dd^cH)^n}{(\eta_\varepsilon + dd^c\rho_\varepsilon)^n} \geq \log\frac{(\theta_t + dd^c\rho_\varepsilon + dd^cH)^n}{(\theta_t + dd^c\rho_\varepsilon)^n}$$

on $X \times [0, T]$, and hence $\inf_{X\times[0,T]} H = \inf_{X\times\{0\}} H$ by Proposition 4.1.3. Since c_ε and the C^0 norm of ρ_ε are both under control, we get the desired lower bound for φ.
Step 2: upper bound. By non-negativity of the relative entropy of the probability measure $\mu/\int\mu$ with respect to

$$\frac{(\theta_t + dd^c\varphi)^n}{\int\theta_t^n} = \frac{e^{\dot\varphi}\mu}{\int\theta_t^n}$$

(or, in other words, by concavity of the logarithm and Jensen's inequality), we have

$$\int\left(\log\left(\frac{\int\theta_t^n}{\int\mu}\right) - \dot\varphi\right)\mu \geq 0.$$

It follows that

$$\frac{d}{dt}\left(\int \varphi_t \mu\right) \le \left(\int \mu\right) \log\left(\int \theta_t^n\right) - \left(\int \mu\right) \log\left(\int \mu\right)$$
$$\le \|f\|_{L^p} \left(\int \omega_X^n\right)^{1-1/p} \log C + e^{-1} =: A_1$$

is under control, and hence

$$\sup_{t\in[0,T]} \frac{\int \varphi_t \mu}{\int \mu} \le \frac{\int \varphi_0 \mu}{\int \mu} + A_1 T \le \sup_X \varphi_0 + A_1 T$$

with $A_1 > 0$ under control. We claim that there exists $B > 0$ under control such that

$$\sup_X \psi \le \frac{\int \psi \mu}{\int \mu} + B$$

for all θ-psh functions ψ. Applying this with $\psi = \varphi_t$ will yield the desired control on its upper bound. By Skoda's integrability theorem in its uniform version [Zer01], there exist $\delta > 0$ and $B > 0$ only depending on θ such that

$$\int e^{-\delta\psi} \omega_X^n \le B$$

for all θ-psh functions ψ normalized by $\sup_X \psi = 0$. By Hölder's inequality, it follows that $\int e^{-\delta'\psi} \mu \le B'$ with δ', B' under control, and the claim follows by Jensen's inequality. ☐

Remark 4.4.3. The proof given above is directly inspired from that of [ST09, Lemma 3.8]. Let us stress, as a pedagogical note to the non expert reader, that the C^0-estimate thus follows from

- The elementary maximum principle (Proposition 4.1.3);
- Kolodziej's L^∞ estimate for solutions of Monge–Ampère equations [Kol98, EGZ09];
- Skoda's exponential integrability theorem for psh functions (which is in fact also an ingredient in the previous item).

4.4.3 Bounding the Time-Derivative and the Laplacian on the Ample Locus

Lemma 4.4.4. *Suppose that $\varphi \in C^\infty(X \times [0,T])$ satisfies (4.10). Assume that the volume form μ is written as*

$$\mu = e^{\psi^+ - \psi^-} \omega_X^n$$

with $\psi^\pm \in C^\infty(X)$, *and let* $C > 0$ *be a constant such that*

(i) $-C\omega_X \le \dot{\theta}_t \le C\omega_X$ *for* $t \in [0, T]$*;*
(ii) $\sup_X \psi^\pm \le C$*;*
(iii) $dd^c \psi^\pm \ge -C\omega_X$.

For each compact set $K \subset \Omega$, *we can then find* $A > 0$ *only depending on* K, θ, T *and* C *such that*

$$\sup_{K\times[0,T]} t\left(|\dot{\varphi}| + \log|\Delta\varphi|\right) \le A\left(1 + \sup_{X\times[0,T]} |\varphi| - \inf_K \psi^-\right),$$

where Δ *denotes the Laplacian with respect to the reference metric* ω_X.

Proof. Since ω_Ω extends to a Kähler form on a compactification of Ω dominating X, there exists a constant $c > 0$ under control such that $\omega_\Omega \ge c\omega_X$, and hence

$$\theta_t + dd^c \psi_\theta \ge c\omega_X \tag{4.16}$$

on $\Omega \times [0, T]$ since $\theta_t \ge \theta$.

Step 1: upper bound for $\dot{\varphi}$. We want to apply the maximum principle to

$$H^+ := t\dot{\varphi} + A(\psi_\theta - \varphi),$$

with a constant $A > 0$ to be specified in a moment. Thanks to (4.12) and (4.13), we get

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^+ \le -(A-1)\dot{\varphi} + \operatorname{tr}_{\omega_t}\left(t\dot{\theta}_t - A\omega_\Omega\right) + An.$$

By (i) and (4.16) we have

$$t\dot{\theta}_t - A\omega_\Omega \le TC\omega_X - Ac\omega_X.$$

Choosing

$$A := c^{-1}TC + 1,$$

we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^+ \le -A_1\dot{\varphi} + A_2 \tag{4.17}$$

with $A_1, A_2 > 0$ under control. We are now in a position to apply the maximum principle. Since $\psi_\theta \to -\infty$ near $\partial\Omega$, H^+ achieves its maximum on $\Omega \times [0, T]$ at some point (x_0, t_0). If $t_0 = 0$ then

$$\sup_{\Omega\times[0,T]} H^+ \le -A \inf_X \varphi,$$

since $\psi_\theta \le 0$. If $t_0 > 0$ then $\left(\frac{\partial}{\partial t} - \Delta_t\right) H^+ \ge 0$ at (x_0, t_0), and (4.17) yields an upper bound for $\dot{\varphi}$ at (x_0, t_0). It follows that

$$\sup_{\Omega\times[0,T]} H^+ \le C_1 - A \inf_X \varphi.$$

with $C_1 > 0$ under control. Since ψ_θ is bounded below on the given compact set $K \subset \Omega$, we get in particular the desired upper bound on $t\dot{\varphi}$.
Step 2: lower bound for $\dot{\varphi}$. We now want to apply the maximum principle to

$$H^- := t(-\dot{\varphi} + 2\psi^-) + A(\psi_\theta - \varphi),$$

which satisfies by (4.12) and (4.13)

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^- \le -(A+1)\dot{\varphi} + 2\psi^- + \mathrm{tr}_{\omega_t}\left(-t\dot{\theta}_t - 2t dd^c \psi^- - A\omega_\Omega\right) + An. \tag{4.18}$$

On the one hand, note that

$$-\dot{\varphi} = \log\left(\frac{\omega_X^n}{\omega_t^n}\right) + \psi^+ - \psi^-,$$

and hence

$$-\dot{\varphi} \le (\dot{\varphi} - 2\psi^-) + 2\log\left(\frac{\omega_X^n}{\omega_t^n}\right) + 2C$$

since $\sup_X \psi^+ \le C$. Using $\psi_\theta \le 0$, we also have $t(\dot{\varphi} - 2\psi^-) \le -(H^- + A\varphi)$, and we get

$$-(A+1)\dot{\varphi} + 2\psi^- \le -t^{-1}(A+1)(H^- + A\varphi) + 2(A+1)\log\left(\frac{\omega_X^n}{\omega_t^n}\right) + (2A+4)C$$

on $X \times (0, T]$, using this time $\sup_X \psi^- \le C$. On the other hand, (i), (iii) and (4.16) show that

$$-t\dot{\theta}_t - 2t dd^c \psi^- - A\omega_\Omega \le (3TC - Ac)\omega_X,$$

which is bounded above by $-\omega_X$ if we choose

$$A := c^{-1}\,(3TC + 1)\,.$$

Plugging these estimates into (4.18), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^- \le -t^{-1}(A+1)(H^- + A\varphi) + 2(A+1)\log\left(\frac{\omega_X^n}{\omega_t^n}\right) - \operatorname{tr}_{\omega_t}(\omega_X) + C_2$$

on $\Omega \times (0,T]$, with $C_2 > 0$ under control. By the arithmetico-geometric inequality and the fact that $2(A+1)\log y - ny^{1/n}$ is bounded above for $y \in (0,+\infty[$, we have

$$2(A+1)\log\left(\frac{\omega_X^n}{\omega_t^n}\right) - \operatorname{tr}_{\omega_t}(\omega_X) \le 2(A+1)\log\left(\frac{\omega_X^n}{\omega_t^n}\right) - n\left(\frac{\omega_X^n}{\omega_t^n}\right)^{1/n} \le C_3$$

and hence

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^- \le -t^{-1}(A+1)(H^- + A\varphi) + C_4 \tag{4.19}$$

with $C_3, C_4 > 0$ under control. We can now apply the maximum principle to obtain as before

$$\sup_{\Omega\times[0,T]} H^- \le C_5 - A\inf_X \varphi.$$

this yields the desired lower bound on $\dot{\varphi}$.

Step 3: Laplacian bound. We are going to apply the maximum principle to

$$H := t(\log \operatorname{tr}_{\omega_\Omega}(\omega_t) + \psi^-) + A(\psi_\theta - \varphi),$$

with $A > 0$ to be specified below. Since ω_Ω extends to a Kähler metric on some compactification of Ω, its holomorphic bisectional curvature is bounded below by $-C_1$ with $C_1 > 0$ under control, and Proposition 4.1.2 yields

$$-\Delta_t \log \operatorname{tr}_{\omega_\Omega}(\omega_t) \le \frac{\operatorname{tr}_{\omega_\Omega}\operatorname{Ric}(\omega_t)}{\operatorname{tr}_{\omega_\Omega}(\omega_t)} + C_1 \operatorname{tr}_{\omega_t}(\omega_\Omega). \tag{4.20}$$

On the one hand, we have $\operatorname{Ric}(\omega_t) = \operatorname{Ric}(\mu) - dd^c\dot{\varphi}$ since $\omega_t^n = e^{\dot{\varphi}}\mu$. On the other hand,

$$\frac{\partial}{\partial t}\log \operatorname{tr}_{\omega_\Omega}(\omega_t) = \frac{\operatorname{tr}_{\omega_\Omega}(\dot{\theta}_t + dd^c\dot{\varphi}_t)}{\operatorname{tr}_{\omega_\Omega}(\omega_t)},$$

and hence

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) \log \operatorname{tr}_{\omega_\Omega}(\omega_t) \le \frac{\operatorname{tr}_{\omega_\Omega}\left(\operatorname{Ric}(\mu) + \dot{\theta}_t\right)}{\operatorname{tr}_{\omega_\Omega}(\omega_t)} + C_1 \operatorname{tr}_{\omega_t}(\omega_\Omega).$$

Now $\dot{\theta}_t \le C\omega_X$ by assumption, and

$$\operatorname{Ric}(\mu) = -dd^c\psi^+ + dd^c\psi^- + \operatorname{Ric}(\omega_X) \le C_2\omega_\Omega + dd^c\psi^-$$

for some $C_2 > 0$ under control, using $dd^c\psi^+ \ge -Cc^{-1}\omega_\Omega$ and

$$\operatorname{Ric}(\omega_X) \le C'\omega_X \le C'c^{-1}\omega_X.$$

It follows that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) \log \operatorname{tr}_{\omega_\Omega}(\omega_t) \le \frac{C_3 + \Delta_{\omega_\Omega}\psi^-}{\operatorname{tr}_{\omega_\Omega}(\omega_t)} + C_1 \operatorname{tr}_{\omega_t}(\omega_\Omega)$$

with $C_3 > 0$ under control. In order to absorb the term involving ψ^- in the left-hand side, we note that $Cc^{-1}\omega_\Omega + dd^c\psi^- \ge 0$, and hence

$$Cc^{-1}\omega_\Omega + dd^c\psi^- \le \operatorname{tr}_{\omega_t}(Cc^{-1}\omega_\Omega + dd^c\psi^-)\omega_t,$$

which yields after taking the trace with respect to ω_Ω

$$0 \le \frac{nCc^{-1} + \Delta_{\omega_\Omega}\psi^-}{\operatorname{tr}_{\omega_\Omega}(\omega_t)} \le Cc^{-1}\operatorname{tr}_{\omega_t}(\omega_\Omega) + \Delta_t\psi^-.$$

Using the trivial inequality $\operatorname{tr}_{\omega_\Omega}(\omega_t)\operatorname{tr}_{\omega_t}(\omega_\Omega) \ge n$ we arrive at

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) (\log \operatorname{tr}_{\omega_\Omega}(\omega_t) + \psi^-) \le C_4 \operatorname{tr}_{\omega_t}(\omega_\Omega) \tag{4.21}$$

with $C_4 > 0$ under control. By (4.21) and (4.11) we thus get

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H \le \log \operatorname{tr}_{\omega_\Omega}(\omega_t) + \psi^- - A\dot{\varphi} + (C_4T - A)\operatorname{tr}_{\omega_t}(\omega_\Omega).$$

Lemma 4.1.1 shows that

$$\log \operatorname{tr}_{\omega_\Omega}(\omega_t) + \psi^- \le \dot{\varphi} + (n-1)\log \operatorname{tr}_{\omega_t}(\omega_\Omega) + C_5, \tag{4.22}$$

since $\sup_X \psi^+ \le C$ and $\omega_X \le C_1\omega_\Omega$. If we choose $A := C_4T + 2$, we finally obtain

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H \leq -\mathrm{tr}_{\omega_t}(\omega_\Omega) - C_6\dot{\varphi} + C_7, \tag{4.23}$$

since $(n-1)\log y - 2y \leq -y + O(1)$ for $y \in (0, +\infty)$.

We are now in a position to apply the maximum principle. Since $\psi_\theta \to -\infty$ near $\partial\Omega$, there exists $(x_0, t_0) \in \Omega \times [0, T]$ such that $H(x_0, t_0) = \sup_{\Omega\times[0,T]} H$. If $t_0 = 0$ then

$$\sup_{\Omega\times[0,T]} H \leq A \sup_X |\varphi_0|,$$

using $\psi_\theta \leq 0$. If $t_0 > 0$ then $\left(\frac{\partial}{\partial t} - \Delta_t\right) H \geq 0$ at (x_0, t_0), and (4.23) yields

$$\mathrm{tr}_{\omega_t}(\omega_\Omega) + C_6\dot{\varphi} \leq C_7,$$

and in particular $\dot{\varphi} < C_7/C_6$, at (x_0, t_0). Plugging this into (4.22), it follows that

$$\log \mathrm{tr}_{\omega_\Omega}(\omega_t) + \psi^- \leq \dot{\varphi} + (n-1)\log(-C_6\dot{\varphi} + C_7) + C_5$$

at (x_0, t_0). Since $y + (n-1)\log(-C_6 y + C_7)$ is bounded above for $y \in]-\infty, C_7/C_6[$, we get $\log \mathrm{tr}_{\omega_\Omega}(\omega_t) + \psi^- \leq C_8$ at (x_0, t_0), and hence

$$\sup_{\Omega\times[0,T]} H = H(x_0, t_0) \leq A \sup_{X\times[0,T]} |\varphi| + TC_8.$$

Finally, there exists a constant $C_K > 0$ such that on the given compact set $K \subset \Omega$ we have $\psi_\theta \geq -C_K$ and $\omega_\Omega \leq C_K\omega$, and the result follows. □

Remark 4.4.5. The arguments used to bound the time-derivative are a combination of the proofs of Lemmas 3.2 and 3.9 in [ST09]. The proof of the Laplacian bound is similar in essence to that of [ST09, Lemma 3.3].

4.4.4 Bounding the Time Derivative in the Affine Case

Lemma 4.4.6. *Under the assumptions of Lemma 4.4.2, suppose that (θ_t) is an* affine *path, so that $\dot{\theta}_t$ is independent of t. Then we have*

$$\sup_{X\times[0,T]} (t\dot{\varphi}) \leq 2 \sup_{X\times[0,T]} |\varphi| + nT,$$

and for each $T' < T$ there exists $A > 0$ only depending on θ, C, p and T' such that

$$\inf_{X\times[0,T']}(t\dot{\varphi}) \geq -A\left(1+\sup_{X\times[0,T]}|\varphi|\right).$$

Proof. The upper bound follows directly from the maximum principle applied to

$$H^+ := t\dot{\varphi} - \varphi - nt,$$

which satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^+ = \operatorname{tr}_{\omega_t}(t\dot{\theta}_t - \theta_t) = \operatorname{tr}_{\omega_t}(-\theta_0) \leq 0$$

on $X \times [0, T]$. To get the lower bound, take ρ_ε as in Step 0 of the proof of Lemma 4.4.2, and set

$$H^- := -t\dot{\varphi} - A\varphi + \rho_\varepsilon.$$

where $A > 0$ will be specified in a moment. We then have

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H^- = -(A+1)\dot{\varphi} + \operatorname{tr}_{\omega_t}(-t\dot{\theta}_t - A\theta_t - dd^c\rho_\varepsilon) + An.$$

Since θ_t is affine, we have

$$A\theta_t + t\dot{\theta}_t = A\left(\theta_0 + t\dot{\theta}_t + \frac{t}{A}\dot{\theta}_t\right) = A\theta_{(A+1)t/A},$$

and hence

$$A\theta_t + t\dot{\theta}_t \geq \eta_\varepsilon$$

for $t \in [0, T']$ by (4.14), if we fix $A \gg 1$ such that $(A+1)T'/A < T$. With this choice of A we get on $X \times [0, T']$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_t\right) H^- &\leq (A+1)\log\left(\frac{\mu}{\omega_t^n}\right) - \operatorname{tr}_{\omega_t}(\eta_\varepsilon + dd^c\rho_\varepsilon) + An \\ &\leq (A+1)\log\left(\frac{\mu}{\omega_t^n}\right) - ne^{c_\varepsilon/n}\left(\frac{\mu}{\omega_t^n}\right)^{1/n} \leq A_1\end{aligned}$$

with $A_1 > 0$ under control, using the arithmetico-geometric inequality, (4.15) and the fact that $(A+1)\log y - ne^{c_\varepsilon/n}y^{1/n}$ is bounded above for $y \in (0, +\infty)$. The lower bound for $t\dot{\varphi}$ follows from the maximum principle, since $\sup_X|\rho_\varepsilon|$ is under control. □

Remark 4.4.7. This result corresponds to [ST09, Lemma 3.21].

4.4.5 Proof of Theorem 4.4.1

We are now in a position to prove Theorem 4.4.1. Since $\sup_X \psi^+$ is assumed to be bounded below, the mean value inequality for $C\omega_X$-psh functions shows that $\int \psi^+ \omega_X^n$ is also bounded below. By Jensen's inequality and the upper bound on ψ^-, it follows that condition (i) in Lemma 4.4.2 is satisfied. Using the upper bound on ψ^+ and the L^p bound for $e^{-\psi^-}$, we also check condition (ii) of Lemma 4.4.2, which therefore shows that the sup-norm of φ on $X \times [0, T]$ is bounded in terms of C.

By Lemma 4.4.4, on any given neighborhood of $K \times [\varepsilon, T]$ $|\dot{\varphi}|$ and $|\Delta\varphi|$ are bounded in terms of the C, the C^1 norm of (θ_t) on $X \times [0, T]$ and the C^0 norm of ψ^- on the neighborhood in question. We conclude by applying the Evans–Krylov type estimates of Theorem 4.1.4 locally on the ample locus of θ.

4.5 Proof of the Main Theorem

Our goal in this section is to prove Theorems 4.3.3 and 4.3.5.

4.5.1 The Non-degenerate Case

As a first step towards the proof of Theorem 4.3.3, we first consider the non-degenerate case, which amounts to the following result:

Theorem 4.5.1. *Let X be a compact Kähler manifold and $0 < T < +\infty$. Let $(\theta_t)_{t \in [0,T]}$ be a smooth family of Kähler forms on X and let μ be a smooth positive volume form. If $\varphi_0 \in C^\infty(X)$ is strictly θ_0-psh, i.e. $\theta_0 + dd^c\varphi_0 > 0$, then there exists a unique $\varphi \in C^\infty(X \times [0, T])$ such that $\varphi|_{X \times \{0\}} = \varphi_0$ and*

$$\frac{\partial \varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right]$$

on $X \times [0, T]$.

At least in the case of the Kähler–Ricci flow, this result goes back to [Cao85, Tsu88, Tzha06], see Theorem 3.3.1 in Chap. 3 of the present volume. But since the above statement follows directly from the a priori estimates we have proved so far (Theorem 4.4.1), we may as well provide a proof for completeness.

Proof. Uniqueness follows from the maximum principle (Proposition 4.1.3). By the general theory of non-linear parabolic equations, the solution φ is defined on a maximal half-open interval $[0, T')$ with $T' \le T$. Since $(\theta_t)_{t\in[0,T]}$ is a smooth path of Kähler metrics, we have $\theta_t \ge c\omega_X$ for all $t \in [0, T]$ if $c > 0$ is small enough, and we may thus apply Theorem 4.4.1 with $\theta = c\omega_X$ and $K = X = \mathrm{Amp}\,(\theta)$ to get that all C^k norms of φ are bounded on $X \times [\varepsilon, T')$ for any fixed $\varepsilon > 0$. It follows that φ extends to a C^∞ function on $X \times [0, T']$. Since $\dot{\varphi}$ is in particular bounded below, the smooth function $\varphi|_{X\times\{T'\}}$ is strictly $\theta_{T'}$-psh. By the local existence result, we conclude that $T' = T$, since φ could otherwise be extended beyond the maximal existence time T'. □

4.5.2 A Stability Estimate

If $\varphi, \varphi' \in C^\infty(X \times [0, T])$ are two solutions as in Theorem 4.5.1 corresponding to two initial data $\varphi_0, \varphi_0' \in C^\infty(X)$, the maximum principle of Proposition 4.1.3 immediately implies that

$$\|\varphi - \varphi'\|_{C^0(X\times[0,T])} \le \|\varphi_0 - \varphi_0'\|_{C^0(X)}.$$

In order to treat the general case of Theorem 4.3.3, we need to generalize this estimate when the path of Kähler metrics (θ_t) is allowed to vary as well.

Proposition 4.5.2. *Let $(\theta_t)_{t\in[0,T]}$ and $(\theta_t')_{t\in[0,T]}$ be two smooth paths of Kähler metrics on X, and suppose that $\varphi, \varphi' \in C^\infty(X \times [0, T])$ satisfy*

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right]$$

and

$$\frac{\partial\varphi'}{\partial t} = \log\left[\frac{(\theta_t' + dd^c\varphi)^n}{\mu}\right],$$

with the same volume form μ in both cases. As in Theorem 4.4.1, write μ as

$$\mu = e^{\psi^+ - \psi^-}\omega_X^n$$

with $\psi^\pm \in C^\infty(X)$, and assume given $C > 0$ and $p > 1$ such that

(i) $\theta_t \le C\omega_X$ and $\theta_t' \le C\omega_X$ for $t \in [0, T]$;
(ii) $-C \le \sup_X \psi^\pm \le C$;
(iii) $dd^c\psi^\pm \ge -C\omega_X$;
(iv) $\|e^{-\psi^-}\|_{L^p} \le C$.

Finally, let θ be a semipositive $(1,1)$-form with big cohomology class such that

(v) $\theta_t \geq \theta$ *and* $\theta'_t \geq \theta$ *for* $t \in [0,T]$.

For each compact subset $K \Subset \mathrm{Amp}\,(\theta)$*, we can then find a constant* $A > 0$ *only depending on* K*,* θ*,* C*,* p *and* T *such that*

$$\|\varphi - \varphi'\|_{C^0(K\times[0,T])} \leq \|\varphi_0 - \varphi'_0\|_{C^0(X)} + A\left(1 + \|\varphi_0\|_{C^0(X)} + \|\varphi'_0\|_{C^0(X)} - \inf_K \psi^-\right) \|\theta_t - \theta'_t\|_{C^0(X\times[0,T])}.$$

Proof. Set $N := \|\varphi_0 - \varphi'_0\|_{C^0(X)}$ and $M = \|\theta_t - \theta'_t\|_{C^0(X\times[0,T])}$. We may assume that $M > 0$, and we set for $\lambda \in [0, M]$

$$\theta_t^\lambda := \left(1 - \tfrac{\lambda}{M}\right)\theta_t + \tfrac{\lambda}{M}\theta'_t.$$

For each λ fixed, $(\theta_t^\lambda)_{t\in[0,T]}$ is a smooth path of Kähler forms, and Theorem 4.5.1 yields a unique solution $\varphi^\lambda \in C^\infty(X \times [0,T])$ to the parabolic complex Monge–Ampère equation

$$\begin{cases} \dfrac{\partial \varphi^\lambda}{\partial t} = \log\left[\dfrac{\left(\theta_t^\lambda + dd^c\varphi^\lambda\right)^n}{\mu}\right] \\ \varphi^\lambda|_{X\times\{0\}} = \left(1 - \tfrac{\lambda}{M}\right)\varphi_0 + \tfrac{\lambda}{M}\varphi'_0 \end{cases} \tag{4.24}$$

By the local existence theory, φ^λ depends smoothly on the parameter λ. If we denote by Δ_t^λ the Laplacian with respect to the Kähler form

$$\omega_t^\lambda := \theta_t^\lambda + dd^c\varphi_t^\lambda,$$

then we have

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right)\left(\frac{\partial \varphi^\lambda}{\partial \lambda}\right) = \mathrm{tr}_{\omega_t^\lambda}\left(\frac{\partial \theta_t^\lambda}{\partial \lambda}\right),$$

and hence

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right)\left(\frac{\partial \varphi^\lambda}{\partial \lambda}\right) = M^{-1}\mathrm{tr}_{\omega_t^\lambda}\left(\theta'_t - \theta_t\right) \leq \mathrm{tr}_{\omega_t^\lambda}(\omega_X), \tag{4.25}$$

by definition of M.

Using the notation of Sect. 4.4.1, we are going to apply the maximum principle to the function $H \in C^\infty(\Omega \times [0,T])$ defined by

$$H := e^{-At}\left(\frac{\partial \varphi^\lambda}{\partial \lambda}\right) + A\psi^- + A^2(\psi_\theta - \varphi^\lambda),$$

where $A > 0$ will be specified below. Using (4.13), (4.25) and $dd^c\psi^- \geq -C\omega_X$, we compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right) H &\leq -Ae^{-At}\left(\frac{\partial \varphi^\lambda}{\partial \lambda}\right) + \operatorname{tr}_{\omega_t^\lambda}\left(e^{-At}\omega_X + AC\omega_X\right)\\ &\quad + A^2 \log\left[\frac{\mu}{(\omega_t^\lambda)^n}\right] - A^2 \operatorname{tr}_{\omega_t^\lambda}(\omega_\Omega) + A^2 n\\ &= -AH + A^2\psi^- + A^3(\psi_\theta - \varphi^\lambda) + (1 + AC)\operatorname{tr}_{\omega_t^\lambda}(\omega_X)\\ &\quad + A^2\psi^+ - A^2\psi^- + A^2 \log\left[\frac{\omega_X^n}{(\omega_t^\lambda)^n}\right] - A^2\operatorname{tr}_{\omega_t^\lambda}(\omega_\Omega) + A^2 n.\end{aligned}$$

Since $\omega_\Omega \geq c\omega_X$ for some $c > 0$ under control, the arithmetico-geometric inequality allows us, just as before, to choose $A \gg 1$ under control such that

$$(1 + AC)\operatorname{tr}_{\omega_t^\lambda}(\omega_X) + A^2 \log\left[\frac{\omega_X^n}{(\omega_t^\lambda)^n}\right] - A^2\operatorname{tr}_{\omega_t^\lambda}(\omega_\Omega) \leq A_1$$

with $A_1 > 0$ under control. By Lemma 4.4.2, there exists $A_2 > 0$ under control such that

$$\sup_{X\times[0,T]} |\varphi^\lambda| \leq \|\varphi_0\|_{C^0(X)} + \|\varphi_0'\|_{C^0(X)} + A_2. \tag{4.26}$$

Using $\psi^+ \leq C$ and $\psi_\theta \leq 0$, we finally get

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right) H \leq -AH + A^3\left(\|\varphi_0\|_{C^0(X)} + \|\varphi_0'\|_{C^0(X)}\right) + A_3$$

with $A_3 > 0$ under control. Now

$$H|_{\Omega\times\{0\}} \leq M^{-1}N + A\psi^- + A^2\left(\|\varphi_0\|_{C^0(X)} + \|\varphi_0'\|_{C^0(X)}\right),$$

and the maximum principle thus yields

$$\sup_{\Omega\times[0,T]} H \leq M^{-1}N + A_4\left(1 + \|\varphi_0\|_{C^0(X)} + \|\varphi_0'\|_{C^0(X)}\right).$$

Since ψ_θ is bounded below on the given compact set K, we get using again (4.26)

$$\sup_{K\times[0,T]} \frac{\partial \varphi^\lambda}{\partial \lambda} \le M^{-1}N + A_5\left(1 + \|\varphi_0\|_{C^0(X)} + \|\varphi_0'\|_{C^0(X)} - \inf_K \psi^-\right).$$

Integrating with respect to $\lambda \in [0, M]$ and exchanging the roles of φ and φ' yields the desired result. □

Remark 4.5.3. The proof of Proposition 4.5.2 is directly adapted from that of [ST09, Lemma 3.14].

4.5.3 The General Case

We now consider as in Theorem 4.3.3 a smooth path $(\theta_t)_{t\in[0,T]}$ of closed semipositive $(1,1)$-forms such that $\theta_t \ge \theta$ for a fixed closed semipositive $(1,1)$-form θ with big cohomology class. Let μ be a positive measure on X of the form

$$\mu = e^{\psi^+ - \psi^-}\omega_X^n$$

where

- $\psi^\pm$ are quasi-psh functions on X (i.e. there exists $C > 0$ such that $\psi^\pm$ are both $C\omega_X$-psh);
- $e^{-\psi^-} \in L^p$ for some $p > 1$;
- $\psi^\pm$ are smooth on a given Zariski open subset $U \subset \mathrm{Amp}(\theta)$.

Given $\varphi_0 \in C^0(X) \cap \mathrm{PSH}(X, \theta_0)$, our goal is to prove the existence and uniqueness of

$$\varphi \in C_b^0(U \times [0, T]) \cap C^\infty(U \times (0, T])$$

such that $\varphi|_{U\times\{0\}} = \varphi_0$ and

$$\frac{\partial \varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right] \tag{4.27}$$

on $U \times (0, T)$.

4.5.3.1 Existence

We regularize the data. By Demailly [Dem92], there exist two sequences $\psi_k^\pm \in C^\infty(X)$ such that

- $\psi_k^\pm$ decreases pointwise to $\psi^\pm$ on X, and the convergence is in C^∞ topology on U;

- $dd^c\psi_k^{\pm} \geq -C\omega_X$ for a fixed constant $C > 0$.

Note that $\left|\sup_X \psi_k^{\pm}\right|$ is bounded independently of k, while we have for all k

$$\|e^{-\psi_k^-}\|_{L^p} \leq \|e^{-\psi^-}\|_{L^p}.$$

By Richberg's theorem, we similarly get a decreasing sequence $\varphi_0^j \in C^\infty(X)$ such that $\delta_j := \sup_X \left|\varphi_0^j - \varphi_0\right| \to 0$ and $\theta_0 + dd^c\varphi_0^j > -\varepsilon_j\omega$ with $\varepsilon_j \to 0$. We then set

- $\theta_t^j := \theta_t + \varepsilon_j\omega_X$;
- $\mu_{k,l} = e^{\psi_k^+ - \psi_l^-}\omega_X^n$.

Since $(\theta_t^j)_{t\in[0,T]}$ is a smooth path of Kähler forms, $\mu_{k,l}$ is a smooth positive volume form and φ_0^j is smooth and strictly θ_0^j-psh, Theorem 4.5.1 shows that there exists a unique function $\varphi^{j,k,l} \in C^\infty(X \times [0,T])$ such that

$$\begin{cases} \dfrac{\partial\varphi^{j,k,l}}{\partial t} = \log\left[\dfrac{\left(\theta_t^j + dd^c\varphi^{j,k,l}\right)^n}{\mu_{k,l}}\right] \\ \varphi^{j,k,l}|_{X\times\{0\}} = \varphi_0^j. \end{cases} \tag{4.28}$$

By Theorem 4.4.1, $\varphi^{j,k,l}$ is uniformly bounded on $X \times [0,T]$, and bounded in C^∞ topology on $U \times (0,T]$.

Furthermore, the maximum principle (Proposition 4.1.3) shows that for each j fixed the sequence $\varphi^{j,k,l}$ is increasing (resp. decreasing) with respect to k (resp. l). As a consequence,

$$\varphi^{j,k} = \lim_{l\to\infty}\varphi^{j,k,l},\ \varphi^j = \lim_{k\to\infty}\varphi^{j,k}$$

define bounded functions on $X \times [0,T]$ that are uniformly bounded in C^∞ topology on $U \times (0,T]$. Note also that $\varphi^j|_{X\times\{0\}} = \varphi_0^j$ by construction. The stability estimate of Proposition 4.5.2 shows that for each compact $K \subset U$ there exists $A_K > 0$ such that

$$\sup_{K\times[0,T]}\left|\varphi^{i,k,l} - \varphi^{j,k,l}\right| \leq A_K\left(\delta_i + \delta_j + \varepsilon_i + \varepsilon_j\right) \tag{4.29}$$

for all i, j, k, l, and hence

$$\sup_{K\times[0,T]}\left|\varphi^i - \varphi^j\right| \leq A_K\left(\delta_i + \delta_j + \varepsilon_i + \varepsilon_j\right)$$

for all i, j. As a consequence, (φ^i) is a Cauchy sequence in the Fréchet space $C^0(U \times [0,T])$, and hence converges uniformly on compact sets of $U \times [0,T]$ to a bounded function $\varphi \in C^0(U \times [0,T])$, the convergence being in C^∞ topology on $U \times (0,T]$. Passing to the limit in (4.28) shows that φ satisfies (4.27) on $U \times (0,T)$, and φ coincides with $\varphi_0 = \lim_j \varphi_0^j$ on $U \times \{0\}$.

4.5.3.2 Uniqueness

Let φ be the function just constructed, and suppose that

$$\varphi' \in C_b^0(U \times [0,T]) \cap C^\infty(U \times (0,T))$$

satisfies (4.27). We are going to show that

$$\sup_{U \times [0,T]} |\varphi - \varphi'| = \sup_{U \times \{0\}} |\varphi - \varphi'|,$$

which will in particular imply the desired uniqueness statement.

By Lemma 4.3.2, we can choose a θ-psh function $\tau_U \le 0$ that is smooth on U and tends to $-\infty$ near ∂U. Fix $0 < c \ll 1$ with $c\,\theta \le \omega_X$, so that $\omega_X + c\,dd^c\tau_U \ge 0$.

For a given index j define $H^j \in C^0(U \times [0,T)) \cap C^\infty(U \times (0,T))$ by

$$H^j := \varphi^j - \varphi' - c\varepsilon_j \tau_U,$$

using the same notation as in the proof of the existence of φ. On $U \times (0,T)$ we have

$$\begin{aligned}
\frac{\partial H^j}{\partial t} &= \log\left[\frac{(\theta_t + \varepsilon_j \omega_X + dd^c\varphi^j)^n}{(\theta_t + dd^c\varphi')^n}\right] \\
&= \log\left[\frac{\left(\theta_t + dd^c\varphi' + dd^c H^j + \varepsilon_j(\omega_X + c\,dd^c\tau_U)\right)^n}{(\theta_t + dd^c\varphi')^n}\right] \\
&\ge \log\left[\frac{\left(\theta_t + dd^c\varphi' + dd^c H^j\right)^n}{(\theta_t + dd^c\varphi')^n}\right],
\end{aligned}$$

and hence

$$\inf_{U \times [0,T)} H^j = \inf_{U \times \{0\}} H^j$$

by Proposition 4.1.3. Since $\varphi_0^j \to \varphi_0$ uniformly on $U \times \{0\}$ and $\tau_U \le 0$, we get in the limit as $j \to \infty$

$$\inf_{U \times [0,T)} (\varphi - \varphi') \ge \inf_{U \times \{0\}} (\varphi - \varphi').$$

In order to prove the similar inequality with the roles of φ and φ' exchanged, we need to introduce yet another parameter in the construction of φ, in order to allow more flexibility. For each $\delta \in [0,1)$, $(1-\delta)\varphi^j$ is smooth and strictly $(1-\delta)\theta_0^j$-psh, and Theorem 4.5.1 thus yields a unique function $\varphi^{\delta,j,k,l} \in C^\infty(X \times [0,T])$ such that

$$\begin{cases} \dfrac{\partial \varphi^{\delta,j,k,l}}{\partial t} = \log\left[\dfrac{\left((1-\delta)\theta_t^j + dd^c\varphi^{\delta,j,k,l}\right)^n}{\mu_{k,l}}\right] \\ \varphi^{\delta,j,k,l}|_{X\times\{0\}} = (1-\delta)\varphi^j. \end{cases} \tag{4.30}$$

If we further require that $\delta \in [0,1/2]$, then $(1-\delta)\theta_t^j \geq \frac{1}{2}\theta$ for all j and t, and we thus see just as before that $\varphi^{\delta,j,k,l}$ is monotonic with respect to k and l, uniformly bounded on $X \times [0,T]$ and bounded in C^∞ topology on $U \times (0,T]$, and that for each compact $K \Subset U$ we have an estimate

$$\sup_{K\times[0,T)} \left|\varphi^{\delta,i,k,l} - \varphi^{\delta,j,k,l}\right| \leq A_K(\varepsilon_i + \varepsilon_j + \delta_i + \delta_j).$$

We may thus consider

$$\varphi^\delta = \lim_{j\to\infty}\lim_{k\to\infty}\lim_{l\to\infty}\varphi^{\delta,j,k,l},$$

which belongs to $C^0(U \times [0,T]) \cap C^\infty(U \times (0,T))$ and satisfies

$$\frac{\partial \varphi^\delta}{\partial t} = \log\left[\frac{((1-\delta)\theta_t + dd^c\varphi^\delta)^n}{\mu}\right].$$

Since $\|\varphi_0^{j,k,l} - \varphi_0^{\delta,j,k,l}\|_{C^0(X)}$ and $\|\theta_t^j - (1-\delta)\theta_t^j\|_{C^0(X\times[0,T])}$ are both $O(\delta)$ uniformly with respect to j,k,l, Proposition 4.5.2 shows that

$$\sup_{K\times[0,T]} \left|\varphi^{\delta,j,k,l} - \varphi^{j,k,l}\right| \leq C_K\delta$$

for each compact $K \Subset U$, with $C_K > 0$ independent of δ, j, k, l, and hence in the limit

$$\sup_{K\times[0,T]} \left|\varphi^\delta - \varphi\right| \leq C_K\delta \tag{4.31}$$

for all $\delta \in [0,1/2]$. Now define for each $\delta \in [0,1/2]$ a function $H^\delta \in C^0(U \times [0,T]) \cap C^\infty(U \times (0,T])$ by

$$H^\delta := \varphi' - \varphi^\delta - \delta\tau_U.$$

We have

$$\frac{\partial H^\delta}{\partial t} = \log\left[\frac{\left((1-\delta)\theta_t + dd^c\varphi^\delta + \delta(\theta_t + dd^c\tau_U) + dd^c H^\delta\right)^n}{\left((1-\delta)\theta_t + dd^c\varphi^\delta\right)^n}\right]$$
$$\geq \log\left[\frac{\left((1-\delta)\theta_t + dd^c\varphi^\delta + dd^c H^\delta\right)^n}{\left((1-\delta)\theta_t + dd^c\varphi^\delta\right)^n}\right],$$

and hence

$$\inf_{U\times[0,T]} H^\delta = \inf_{U\times\{0\}} H^\delta.$$

Since $\tau_U \leq 0$, we get

$$\varphi' - \varphi^\delta \geq \delta\tau_U + \inf_{U\times\{0\}}(\varphi' - (1-\delta)\varphi)$$

on $U \times [0, T]$, and hence

$$\inf_{U\times[0,T]}(\varphi' - \varphi) \geq \inf_{U\times\{0\}}(\varphi' - \varphi)$$

in the limit as $\delta \to 0$, using (4.31) and the fact that φ is bounded.

4.5.4 The Affine Case: Proof of Theorem 4.3.5

We use the notation of the existence proof above. If (θ_t) is an affine path, then so is $\theta_t^j = \theta_t + \varepsilon_j\omega_X$. We may thus apply Lemma 4.4.6 to conclude that for each $\varepsilon > 0$, $\frac{\partial\varphi^{j,k,l}}{\partial t}$ is uniformly bounded above on $X \times [\varepsilon, T]$, and uniformly bounded below on $X \times [\varepsilon, T - \varepsilon]$. Since φ is a limit of $\varphi^{j,k,l}$ in C^∞-topology on $U \times (0, T]$, we conclude as desired that $\frac{\partial\varphi}{\partial t}$ is bounded above on $U \times [\varepsilon, T]$ and bounded below on $U \times [\varepsilon, T - \varepsilon]$.

Denote also by φ the quasi-psh extension to $X \times [0, T]$ and let $\varepsilon > 0$. Since the time derivative is bounded above on $X \times [\varepsilon, T]$, there exists $C_\varepsilon > 0$ such that

$$(\theta_t + dd^c\varphi)^n \leq C_\varepsilon\mu$$

on $X \times \{t\}$ for each $t \in [\varepsilon, T]$. By the results of [EGZ11] (which rely on viscosity techniques), if follows that φ is continuous on $X \times \{t\}$ for each $t \in [\varepsilon, T]$. Since the time derivative is bounded on $U \times [\varepsilon, T - \varepsilon]$, φ is also uniformly Lipschitz continuous in the time variable on $X \times [\varepsilon, T - \varepsilon]$, and it follows as desired that φ is continuous on $X \times (0, T)$.

Remark 4.5.4. It is of course reasonable to expect that φ is in fact continuous on the whole of $X \times [0, T]$.

4.6 The Kähler–Ricci Flow on a Log Terminal Variety

4.6.1 Forms and Currents with Local Potentials

Let X be a complex analytic space with normal singularities, and denote by n its dimension. Since closed $(1,1)$-forms and currents on X are not necessarily locally dd^c-exact in general, we need to rely on a specific terminology (compare [EGZ09, Sect. 5.2]). We refer for instance to [Dem85] for the basic facts on smooth functions, distributions and psh functions on a complex analytic space. The main point for us is that any psh function on X_{reg} uniquely extends to a psh function on X by normality, see [GR56]. For lack of a proper reference, we include:

Lemma 4.6.1. *Any pluriharmonic distribution on X is locally the real part of a holomorphic function, i.e. the kernel of the dd^c operator on the sheaf $\mathcal{D}'_X$ of germs of distributions coincides with the sheaf $\Re\mathcal{O}_X$ of real parts of holomorphic germs.*

Proof. If $u \in \mathcal{D}'_X$ satisfies $dd^c u = 0$, then $\pm u$ is psh on X_{reg}, and hence extends to a psh function on X by Grauert and Remmert [GR56]. In particular, $\pm u$ is usc and bounded above, which means that u is the germ of a continuous (finite valued) function.

From there on, the proof is basically the same as [FS90, Proposition 1.1]. Let $\pi : X' \to X$ be a proper bimeromorphic morphism with X' smooth. Since $\pm(u \circ \pi)$ is psh on the complex manifold X', we have $u \in \pi_* (\Re\mathcal{O}_{X'})$. We will thus be done if we prove that

$$\pi_* (\Re\mathcal{O}_{X'}) = \Re\mathcal{O}_X.$$

Since X is normal, Zariski's main theorem implies that $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$, and hence π has connected fibers. The claim is thus that the coboundary morphism

$$\pi_* (\Re\mathcal{O}_{X'}) \to R^1\pi_*(i\mathbb{R}) \tag{4.32}$$

associated to the short exact sequence

$$0 \to i\mathbb{R} \to \mathcal{O}_{X'} \to \Re\mathcal{O}_{X'} \to 0$$

is zero. For each $x \in X$, the composition of (4.32) with the restriction morphism

$$R^1\pi_* (i\mathbb{R})_x \to H^1\left(\pi^{-1}(x), i\mathbb{R}\right) \tag{4.33}$$

is zero, because it factors through

$$H^0\left(\pi^{-1}(x), \Re\mathcal{O}_{\pi^{-1}(x)}\right) = \mathbb{R}$$

by the maximum principle, $\pi^{-1}(x)$ being compact and connected. But (4.33) is in fact an isomorphism, just because π is a proper map between locally compact spaces (cf. for instance [Dem09, Theorem 9.10, p. 223]), and it follows as desired that (4.32) is zero. □

Definition 4.6.2. A $(1,1)$-*form (resp.* $(1,1)$-*current) with local potentials* on X is defined to be a section of the quotient sheaf $\mathcal{C}^\infty_X/\Re\mathcal{O}_X$ (resp. $\mathcal{D}'_X/\Re\mathcal{O}_X$). We also introduce the *Bott–Chern cohomology space*

$$H^{1,1}_{\mathrm{BC}}(X) := H^1(X, \Re\mathcal{O}_X).$$

Thanks to Lemma 4.6.1, a $(1,1)$-form with local potentials can be more concretely described as a closed $(1,1)$-form θ on X that is locally of the form $\theta = dd^c u$ for a smooth function u. We say that θ is a *Kähler form* if u is strictly psh. Similarly, a closed $(1,1)$-current T with local potentials is locally of the form $dd^c\varphi$ where φ is a distribution. Since X is normal, and hence locally irreducible, $dd^c\varphi$ is a positive current iff φ is a psh function.

The sheaves $\mathcal{C}^\infty_X$ and $\mathcal{D}'_X$ being soft, hence acyclic, the cohomology long exact sequence shows that $H^{1,1}_{\mathrm{BC}}(X)$ is isomorphic to the quotient of the space of $(1,1)$-forms (resp. currents) with local potentials by $dd^c C^\infty(X)$ (resp. $dd^c\mathcal{D}'(X)$). In particular, any $(1,1)$-current T with local potentials can be (globally) written as

$$T = \theta + dd^c\varphi$$

where θ is a $(1,1)$-form with local potentials and φ is a distribution.

Note also that $H^{1,1}_{\mathrm{BC}}(X)$ is finite dimensional when X is compact, as follows from the cohomology long exact sequence associated to

$$0 \to i\mathbb{R} \to \mathcal{O}_X \to \Re\mathcal{O}_X \to 0$$

and the finite dimensionality of $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathbb{R})$.

Proposition 4.6.3. *Let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be a $(1,1)$-class on a normal complex space X, and let T be a closed positive $(1,1)$-current on X_{reg} representing the restriction $\alpha|_{X_{\mathrm{reg}}}$ to the regular part of X. Then:*

(i) T uniquely extends as a positive $(1,1)$-current with local potentials on X, and the dd^c-class of the extension coincides with α;
(ii) if X is compact Kähler and if T has locally bounded potentials on an open subset U of X_{reg}, then the positive measure T^n, defined on U in the sense of Bedford–Taylor, has finite total mass on U.

Proof. Let θ be a $(1,1)$-form with local potentials representing α. On X_{reg} we then have $T = \theta|_{X_{\text{reg}}} + dd^c\varphi$, where φ is a θ-psh function on X_{reg}. If U is a small enough neighborhood of a given point of X, then $\theta = dd^c u$ for some smooth function u on U, and $u + \varphi$ is a psh function on U_{reg}. By the Riemann-type extension property for psh functions [GR56], $u + \varphi$ uniquely extends to a psh function on U, and (i) easily follows.

Point (ii) follows from [BEGZ10, Proposition 1.16] (which is in turn an easy consequence of Demailly's regularization theorem [Dem92]). More precisely, choose a resolution of singularities $\pi : X' \to X$, where X' can be taken to be a compact Kähler manifold and π is an isomorphism above X_{reg}. Denoting by $\langle \pi^* T^n \rangle$ the top-degree non-pluripolar product of $\pi^* T$ on X' (in the sense of [BEGZ10]), we then have

$$\int_U T^n = \int_{\pi^{-1}(U)} \pi^* T^n \leq \int_{X'} \langle \pi^* T^n \rangle < +\infty. \qquad \square$$

We will also use the following simple fact.

Lemma 4.6.4. *Let $\pi : X' \to X$ be a bimeromorphic morphism between normal compact complex spaces, let $A \subset X$ and $A' \subset X'$ be closed analytic subsets of codimension at least 2, and let u be a psh function on $(X' \setminus A') \cap \pi^{-1}(X \setminus A)$. Then u is constant.*

Proof. Since A' has codimension at least 2, u extends to a psh function on $\pi^{-1}(X \setminus A)$ by Grauert and Remmert [GR56]. By Zariski's main theorem, π has connected fibers, and u therefore descends to a psh function v on $X \setminus A$, which extends to a psh function on X since A has codimension at least 2. It follows that v is constant. $\square$

4.6.2 Log Terminal Singularities

Recall that a complex space X is $\mathbb{Q}$-*Gorenstein* if it has normal singularities and if its canonical bundle K_X exists as a $\mathbb{Q}$-line bundle, which means that there exists $r \in \mathbb{N}$ and a line bundle L on X such that $L|_{X_{\text{reg}}} = rK_{X_{\text{reg}}}$.

Let X be a $\mathbb{Q}$-Gorenstein space, and choose a *log resolution* of X, i.e. a projective bimeromorphic morphism $\pi : X' \to X$ which is an isomorphism over X_{reg} and whose exceptional divisor $E = \sum_i E_i$ has simple normal crossings. There is a unique collection of rational numbers a_i, called the *discrepancies* of X (with respect to the chosen log resolution) such that

$$K_{X'} \sim_{\mathbb{Q}} \pi^* K_X + \sum_i a_i E_i,$$

where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence. By definition, X has *log terminal singularities* iff $a_i > -1$ for all i. This definition is independent of the choice of a log resolution; this will be a consequence of the following analytic interpretation of log terminal singularities as a *finite volume* condition. As an example, quotient singularities are log terminal, and conversely every two-dimensional log terminal singularity is a quotient singularity (see for instance [KolMori98] for more information on log terminal singularities).

After replacing X with a small open subset, we may choose a local generator σ of the line bundle rK_X for some $r \in \mathbb{N}^*$. Restricting to X_{reg}, we define a smooth positive volume form by setting

$$\mu_\sigma := \left(i^{rn^2}\sigma \wedge \bar{\sigma}\right)^{1/r}. \tag{4.34}$$

Such measures are called *adapted measures* in [EGZ09]. The key fact is then the following analytic interpretation of the discrepancies:

Lemma 4.6.5. *Let z_i be a local equation of E_i, defined on a neighborhood $U \subset X'$ of a given point of E. Then we have*

$$\left(\pi^*\mu_\sigma\right)_{U \setminus E} = \prod_i |z_i|^{2a_i}\, dV$$

for some smooth volume form dV on U.

This result is a straightforward consequence of the change of variable formula. As a consequence, a $\mathbb{Q}$-Gorenstein variety X has log terminal singularities iff every (locally defined) adapted measure μ_σ has locally finite mass near each singular point of X. The construction of adapted measures can be globalized as follows: let ϕ be a smooth metric on the $\mathbb{Q}$-line bundle K_X. Then

$$\mu_\phi := \left(\frac{i^{rn^2}\sigma \wedge \bar{\sigma}}{|\sigma|_{r\phi}}\right)^{1/r} \tag{4.35}$$

becomes independent of the choice of a local generator σ of rK_X, and hence defines a smooth positive volume form on X_{reg}, which has locally finite mass near points of X_{sing} iff X is log terminal.

Remark 4.6.6. In [ST09], an adapted measure of the form μ_ϕ for a smooth metric ϕ on K_X is called a smooth volume form. We prefer to avoid this terminology, which has the drawback that ω^n is in general not smooth in this sense even when ω is a smooth positive $(1,1)$-form on X. This is in fact already the case for quotient singularities.

The following result illustrates why log terminal singularities are natural in the context of Kähler–Einstein geometry.

Proposition 4.6.7. *Let X be a $\mathbb{Q}$-Gorenstein compact Kähler space, and let ω be a Kähler form on X_{reg} with non-negative Ricci curvature. Assume also that $[\omega] \in H^{1,1}_{\mathrm{BC}}(X_{\mathrm{reg}})$ extends to X. Then X necessarily has log terminal singularities.*

Recall that $[\omega]$ extends to X iff ω extends as a positive $(1,1)$-current with local potentials, by Proposition 4.6.3.

Proof. The volume form ω^n, defined on X_{reg}, induces a Hermitian metric on $K_{X_{\mathrm{reg}}}$. If σ is a local generator of the line bundle rK_X near a given singular point of X, we can consider its pointwise norm $|\sigma|$ on X_{reg}, and it is easy to check from the definitions that

$$\mu_\sigma = |\sigma|^{2/r}\omega^n.$$

Since σ is local generator, $dd^c \log|\sigma|$ is equal to minus the curvature form of the metric on $rK_{X_{\mathrm{reg}}}$, i.e.

$$dd^c \log|\sigma| = r\mathrm{Ric}(\omega).$$

The assumption therefore implies that $\log|\sigma|$ is a local psh function on X_{reg}, and hence extends to a local psh function on X by Grauert and Remmert [GR56]. In particular, $\log|\sigma|$ is locally bounded above on X, and we thus get near each singular point of X $\mu_\sigma \le C\omega^n$ for some constant $C > 0$. Since $[\omega]$ extends to X, ω^n has finite total mass on X_{reg} by Proposition 4.6.3, and it follows as desired that μ_σ has locally finite mass on X. □

4.6.3 The Kähler–Ricci Flow on a Log Terminal Variety

Given an initial projective variety X_0 with log terminal singularities and K_{X_0} pseudoeffective, each step of the Minimal Model Program produces a birational morphism $f : X \to Y$ with X, Y projective and normal, X log terminal and K_X f-ample. The following result, due to Song and Tian [ST09], shows that it is then possible to run the (unnormalized) Kähler–Ricci flow on X, starting from an initial positive current with continuous local potentials coming from Y (the actual assumption on the initial current in [ST09] being in fact slightly more demanding).

Theorem 4.6.8. *Let $f : X \to Y$ be a bimeromorphic morphism between two normal compact Kähler spaces such that X is log terminal and K_X is f-ample. Let also $\alpha \in H^{1,1}_{\mathrm{BC}}(Y)$ be a Kähler class on Y, so that $f^*\alpha + t[K_X]$ is a Kähler class in $H^{1,1}_{\mathrm{BC}}(X)$ for $0 < t \ll 1$, and set*

$$T_0 := \sup\{t \in (0, +\infty) \mid f^*\alpha + t[K_X] \text{ is Kähler on } X\}.$$

Given a positive $(1,1)$*-current* ω_0 *with continuous local potentials on* X *and* $[\omega_0] = f^*\alpha$*, there is a unique way to include* ω_0 *in a family* $(\omega_t)_{t\in[0,T_0)}$ *of positive* $(1,1)$*-currents with continuous local potentials on* X *such that*

(i) $[\omega_t] = f^*\alpha + t[K_X]$ *for all* $t \in [0,T_0)$*;*
(ii) setting $\Omega := X_{\text{reg}} \setminus \text{Exc}(f)$*, the local potentials of* ω_t *are continuous on* $\Omega \times [0,T_0)$*, and locally bounded on* $X \times [0,T]$ *for each* $T < T_0$*;*
(iii) $(\omega_t)_{t\in(0,T_0)}$ *restricts to a smooth path of Kähler forms on* Ω *satisfying*

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t).$$

Moreover, the measures ω_t^n *are uniformly comparable to any given adapted measure as long as* t *stays in a compact subset of* $(0,T)$*.*

This result of course applies in particular when f is the identity map and α is any Kähler class on X. This special case of Theorem 4.6.8 yields the following result for the normalized Kähler–Ricci flow:

Corollary 4.6.9. *Let* X *be a projective complex variety with log terminal singularities such that* $\pm K_X$ *ample. Then each positive* $(1,1)$*-current with continuous local potentials* ω_0 *such that* $[\omega_0] = [\pm K_X]$ *extends in a unique way to a family* $(\omega_t)_{t\in[0,+\infty)}$ *of positive* $(1,1)$*-currents with continuous local potentials such that*

(i) $[\omega_t] = [\pm K_X]$ *for all* $t \in [0,+\infty)$*;*
(ii) the local potentials of ω_t *are continuous on* $X_{\text{reg}} \times [0,+\infty)$*, and bounded on* $X \times [0,T]$ *for each* $T \in (0,+\infty)$*;*
(ii) $(\omega_t)_{t\in(0,+\infty)}$ *restricts to a smooth path of Kähler forms on* X_{reg} *satisfying*

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) \mp \omega_t.$$

Moreover, the volume forms ω_t^n *are uniformly comparable to any given adapted measure as long as* t *stays in a compact subset of* $(0,T)$*.*

Indeed, as is well-known, setting

$$\omega'_s := (1 \pm s)\omega_{\pm \log(1\pm s)}$$

defines a bijection between the solutions of

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) \mp \omega_t$$

on $X_{\text{reg}} \times (0,+\infty)$ and those of

$$\frac{\partial \omega'_s}{\partial s} = -\text{Ric}(\omega'_s)$$

on $X_{\mathrm{reg}} \times (0, T_0)$, with $[\omega_s'] = [\pm K_X] + s[K_X]$ and $T_0 = +\infty$ when $+K_X$ is ample (resp. $T_0 = 1$ when $-K_X$ is ample).

Proof of Theorem 4.6.8. Since α can be represented by a Kähler form on Y, we may choose a closed semipositive $(1,1)$-form θ_0 with local potentials on X such that $[\theta_0] = f^*\alpha$. We thus have $\omega_0 = \theta_0 + dd^c\varphi_0$ with φ_0 a continuous θ_0-psh function on X. Given $T \in (0, T_0)$, we can choose a Kähler form θ_T representing $f^*\alpha + T[K_X]$, by definition of T_0. For $t \in [0, T]$ set

$$\theta_t := \theta_0 + t\chi$$

with $\chi := T^{-1}(\theta_T - \theta_0)$, which defines an affine path of semipositive $(1,1)$-forms with local potentials. For $t \in [0, T]$, the path of currents we are looking is of the form $\omega_t = \theta_t + dd^c\varphi_t$ with

$$\varphi \in C_b^0(\Omega \times [0, T]) \cap C^\infty(\Omega \times (0, T])$$

and $\varphi|_{\Omega\times\{0\}} = \varphi_0$. Since $\chi = T^{-1}(\theta_T - \theta_0)$ is a representative of $[K_X]$, we can find a smooth metric ϕ on the $\mathbb{Q}$-line bundle K_X having χ as its curvature form. If we denote by $\mu := \mu_\phi$ the corresponding adapted measure, it follows from the definitions that for any Kähler form ω on an open subset U of X_{reg} we have

$$-dd^c \log\left[\frac{\omega^n}{\mu}\right] = \chi + \mathrm{Ric}(\omega) \tag{4.36}$$

On $\Omega \times (0, T]$, the equation $\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$ is thus equivalent to

$$dd^c\left(\frac{\partial\varphi}{\partial t}\right) = dd^c \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right].$$

By Lemma 4.6.4, this amounts to

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right] + c(t)$$

for some smooth function $c : [0, T] \to \mathbb{R}$, since

$$\Omega = f^{-1}(Y \setminus Y_{\mathrm{sing}}) \cap (X \setminus X_{\mathrm{sing}})$$

and X, Y each have a singular locus of codimension at least 2 by normality. After choosing a primitive of $c(t)$, we can absorb it in the left-hand side, and we end up with showing the existence and uniqueness of

$$\varphi \in C_b^0(\Omega \times [0, T]) \cap C^\infty(\Omega \times (0, T])$$

such that $\varphi|_{\Omega\times\{0\}} = \varphi_0$ and

$$\frac{\partial \varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi)^n}{\mu}\right]$$

on $\Omega \times (0, T]$. Since θ_T is a Kähler form, we have

$$\theta_T \geq \theta := c\,\theta_0$$

for $0 < c \ll 1$, and hence $\theta_t \geq \theta$ for all $t \in [0, T]$. Now let $\pi : X' \to X$ be a log resolution, which is thus in particular an isomorphism above X_{reg}, and pick a Kähler form $\omega_{X'}$ on X'. Since X has log terminal singularities, by Lemma 4.6.5 the measure $\mu' := \pi^*\mu$ is of the form

$$\mu' := e^{\psi^+ - \psi^-}\omega_{X'}^n$$

where $\psi^\pm$ are quasi-psh functions on X' with logarithmic poles along the exceptional divisor E, smooth on $X' \setminus E = \pi^{-1}(X_{\text{reg}})$, and such that $e^{-\psi^-} \in L^p$ for some $p > 1$. We also have $\theta'_t := \pi^*\theta_t \geq \theta' := \pi^*\theta$. Finally, since $[\theta']$ is the pull-back by $f \circ \pi$ of a Kähler class on Y, we have

$$\text{Amp}\,(\theta') = X' \setminus \text{Exc}(f \circ \pi) = \pi^{-1}(\Omega) \simeq \Omega.$$

Using Theorems 4.3.3 and 4.3.5, it is now easy to conclude the proof of Theorem 4.6.8. □

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