DUALITY BETWEEN THE PSEUDOEFFECTIVE AND THE MOVABLE CONE ON A PROJECTIVE MANIFOLD

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WITH AN APPENDIX BY SÉBASTIEN BOUCKSOM

ABSTRACT. We prove a conjecture of Boucksom-Demailly-Păun-Peternell, namely that on a projective manifold X the cone of pseudoeffective classes in $H^{1,1}_{\mathbb{R}}(X)$ is dual to the cone of movable classes in $H^{n-1,n-1}_{\mathbb{R}}(X)$ via the Poincaré pairing. This is done by establishing a conjectured transcendental Morse inequality for the volume of the difference of two nef classes on a projective manifold. In an appendix by Boucksom it is shown that the Morse inequality also implies that the volume function is differentiable on the big cone, and one also gets a characterization of the prime divisors in the non-Kähler locus of a big class via intersection numbers.

1. INTRODUCTION

In [BDPP13] Boucksom-Demailly-Păun-Peternell proved that a line bundle on a projective manifold is pseudoeffective iff its degree along any member of a covering family of curves is non-negative. As in explained in [BDPP13] this result should be understood in terms of duality of cones.

Let X be a compact Kähler manifold. Recall that a class in $H^{1,1}_{\mathbb{R}}(X)$ is called pseudoeffective if it contains a closed positive current. The set of such classes form a closed convex cone in $H^{1,1}_{\mathbb{R}}(X)$ called the pseudoeffective cone and is usually denoted by $\mathcal{E}$. A line bundle L is pseudoeffective iff $c_1(L) \in \mathcal{E}$.

In $H^{n-1,n-1}_{\mathbb{R}}(X)$ there is a cone called the the movable cone $\mathcal{M}$. It is defines as the closure of the convex cone generated by classes of currents of the form

$$\mu_*(\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}),$$

where $\mu : \tilde{X} \to X$ is some modification and ω_i are Kähler forms on $\tilde{X}$. The cohomology class associated to a curve in X will lie in $\mathcal{M}$ iff it moves in an analytic family which covers X: such a curve is called movable.

Furthermore, when X is projective we let $\mathcal{E}_{NS} := \mathcal{E} \cap NS_{\mathbb{R}}(X)$ where

$$NS_{\mathbb{R}}(X) := (H^{1,1}_{\mathbb{R}}(X) \cap H^2(X, \mathbb{Z})/\text{tors}) \otimes_{\mathbb{Z}} \mathbb{R},$$

and similarly $\mathcal{M}_{NS} := \mathcal{M} \cap N_1(X)$, where

$$N_1(X) := (H^{n-1,n-1}_{\mathbb{R}}(X) \cap H^{2n-2}(X, \mathbb{Z})/\text{tors}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

One can now formulate the result of Boucksom-Demailly-Păun-Peternell in [BDPP13] in the following way:

Theorem 1.1. *On a projective manifold X the cones $\mathcal{E}_{NS}$ and $\mathcal{M}_{NS}$ are dual via the Poincaré pairing of $NS_{\mathbb{R}}(X)$ with $N_1(X)$.*

They also formulated a conjecture:

Conjecture 1.1. On any compact Kähler manifold X the cones $\mathcal{E}$ and $\mathcal{M}$ are dual via the Poincaré pairing of $H^{1,1}_{\mathbb{R}}(X)$ with $H^{n-1,n-1}_{\mathbb{R}}(X)$.

More concretely the conjecture says that a class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ contains a closed positive current iff for all modifications $\mu : \tilde{X} \to X$ of X and Kähler classes $\tilde{\beta}_i$ on $\tilde{X}$ we have that

$$\int_{\tilde{X}} \mu^*(\alpha) \wedge \tilde{\beta}_1 \wedge ... \wedge \tilde{\beta}_{n-1} \geq 0. \tag{1.1}$$

The if part follows almost immediately from the fact that one can pull back a closed positive $(1,1)$-current by μ to get a closed positive $(1,1)$-current on $\tilde{X}$. The (very) hard part is establishing the existence of a closed positive current using the numerical data (1.1).

Our main result confirms the conjecture when X is projective.

Theorem A. When X is projective then $\mathcal{E}$ and $\mathcal{M}$ are dual via the Poincaré pairing of $H^{1,1}_{\mathbb{R}}(X)$ with $H^{n-1,n-1}_{\mathbb{R}}(X)$.

It was observed already in [BDPP13] that to prove Conjecture 1.1 it is enough to establish a certain lower bound on the volume of the difference of two nef classes α and β, namely

$$\mathrm{vol}(\alpha - \beta) \geq (\alpha^n) - n(\alpha^{n-1} \cdot \beta). \tag{1.2}$$

This inequality is known as a transcendental Morse inequality. The case when α and β lies in $NS_{\mathbb{R}}(X)$ is well-known and not hard to prove (see [BDPP13]), and it is used in a crucial way in the proof of Theorem 1.1. Indeed given the transcendental Morse inequality (1.2) the rest of the proof of Theorem 1.1 in [BDPP13] extends to the general case.

It was also recognized in [BDPP13] that to prove Conjecture 1.1 in the case when X is projective, it is enough to establish (1.2) for pairs of nef classes α, β where $\beta \in NS_{\mathbb{R}}(X)$. This is what we set out to do in this paper.

Theorem B. Let α and β be two nef classes on a projective manifold X and assume that $\beta \in NS_{\mathbb{R}}(X)$. Then

$$\mathrm{vol}(\alpha - \beta) \geq (\alpha^n) - n(\alpha^{n-1} \cdot \beta).$$

In [BFJ09] Boucksom-Favre-Jonsson proved, using the estimate (1.2), the following two theorems:

Theorem 1.2. *The volume function is C^1 on $\mathcal{E}^{\circ}_{NS}$ and the partial derivatives are given by*

$$\frac{d}{dt}\bigg|_{t=0} vol(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma.$$

Theorem 1.3. *For any class big class $\alpha \in NS_{\mathbb{R}}$ we have that*

$$vol(\alpha) = \langle \alpha^{n-1} \rangle \cdot \alpha.$$

In particular a prime divisor D will lie in the augmented base locus of L iff

$$\langle c_1(L)^{n-1} \rangle \cdot c_1(D) = 0.$$

Here $\langle \alpha^{n-1} \rangle$ denotes a positive selfintersection of α, which is equal to α^{n-1} when α is nef but not in general. This last result can be thought of as an orthogonality relation (see [BFJ09]).

After learning of our results Boucksom produced a note [Bou16] in which he not only explains how Theorem A is derived from Theorem B, but also proves that Theorem B implies the analogues of Theorem 1.2 and Theorem 1.3:

Theorem C. On a projective manifold X the volume function is continuously differentiable on the big cone and

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle \alpha^{n-1}\rangle \cdot \gamma.$$

Theorem D. For any class $\alpha \in \mathcal{E}^{\circ}_{NS}$ on a projective manifold X we have that

$$\mathrm{vol}(\alpha) = \langle \alpha^{n-1}\rangle^{n-1} \cdot \alpha.$$

In particular a prime divisor D will lie in the non-Kähler locus of α iff

$$\langle \alpha^{n-1}\rangle \cdot c_1(D) = 0.$$

A striking consequence of Theorem C is that the estimate in Theorem B in fact holds for all pairs of nef classes α and β, irrespective of β lying in $NS_{\mathbb{R}}(X)$ or not. We thus get:

Theorem B'. Let α and β be two nef classes on a projective manifold X. Then

$$\mathrm{vol}(\alpha - \beta) \geq (\alpha^n) - n(\alpha^{n-1} \cdot \beta).$$

Boucksom's note [Bou16] now forms an appendix to our paper (Appendix A).

1.1. **Outline of proof of Theorem B.** First we note that by continuity we can assume α to be nef and big and β to Kähler and integral, i.e. $\beta = c_1(L)$ for some line bundle L. To prove Theorem B in that case we will establish an infinitesimal version, which then easily implies the theorem. The infinitesimal estimate we want to prove states that if α is big and β is Kähler and integral then

$$\liminf_{t\to 0+} \frac{\mathrm{vol}(\alpha - t\beta) - \mathrm{vol}(\alpha)}{t} \geq -n\langle \alpha^{n-1}\rangle \cdot \beta. \tag{1.3}$$

Let θ be a smooth representative of α and let ω be a Kähler form in β. Without loss of generality we can assume that $\beta = c_1(L)$ where L is effective. We call the associated divisor Y. There is an ω-psh function g such that $dd^c g = [Y] - \omega$; g will have a logarithmic singularity along Y.

Now we pick a $t > 0$ such that $\alpha - t\beta$ is big. Adding tg turns any $(\theta - t\omega)$-psh function into a θ-psh function with logarithmic singularity of order at least t along D.

One finds the volume of $\alpha - t\beta$ by integrating the Monge-Ampère measure of any $(\theta - t\omega)$-psh function with minimal singularities. This trick of adding tg does not change the Monge-Ampère measure, so we see that we want to find a θ-psh function with the right singularity along D, whose total Monge-Ampère mass can be appropriately bounded from below.

We create such a function by considering a family of upper envelopes. It is well-known that a good way to construct θ-psh functions with minimal singularities is by envelopes: i.e. we specify some function f on X and then consider the envelope

$$\phi := \sup\{\psi \leq f : \psi \ \theta\text{-psh}\}.$$

So one way to get a θ-psh function with correct behavior along D is to take

$$\phi_\infty := \sup\{\psi \leq tg : \psi \ \theta\text{-psh}\}.$$

To estimate the Monge-Ampère mass of ϕ_∞ we will write it as the limit of a decreasing sequence of θ-psh functions ϕ_R, all having minimal singularities. Since each ϕ_R has minimal singularities we know that for each R,

$$\int_X MA_\theta(\phi_R) = \mathrm{vol}(\alpha).$$

The Monge-Ampère measures $MA_\theta(\phi_R)$ do not converge weakly to $MA_\theta(\phi_\infty)$ on the whole of X because then α and $\alpha - t\beta$ would have the same volume, which is not true. By a crucial continuity property of the Monge-Ampère operator though we have weak convergence on any open set on which ϕ_∞ is locally bounded. Thus we want to pick a large open set U like that and see what happens to the measures $\mathbb{1}_U MA_\theta(\phi_R)$.

To be able to control these measures we first let $\phi := \sup\{\psi \leq 0 : \psi\ \theta\text{-psh}\}$ and then define

$$\phi_R := \sup\{\psi \leq \phi + tg_R : \psi\ \theta\text{-psh}\},$$

where g_R is a certain regularization of g which decreases to g as R tends to infinity. A deep result of Berman-Demailly [BD12] says that ϕ is particularly well-behaved: $dd^c\phi$ has L^∞-coefficients away from the non-Kähler locus, and the Monge-Ampère measure is simply given by $\mathbb{1}_D \theta^n$ where $D := \{\phi = 0\}$. We can use this to prove that the Monge-Ampère measure of each ϕ_R also is well-behaved, it is given by $\mathbb{1}_{D_R}(\theta + dd^c\phi + tdd^c g_R)^n$ where $D_R := \{\phi_R = \phi + tg_R\}$.

Let U be an open set at a given distance from Y. We choose our regularizations g_R such that for any such set $g_R = g$ for R large enough. Thus on U the obstacle $\phi + tg_R$ does not change for large R which for one thing means that

$$\mathbb{1}_{D_R \cap U}(\theta + dd^c\phi + tdd^c g_R)^n = \mathbb{1}_{D_R \cap U}(\theta + dd^c\phi + tdd^c g)^n,$$

but even more importantly makes $D_R \cap U$ decrease with R. It follows that the measures $\mathbb{1}_U MA_\theta(\phi_R)$ converge to $\mathbb{1}_{D_R \cap U}(\theta + dd^c\phi + tdd^c g)^n$. If we also assume that ϕ_∞ is locally bounded on U then we get that

$$\begin{aligned}\mathrm{vol}(\alpha - t\beta) = \int_X MA_\theta(\phi_\infty) \geq \int_{D_R \cap U} (\theta + dd^c\phi + tdd^c g)^n = \\ = \lim_{R\to\infty} \int_U MA_\theta(\phi_R) = \mathrm{vol}(\alpha) - \lim_{R\to\infty}\int_{U^c} MA_\theta(\phi_R).\end{aligned}$$

Thus we need to estimate $\int_{U^c} MA_\theta(\phi_R)$ from above. We again use that $MA_\theta(\phi_R)$ is of a nice form, and get, using the multilinearity of the non-pluripolar product, the following:

$$\begin{aligned}MA_\theta(\phi_R) = \mathbb{1}_{D_R}(\theta + dd^c\phi + tdd^c g_R)^n \leq \mathbb{1}_{D_R}(\theta + dd^c\phi + t(\omega + dd^c g_R))^n \leq \\ \leq MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \langle(\theta + dd^c\phi)^{n-k}\rangle \wedge (\omega + dd^c g_R)^k.\end{aligned}$$

Finally integrating over U^c yields

$$\begin{aligned}\int_{U^c} MA_\theta(\phi_R) \leq \\ \leq \int_{U^c} MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \int_{U^c} \langle(\theta + dd^c\phi)^{n-k}\rangle \wedge (\omega + dd^c g_R)^k \leq \\ \leq \int_{U^c} MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \langle \alpha^{n-k}\rangle \cdot \beta^k.\end{aligned}$$

By enlarging U we can make $\int_{U^c} MA_\theta(\phi)$ as small as we want. This then implies the infinitesimal estimate (1.3).

1.2. **Related work.** As has already been said Boucksom-Demailly-Păun-Peternell proved the integral version of Conjecture 1.1 in [BDPP13]. They also settled the conjecture in the case when X is compact hyperkähler, or more generally, a limit by deformation of projective manifolds with Picard number $\rho = h^{1,1}$ (see [BDPP13], Corollary 10). In the projective case they established a weaker version of Theorem B, namely that if α is nef and β is nef and lies in $NS_{\mathbb{R}}(X)$ then

$$\mathrm{vol}(\alpha - \beta) \geq (\alpha^n) - \frac{(n+1)^2}{4}(\alpha^{n-1} \cdot \beta).$$

Their proof is different from ours. As ours it uses a family of θ-psh functions that converge to something with a logarithmic singularity along the divisor of ω. But instead of being envelopes these functions solve a Monge-Ampère equation, which concentrates the mass along the divisor.

In [Xiao13] Xiao proved a kind of weaker qualitative version of (1.2), namely given two nef classes α and β on a compact Kähler manifold X then if

$$(\alpha^n) > 4n(\alpha^{n-1} \cdot \beta)$$

it follows that $\alpha - \beta$ is big. Later, using the same kind of techniques as Xiao, Popovici improved on this, showing that

$$(\alpha^n) > n(\alpha^{n-1} \cdot \beta)$$

implies $\alpha - \beta$ to be big. Then Xiao refining the techniques further established in [Xiao14] that if α is big and β movable then

$$\mathrm{vol}(\alpha) > n(\langle \alpha^{n-1} \rangle \cdot \beta)$$

implies $\alpha - \beta$ to be big.

The differentiability of the volume of big line bundles was proved independently and at the same time as Boucksom-Favre-Jonsson by Lazarsfeld-Mustaţă [LM09] using the theory of Okounkov bodies. They expressed the derivative in terms of restricted volumes, and as a consequence the restricted volume of a big line bundle L along a prime divisors D coincides with the pairing $\langle c_1(L)^{n-1} \rangle \cdot c_1(D)$. The restricted volume is only really defined along subvarieties that are not contained in the augmented base locus, while the pairing $\langle c_1(L)^{n-1} \rangle \cdot c_1(D)$ always is defined and furthermore depend continuously on L. It thus follows from Theorem 1.3 that if a prime divisor D is not contained in the augmented base locus for small ample perturbations $L + \epsilon A$ of L and furthermore the restricted volume of $L + \epsilon A$ along D remains bounded from below by some positive number then D cannot be contained in the augmented base locus of L. A deep result of Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa [ELMNP09] states that the restricted volume can be used to characterize the whole augmented base locus. Whether the analogous result pertaining to the non-Kähler locus of a big class is true is still non known, but we note that our Theorem D can be seen as a partial result in that direction. The nef case though was recently completely settled by Collins-Tosatti [CT15]. They prove that if α is a nef and big class on a compact Kähler manifold then the non-Kähler locus of α is equal to the null-locus, i.e. union of all irreducible analytic subspaces V such that

$$\int_V \alpha^{\dim V} = 0.$$

Apart from the pseudoeffective and the big cone in $H^{1,1}_{\mathbb{R}}(X)$ there are two other very important cones, namely the Kähler cone $\mathbb{K}$ and its closure the nef cone $\mathcal{N}$. In the deep and important work [DP04] (indeed it is used in a crucial way in [BDPP13] and hence also in the proof of our results) Demailly-Păun proves that the nef cone $\mathcal{N}$ is dual to the pseudoeffective cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ (i.e. those classes that contain a closed positive $(n-1,n-1)$-current). See also the simpler proof by Chiose in the recent [Chi16].

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2. Preliminaries

2.1. **θ-psh functions.** Let (X,ω) be a compact Kähler manifold.

Let θ be a closed smooth real $(1,1)$-form on X and $\alpha := [\theta] \in H^{1,1}_{\mathbb{R}}(X)$ its cohomology class. We say that a function $u : X \to [-\infty,\infty)$ is θ-psh if whenever locally $\theta = dd^c v$ for some smooth function v we have that $u+v$ is plurisubharmonic and not identically equal to $-\infty$. Thus $\theta + dd^c u$ is a closed positive $(1,1)$-current. Conversely, if T is a closed positive $(1,1)$-current in α then there exists a θ-psh function u such that $T = \theta + dd^c u$, and this u is unique up to a constant.

A function u which is θ-psh for some θ is called almost psh.

If θ' is another closed smooth real $(1,1)$-form cohomologuous to θ, then by the dd^c-lemma there exists a smooth function f such that $\theta' = \theta + dd^c f$. Thus one sees that u is θ-psh iff $u - f$ is θ'-psh.

The set of θ-psh functions is denoted by $PSH(X,\theta)$. The class α is called pseudoeffective if it contains a closed positive current and we note that this is equivalent to $PSH(X,\theta)$ being nonempty. A class is said to be big if for some $\epsilon > 0$ and some Kähler class β we have that $\alpha - \epsilon\beta$ is pseudoeffective.

A θ-psh function u is said to have analytic singularities if locally it can be written as $c\ln(\sum_i |g_i|^2) + f$ where $c > 0$, g_i is a finite collection of local holomorphic functions and f is smooth. A deep regularization result of Demailly states that if α is big then there exists a θ-psh function with analytic singularities.

A θ-psh function u is said to have minimal singularities if for every $v \in PSH(X,\theta)$ we have that $u \geq v + O(1)$. It is easy to show using envelopes that whenever α is pseudoeffective one can find θ-psh functions with minimal singularities. They are far from unique though, in fact this is what we will exploit later in the proof of Theorem B.

2.2. **Lelong numbers and the non-Kähler locus.** Given a θ-psh function u its Lelong number at a point $x \in X$, denoted by $\nu_x(u)$, is defined as the supremum of all λ such that $u(z) \leq \lambda \ln|z|^2 + O(1)$ locally near x, where z_i are local holomorphic coordinates centered at x. When Y is an irreducible analytic subset we define the Lelong number of u along Y as $\nu_Y(u) := \inf_{x\in Y} \nu_x(u)$. A fundamental result of Siu [Siu74] states that for any c the set $E_c(u) := \{x : \nu_x(u) \geq c\}$ is analytic. In particular this implies that $\nu_Y(u)$ equals the Lelong number of u at a generic point of Y.

If α is big we define the Lelong number of α at a point x as $\nu_x(\alpha) := \nu_x(u)$ where u is any θ-psh function with minimal singularities. The set $E_{nn}(\alpha) := \{x : \nu_x(\alpha) > 0\}$ is called the non-nef locus of α, and one can show that it is a countable union of analytic sets. One also defines the non-Kähler locus $E_{nK}(\alpha)$ as

$$\bigcap_{\epsilon>0} E_{nn}([\theta - \epsilon\omega]).$$

Clearly $E_{nn}(\alpha) \subseteq E_{nK}(\alpha)$, but the non-Kähler locus has the advantage of being a proper analytic subset of X (see [Bou02]).

2.3. **Non-pluripolar positive products and Monge-Ampère measures.** A key tool will be the notion of non-pluripolar positive products of closed positive currents. This theory was first developed in the local setting by Bedford-Taylor [BT82] and later in the geometric setting of compact Kähler manifolds by Boucksom-Eyssidieux-Guedj-Zeriahi in [BEGZ10].

If T_i, $i = \{1, ..., p\}$ are closed positive $(1,1)$-currents one defines a closed positive (p,p)-current called the non-pluripolar positive product of T_i, denoted by $\langle T_1 \wedge ... \wedge T_p\rangle$ (see [BEGZ10] for the definition). To avoid some technical issues we will henceforth only consider products when the cohomology classes of the currents T_i are all big. The product is symmetric and multilinear, so in particular if T_1 and T_2 are two closed positive $(1,1)$-currents then

$$\langle (T_1 + T_2)^n\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle T_1^{n-k} \wedge T_2^k\rangle.$$

Also if η is a closed semipositive $(1,1)$-form then $\langle T_1 \wedge ... \wedge T_p \wedge \eta^q\rangle = \langle T_1 \wedge ... \wedge T_p\rangle \wedge \eta^q$. An important property of non-pluripolar products (which also explains the name) is that they never put any mass on pluripolar sets, which includes proper analytic subsets.

When $p = n = \dim_{\mathbb{C}} X$, $\langle T_1 \wedge ... \wedge T_n\rangle$ is a positive measure, and when the n currents are all equal $T_i = \theta + dd^c\psi$, then $\langle (\theta + dd^c\psi)^n\rangle$ is known as the (non-pluripolar) Monge-Ampère measure of ψ, which we also denote by $MA_\theta(\psi)$. A basic fact is that if on some open set U we have that ψ is locally bounded and $dd^c\psi$ has coefficients in L^∞ then

$$\mathbb{1}_U MA_\theta(\psi) = \mathbb{1}_U (\theta + dd^c u)^n.$$

Here the right hand side simply denotes the measure one gets by taking the appropriate determinant of the coefficient functions (typically this only makes sense when the coefficients are L^∞-functions, since in general one cannot multiply measures).

An absolutely fundamental role in this theory is played by the following convergence result for Monge-Ampère measures, proved by Bedford-Taylor in [BT82].

Theorem 2.1. *Let U be an open set and u_k be a decreasing sequence of θ-psh functions such that $u := \lim_{k\to\infty} u_k$ is locally bounded on U (u will then automatically by θ-psh on U). Then the measures $\mathbb{1}_U MA_\theta(u_k)$ converge weakly to $\mathbb{1}_U MA_\theta(u)$.*

Remark 2.1. One can also allow u_k to increase a.e. to their limit u and the convergence still holds, and one is also allowed to restrict to a plurifine open set (see [BT82]) but we will not need that here. One should note that the assumption on u being locally bounded is absolutely vital, without it the statement would be blatantly false.

We cite the following important result from [BEGZ10].

Theorem 2.2. *Assume we have two p-tuples of currents $T_i = \theta_i + dd^c\psi_i$ and $T_i' = \theta_i + dd^c\psi_i'$ such that for each i, $\psi_i \leq \psi_i' + O(1)$ and furthermore each ψ_i is bounded from below by some almost psh function with analytic singularities (or more generally with small unbouded locus). Then the cohomology class of $\langle T_1 \wedge ... \wedge T_p\rangle$ is bounded from above by the cohomology class of $\langle T_1' \wedge ... \wedge T_p'\rangle$.*

This result implies that if $T_i = \theta_i + dd^c\psi_i$, $i \in \{1, ..., p\}$ where $\psi_i \in PSH(\gamma_i)$ all have minimal singularities, then the cohomology class of

$$\langle T_1 \wedge ... \wedge T_p\rangle$$

only depends on the cohomology classes $\alpha_i := [\gamma_i]$ and not on the particular T_i. This class is called the positive intersection of α_i and denoted by

$$\langle \alpha_1 \wedge ... \wedge \alpha_p\rangle.$$

It is important to note that it does not in general depend multilinearly on the cohomology classes. But it is naturally homogeneous in each variable and it is also monotone in the following sense:

$$\langle \alpha_i \wedge ... \wedge \alpha_p \rangle \leq \langle \alpha'_1 \wedge ... \wedge \alpha'_p \rangle$$

if each difference $\alpha'_i - \alpha_i$ is pseudoeffective. This again follows from Theorem 2.2. These properties together imply that $\langle \alpha_1 \wedge ... \wedge \alpha_p \rangle$ at least depends continuously on the classes α_i (see [BEGZ10]).

In particular we see that the number $\int_X MA_\theta(u) = \langle (\theta + dd^c u)^n \rangle$ only depends on α as long as u has minimal singularities. This number is called the volume of α, and is written as $\mathrm{vol}(\alpha)$. We also see from Theorem 2.2 that if $v \in PSH(X, \theta)$ is bounded from below by some almost psh function with analytic singularities then

$$\int_X MA_\theta(v) \leq \mathrm{vol}(\alpha).$$

The following is a deep result of Boucksom in [Bou02] building on work of Demailly-Păun in [DP04] (see also [Chi16]).

Theorem 2.3. *Let α_k be a sequence of big classes that converge to a class α. Then if $\limsup_{k \to \infty}$ vol$(\alpha_k) > 0$ it follows that α is big.*

From this we see that letting $\mathrm{vol}(\alpha) := 0$ for α not big gives a continuous extension of the volume function to the whole of $H^{1,1}_{\mathbb{R}}(X)$.

3. Regularity of envelopes

In our proof of Theorem B a key role will be played by a family of envelopes. The proof will rely on us being able to control the behaviour of their Monge-Ampère measures. For this we need a deep result of Berman-Demailly [BD12].

Theorem 3.1. *Let θ be a smooth closed real $(1,1)$-form on a compact Kähler manifold (X, ω). Assume that the class $\alpha := [\theta]$ is big and let ψ_0 be a strictly θ-psh function with analytic singularities. Let ϕ be defined as*

$$\phi := \sup\{\psi \leq 0 : \psi \in PSH(X, \theta)\},$$

and let $D := \{\phi = 0\}$. Then $\phi \in PSH(X, \theta)$ has minimal singularities and for some constants C and B we have that

$$|dd^c \phi|_\omega \leq C(|\psi_0| + 1)^2 e^{B|\psi_0|}.$$

It follows that

$$MA_\theta(\phi) = \mathbb{1}_D \theta^n$$

and hence

$$\mathit{vol}(\alpha) = \int_X MA_\theta(\phi) = \int_D \theta^n.$$

Remark 3.1. It was remarked in [BD12] that θ in Theorem 3.1 is allowed to have coefficients in L^∞.

This theorem deals with one particular envelope. For our purposes we will need a variation of it where we are allowed to consider more general obstacle functions. For technical reasons we want to use functions of the form $\phi + f$ as obstacles for the envelope, where ϕ is the envelope from Theorem 3.1 and f is smooth. It turns out that we can use Theorem 3.1 combined with the continuity property of the Monge-Ampère operator Theorem 2.1 to get basic control of the Monge-Ampère measures of these particular envelopes.

Proposition 3.2. Let θ and ϕ be as in Theorem 3.1, let f be a smooth function on X, and let

$$\phi_f := \sup\{\psi \leq \phi + f : \psi \in PSH(\theta)\}.$$

Then $\phi_f \in PSH(\theta)$ has minimal singularities and

$$MA_\theta(\phi_f) = \mathbb{1}_{D_f \setminus Z}(\theta + dd^c\phi + dd^c f)^n,$$

where $D_f := \{\phi_f = \phi + f\}$ and $Z := \{\psi_0 = -\infty\}$. We thus get that

$$\mathrm{vol}(\alpha) = \int_{D_f \setminus Z} (\theta + dd^c\phi + dd^c f)^n.$$

Proof. Note that $\phi + \min_X f$ is a candidate for the envelope, showing that $\phi + \min_X f \leq \phi_f$ and hence ϕ_f has minimal singularities.

Let $|\cdot|_{reg}$ be a smooth convex function on $\mathbb{R}$ which coincides with $|\cdot|$ for $|x| \geq 1$ and

$$\max_{reg}(x, y) := \frac{x + y + |x - y|_{reg}}{2}$$

be the corresponding regularized max function. Let

$$\phi_k := \max_{reg}(\phi, -k - \max_{reg}(\psi_0, -k - 1)).$$

Then ϕ_k decreases down to ϕ and $\theta_k := \theta + dd^c\phi_k + dd^c f$ has L^∞-coefficients. Let

$$u_k := \sup\{v \leq 0 : v \in PSH(\theta_k)\}$$

and $D_k := \{u = 0\}$. By Theorem 3.1 (with L^∞-coefficients, see Remark 3.1)

$$MA_{\theta_k}(u_k) = \mathbb{1}_{D_k}\theta_k^n. \tag{3.1}$$

It is easy to see that

$$u_k + \phi_k + f = \sup\{\psi \leq \phi_k + f : \psi \in PSH(\theta)\} =: \psi_k,$$

and so $D_k = \{\psi_k = \phi_k + f\}$ and by (3.1)

$$MA_\theta(\psi_k) = \mathbb{1}_{D_k}(\theta + dd^c\phi_k + dd^c f)^n.$$

Since ϕ_k decreases down to ϕ we get that ψ_k decreases down to ϕ_f.

Without loss of generality we can assume that $\psi_0 \leq 0$ which then implies that $\psi_0 \leq \phi$. Let $U_C := \{\psi_0 > -C\}$, then untangling the definition shows that $\phi_k = \phi$ on U_C as long as $k \geq 2C$. Thus for large k,

$$\mathbb{1}_{U_C} MA_\theta(\psi_k) = \mathbb{1}_{D_k \cap U_C}(\theta + dd^c\phi + dd^c f)^n.$$

We now claim that for $k > 2C$, $D_k \cap U_C$ decreases with k and

$$(\bigcap_{k > 2C} D_k) \cap U_C = D_f \cap U_C.$$

For the first statement, $x \in D_k \cap U_C$ iff $\psi_k(x) = \phi_k(x) + f(x) = \phi(x) + f(x)$, but since $\psi_k(x)$ decreases in k we must have that $D_k \cap U_C$ decreases with k. On the other hand, $x \in D_f \cap U_C$ iff $\phi_f(x) = \phi(x) + f(x)$ iff $\psi_k(x) = \phi(x) + f(x)$ for all $k > 2C$.

If μ is a finite measure and A_k is a decreasing sequence of measurable sets with $A := \cap_k A_k$ then the basic continuity of measures implies that $\mathbb{1}_{A_k}\mu$ converge (strongly) to $\mathbb{1}_A\mu$. Thus we see that $\mathbb{1}_{U_C} MA_\theta(\psi_k)$ converges to $\mathbb{1}_{D_f \cap U_C}(\theta + dd^c\phi + dd^c f)^n$.

We also note that $\phi_f \geq \psi_0 - C'$ for some constant C', so ϕ_f is bounded on U_C. By Theorem 2.1 we thus get that $\mathbb{1}_{U_C} MA_\theta(\psi_k)$ converges weakly to $\mathbb{1}_{U_C} MA_\theta(\phi_f)$ and hence

$$\mathbb{1}_{U_C} MA_\theta(\phi_f) = \mathbb{1}_{D_f \cap U_C}(\theta + dd^c\phi + dd^c f).$$

Letting C tend to infinity proves the proposition since $(\cup_C U_C)^c = Z$ being pluripolar cannot support any part of $MA_\theta(\phi_f)$. $\square$

4. Proof of Theorem B

Our goal is to establish the fundamental volume bound (1.2) for α nef and β nef and lying in $NS_{\mathbb{R}}(X)$. To do this we will prove the following

Theorem 4.1. *Let X be projective, let α be a big class (so non necessarily Kähler) and β an integral Kähler class. Then we have that*

$$\liminf_{t\to 0+} \frac{\mathit{vol}(\alpha - t\beta) - \mathit{vol}(\alpha)}{t} \geq -n\langle \alpha^{n-1}\rangle \cdot \beta.$$

This is the appropriate infinitesimal version of (1.2) extended to arbitrary big classes α.

Before turning to the proof of Theorem 4.1 let us explain how it implies Theorem B.

We use the following elementary lemma:

Lemma 4.2. *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions such that for all $t \in [a, b)$ we have that*

$$\liminf_{h\to 0+} \frac{f(t+h) - f(t)}{h} \geq g(t).$$

Then $f(b) - f(a) \geq \int_a^b g(t)dt$.

Proof. By considering $f(t) - \int_a^t g(s)ds$ we can assume that $g \equiv 0$ and hence we want to show that this implies $f(b) \geq f(a)$. Pick $\epsilon > 0$. Let $t_0 := \sup\{t \in [a, b] : f(s) - f(a) \geq -\epsilon s \text{ for all } s \in [a, t]\}$. If $t_0 \neq b$ then it would follow that $\liminf_{h\to 0+} \frac{f(t_0+h)-f(t_0)}{h} \leq -\epsilon$ which is a contradiction. Hence $t_0 = b$ so $f(b) - f(a) \geq -\epsilon$ and thus $f(b) \geq f(a)$. $\square$

4.1. Proof of Theorem B given Theorem 4.1.

Proof. By continuity and the scaling properties of the volume and the intersection numbers we can assume that α is Kähler while β is Kähler and integral (i.e. $\beta = c_1(L)$ for some holomorphic line bundle L).

Let

$$t_0 := \sup\{t \in [0, 1] : \alpha - t\beta \in \mathcal{E}^0\},$$

$f(t) := \mathrm{vol}(\alpha - t\beta)$ and $g(t) := -n\langle(\alpha - t\beta)^{n-1}\rangle \cdot \beta$. Both f and g are continuous on $[0, t_0)$ (see Section 2) and it is also easy to show that g is increasing. Theorem 4.1 now exactly says that

$$\liminf_{h\to 0+} \frac{f(t+h) - f(t)}{h} \geq g(t),$$

thus by the lemma above we get that for any $t \in [0, t_0)$

$$\mathrm{vol}(\alpha - t\beta) \geq \mathrm{vol}(\alpha) - n\int_{s=0}^{t} \left(\langle(\alpha - s\beta)^{n-1}\rangle \cdot \beta\right) ds \geq$$
$$\geq \mathrm{vol}(\alpha) - nt\langle\alpha^{n-1}\rangle \cdot \beta = (\alpha^n) - nt(\alpha^{n-1} \cdot \beta).$$

Assume that

$$(\alpha^n) - nt(\alpha^{n-1} \cdot \beta) > 0$$

because otherwise the volume estimate is trivially true. We then get that for any $t \in [0, t_0)$

$$\mathrm{vol}(\alpha - t\beta) \geq (\alpha^n) - nt(\alpha^{n-1} \cdot \beta) \geq (\alpha^n) - n(\alpha^{n-1} \cdot \beta) > 0. \tag{4.1}$$

By Theorem 2.3 which as we recall says that the volume tends to zero as one approaches the boundary of the big cone we see that (4.1) implies that $\alpha - t_0\beta$ is big and hence $t_0 = 1$.

The continuity of the volume function in the big cone combined with (4.1) then establishes the desired volume estimate.

□

4.2. **Proof of Theorem 4.1.**

Proof. Let θ be a smooth representative of α and let ω be a Kähler form in β.

Pick $t > 0$ such that $\alpha - t\beta$ is big (which is always possible since $\mathcal{E}^0$ is an open cone). By the homogeneity of the positive intersection we are allowed to multiply both α and θ with the same positive constant. Thus without loss of generality we may assume that L is effective. Let s be a nontrivial holomorphic section of L. There is a positive metric h of L whose curvature form is ω, and we let

$$g := \ln |s|_h^2,$$

where we normalize h so that $\max g = 0$. We thus have $dd^c g = [Y] - \omega$ where Y is the effective divisor defined by s. For any $R > 0$ we let

$$g_R := \max_{reg}(g, -R).$$

Let as in Theorem 3.1

$$\phi := \sup\{\psi \leq 0 : \psi \in PSH(\theta)\}.$$

Let also

$$\phi_R := \sup\{\psi \leq \phi + tg_R : \psi \in PSH(\theta)\}$$

and

$$D_R := \{\phi_R = \phi + tg_R\}.$$

From Proposition 3.2 $\phi_R \in PSH(\theta)$ has minimal singularities, and hence

$$\text{vol}(\alpha) = \int_X MA_\theta(\phi_R). \tag{4.2}$$

We also get from Theorem 3.1 and Proposition 3.2 that

$$MA_\theta(\phi_R) = \mathbb{1}_{D_R \setminus Z}(\theta + dd^c\phi + tdd^c g_R)^n, \tag{4.3}$$

the measure being locally L^∞ on $X \setminus Z$ where Z is the singular set of some strictly θ-psh function ψ_0 with analytic singularities.

Let also

$$\phi_\infty := \sup\{\psi \leq \phi + tg : \psi \in PSH(\theta)\}$$

and

$$D_\infty := \{\phi_\infty = \phi + tg\}.$$

We claim that $\phi_\infty - tg \in PSH(\theta - t\omega)$ and that in fact

$$\phi_\infty - tg = \sup\{\psi \leq 0 : \psi \in PSH(\theta - t\omega)\}. \tag{4.4}$$

Namely, we note that since $dd^c g = [Y] - \omega$ we have that $dd^c(\phi_\infty - tg) \geq -\theta + t\omega$ on $X \setminus Y$. On the other hand $\phi_\infty - tg \leq \phi \leq 0$ so it extends as an $(\theta - t\omega)$-psh function across Y. If ψ is any other $(\theta - t\omega)$-psh function with $\psi \leq 0$, then it is also θ-psh (since $t\omega$ is Kähler) and thus $\psi \leq \phi$ (ϕ being defined as the supremum of all such functions). Since $\psi + tg$ clearly lies in $PSH(\theta)$ it follows that $\psi + tg \leq \phi_\infty$ which shows the validity of (4.4). In particular $\phi_\infty - tg$ has minimal singularities and so

$$\text{vol}(\alpha - t\beta) = \int_X MA_{\theta - t\omega}(\phi_\infty - tg) = \int_X MA_\theta(\phi_\infty). \tag{4.5}$$

Since $\alpha - t\beta$ was assumed to be big we also get that $\phi_\infty \geq \psi_1$ for some quasi-psh function ψ_1 with analytic singularities.

The obstacles $\phi + tg_R$ decreases to $\phi + tg$ and therefore the envelopes ϕ_R decreases to ϕ_∞. Let $U_C := \{\psi_1 > -C\} \cap \{g > -C\}$. Note that for $R > C+1$, $g_R = \max_{reg}(g, -R) = g$ on U_C and so by (4.3)

$$\mathbb{1}_{U_C} MA_\theta(\phi_R) = \mathbb{1}_{D_R \cap U_C \setminus Z}(\theta + dd^c\phi - t\omega)^n.$$

We now claim that for $R > C+1$, $D_R \cap U_C$ decreases with R and

$$(\bigcap_{R>C} D_R) \cap U_C = D_\infty \cap U_C.$$

It is exactly the same situation as in the proof of Proposition 3.2. For the first statement, $x \in D_R \cap U_C$ ($R > C+1$) iff $\phi_R(x) = \phi(x) + tg_R(x) = \phi(x) + tg(x)$, but since $\phi_R(x)$ decreases in R we must have that $D_R \cap U_C$ decreases with R. On the other hand, $x \in D_\infty \cap U_C$ iff $\phi_\infty(x) = \phi(x) + tg(x)$ iff $\phi_R(x) = \phi(x) + tg(x)$ for all $R > C+1$.

We also note as in Proposition 3.2 that this implies that $\mathbb{1}_{D_R \cap U_C \setminus Z}(\theta + dd^c\phi - t\omega)^n$ converge (strongly) to $\mathbb{1}_{D_\infty \cap U_C \setminus Z}(\theta + dd^c\phi - t\omega)^n$. But since $\phi_\infty \geq \psi_1$ we have that ϕ_∞ is bounded on U_C and thus Theorem 2.1 implies that $\mathbb{1}_{U_C} MA_\theta(\phi_R)$ converge weakly to $\mathbb{1}_{U_C} MA_\theta(\phi_\infty)$. Put together this shows that

$$\mathbb{1}_{U_C} MA_\theta(\phi_\infty) = \mathbb{1}_{D_\infty \cap U_C \setminus Z}(\theta + dd^c\phi - t\omega)^n$$

and hence

$$\lim_{R\to\infty} \int_{U_C} MA_\theta(\phi_R) = \int_{U_C} MA_\theta(\phi_\infty). \tag{4.6}$$

Combining (4.2),(4.5) and (4.6) we get that

$$\begin{aligned} \text{vol}(\alpha - t\beta) = \int_X MA_\theta(\phi_\infty) \geq \int_{U_C} MA_\theta(\phi_\infty)^n = \\ = \lim_{R\to\infty} \int_{U_C} MA_\theta(\phi_R) = \text{vol}(\theta) - \lim_{R\to\infty} \int_{U_C^c} MA_\theta(\phi_R). \end{aligned} \tag{4.7}$$

We thus need to estimate $\int_{U_C^c} MA_\theta(\phi_R)$ from above. But using (4.3) we clearly have that

$$\begin{aligned} MA_\theta(\phi_R) = \mathbb{1}_{D_R \setminus Z}(\theta + dd^c\phi + tdd^c g_R)^n \leq \\ \leq \mathbb{1}_{D_R \setminus Z}(\theta + dd^c\phi + t(\omega + dd^c g_R))^n \leq \\ \leq \mathbb{1}_{X \setminus Z}(\theta + dd^c\phi + t(\omega + dd^c g_R))^n = MA_{\theta + t\omega}(\phi + tg_R). \end{aligned} \tag{4.8}$$

Here we used that adding $t\omega$ only increases the mass on $D_R \setminus Z$ (it being Kähler), and then that g_R is ω-psh, making $MA_{\theta+t\omega}(\phi + tg_R)$ a well-defined positive measure on X.

Since both $\theta + dd^c\phi$ and $t(\omega + dd^c g_R)$ are closed positive $(1,1)$-currents we get that

$$\begin{aligned} MA_{\theta+t\omega}(\phi + tg_R) = \langle(\theta + dd^c\phi + t(\omega + dd^c g_R))^n\rangle = \\ = \sum_{k=0}^n t^k \binom{n}{k} \langle(\theta + dd^c\phi)^{n-k} \wedge (\omega + dd^c g_R)^k\rangle = \\ = MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \langle(\theta + dd^c\phi)^{n-k}\rangle \wedge (\omega + dd^c g_R)^k, \end{aligned} \tag{4.9}$$

simply using the multilinearity of the non-pluripolar product (see Section 2).

We can now use (4.8) and (4.9) to estimate $\int_{U_C^c} MA_\theta(\phi_R)$, namely

$$\int_{U_C^c} MA_\theta(\phi_R) \le \int_{U_C^c} MA_{\theta+t\omega}(\phi + tg_R) =$$
$$= \int_{U_C^c} \left(MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \langle (\theta + dd^c\phi)^{n-k} \rangle \wedge (\omega + dd^c g_R)^k \right) \le$$
$$\le \int_{U_C^c} MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \int_X \langle (\theta + dd^c\phi)^{n-k} \rangle \wedge (\omega + dd^c g_R)^k =$$
$$= \int_{U_C^c} MA_\theta(\phi) + \sum_{k=1}^n t^k \binom{n}{k} \langle \alpha^{n-k} \rangle \cdot \beta^k,$$

where we in the last step used that both ϕ and g_R have minimal singularities (see Section 2).

Since this estimate is independent of R we conclude from (4.7) that

$$\text{vol}(\alpha - t\beta) \ge \text{vol}(\alpha) - \int_{U_C^c} MA_\theta(\phi) - \sum_{k=1}^n t^k \binom{n}{k} \langle \alpha^{n-k} \rangle \cdot \beta^k$$

while letting C tend to infinity yields

$$\text{vol}(\alpha - t\beta) \ge \text{vol}(\alpha) - \sum_{k=1}^n t^k \binom{n}{k} \langle \alpha^{n-k} \rangle \cdot \beta^k.$$

In particular

$$\liminf_{t \to 0+} \frac{\text{vol}(\alpha - t\beta) - \text{vol}(\alpha)}{t} \ge -n\langle \alpha^{n-1} \rangle \cdot \beta,$$

which was to be proved.

□

Appendix A. Remarks on orthogonality, differentiabiltity and duality – S. Boucksom

A.1. **Differentiability and duality in the projective case.** Demailly conjectures that the following 'transcendental Morse inequality'

$$\text{vol}(\alpha - \beta) \ge (\alpha^n) - n(\alpha^{n-1} \cdot \beta) \tag{A.1}$$

holds for any two nef classes $\alpha, \beta \in H^{1,1}_{\mathbb{R}}(X)$ on a compact Kähler manifold X of complex dimension n.

In the main paper the following was proved:

Theorem A.1. *The Morse inequality (A.1) holds when X is projective and $\beta \in NS_{\mathbb{R}}(X)$.*

As we shall see, this result implies the following general statements.

Theorem A.2. *Let X be a projective manifold.*

(ii) *The Morse inequality (A.1) holds for arbitrary nef $(1,1)$-classes.*
(ii) *The differentiability theorem of* [BFJ09] *holds for all $(1,1)$-classes: for each $\alpha, \gamma \in H^{1,1}_{\mathbb{R}}(X)$ with α big, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \text{vol}(\alpha + t\gamma) = n\,\gamma \cdot \langle \alpha^{n-1} \rangle.$$

(iii) *The duality theorem of* [BDPP13] *holds for all* $(1,1)$*-classes: a class* $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ *is psef iff*

$$\alpha \cdot \mu_*(\omega^{n-1}) \geq 0$$

for any modification $\mu : X' \to X$ *and any Kähler class* ω *on* X'.

A.2. **From orthogonality to differentiability.** As we next show, the orthogonality property of [BDPP13] is equivalent to the differentiability property of [BFJ09]. Our argument is inspired by the simplified proof of [BB10, Theorem B] provided in [LN, Lemma 6.13] (see also [Xiao14, Proposition 1.1] for a related result).

Theorem A.3. *For a given compact Kähler manifold* X, *the following properties are equivalent:*

(i) *Orthogonality: each big class* $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ *satisfies*

$$\mathrm{vol}(\alpha) = \alpha \cdot \langle \alpha^{n-1} \rangle.$$

(ii) *Differentiability: for each* $\alpha, \gamma \in H^{1,1}_{\mathbb{R}}(X)$ *with* α *big, we have*

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\, \gamma \cdot \langle \alpha^{n-1} \rangle.$$

Further, these properties imply the transcendental Morse inequality (A.1) for all nef classes, as well as the duality theorem.

Lemma A.4. *The differentiability property (ii) holds if and only if*

$$\mathrm{vol}(\alpha)^{1/n} - \mathrm{vol}(\beta)^{1/n} \geq \frac{(\alpha - \beta) \cdot \langle \alpha^{n-1} \rangle}{\mathrm{vol}(\alpha)^{1-1/n}} \tag{A.2}$$

for any two big classes $\alpha, \beta \in H^{1,1}_{\mathbb{R}}(X)$.

Proof. Since vol is positive on the big cone, (ii) is equivalent to

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\alpha + t\gamma)^{1/n} = \frac{\gamma \cdot \langle \alpha^{n-1} \rangle}{\mathrm{vol}(\alpha)^{1-1/n}}. \tag{A.3}$$

By concavity of $\mathrm{vol}^{1/n}$ on the big cone [Bou02], we thus see that (ii) implies (A.2). Assume conversely that the latter holds. Then

$$\frac{t\gamma \cdot \langle \alpha^{n-1} \rangle}{\mathrm{vol}(\alpha)^{1-1/n}} \geq \mathrm{vol}(\alpha + t\gamma)^{1/n} - \mathrm{vol}(\alpha)^{1/n} \geq \frac{t\gamma \cdot \langle (\alpha + t\gamma)^{n-1} \rangle}{\mathrm{vol}(\alpha + t\gamma)^{1-1/n}},$$

for $|t| \ll 1$, which yields (A.3) by continuity of positive intersection products on the big cone [BFJ09]. □

Since $\mathrm{vol}(\alpha) = (\alpha^n)$ is differentiable when α is nef, the same argument shows that (A.2) holds when α, β are nef and big.

Proof of Theorem A.3. Assume that (i) holds, and pick big classes $\alpha, \beta \in H^{1,1}_{\mathbb{R}}(X)$. By Lemma A.4, it will be enough to establish (A.2). By definition of positive intersection numbers, there exists a sequence of modifications $\mu_k : X_k \to X$ and Kähler classes α_k, β_k on X_k with

- $\alpha_k \leq \mu_k^* \alpha$ (i.e. the difference is psef);
- $\beta_k \leq \mu_k^* \beta$;
- $\mathrm{vol}(\alpha_k) \to \mathrm{vol}(\alpha)$;
- $\mathrm{vol}(\beta_k) \to \mathrm{vol}(\beta)$;
- $(\mu_k^* \beta \cdot \alpha_k^{n-1}) \to \beta \cdot \langle \alpha^{n-1} \rangle$.

As noted above, (A.2) holds when the classes are nef, and hence

$$\mathrm{vol}(\alpha_k)^{1/n} - \mathrm{vol}(\beta_k)^{1/n} \geq \frac{(\alpha_k - \beta_k)\cdot\alpha_k^{n-1}}{\mathrm{vol}(\alpha_k)^{1-1/n}}$$

$$\geq \frac{(\alpha_k^n) - (\mu_k^*\beta\cdot\alpha_k^{n-1})}{\mathrm{vol}(\alpha_k)^{1-1/n}}$$

since $\beta_k \leq \mu_k^*\beta$ and α_k is nef. In the limit as $k\to\infty$ we infer

$$\mathrm{vol}(\alpha)^{1/n} - \mathrm{vol}(\beta)^{1/n} \geq \frac{\mathrm{vol}(\alpha) - \beta\cdot\langle\alpha^{n-1}\rangle}{\mathrm{vol}(\alpha)^{1-1/n}},$$

which is (A.2) since $\mathrm{vol}(\alpha) = \alpha\cdot\langle\alpha^{n-1}\rangle$. This proves (i)$\Longrightarrow$(ii). Conversely, applying (ii) with $\gamma=\alpha$ yields (i), as already observed in [BFJ09].

Assume now that (i) and (ii) hold. As observed in [BFJ09], the Morse inequality (A.1) holds for any two nef classes α,β, because

$$\mathrm{vol}(\alpha-\beta) - (\alpha^n) = -n\int_0^1 \beta\cdot\langle(\alpha-t\beta)^{n-1}\rangle dt \geq -n(\beta\cdot\alpha^{n-1})$$

since $\alpha - t\beta \leq \alpha$ and β is nef.

We next turn to the duality theorem. It is enough to show that any psef class α in the interior of the closed convex cone generated by classes of the form $\mu_*\omega^{n-1}$ is big. For such a class, there exists a Kähler class ω on X such that

$$\alpha\cdot\langle\beta^{n-1}\rangle \geq \omega\cdot\langle\beta^{n-1}\rangle \tag{A.4}$$

for all big classes $\beta\in H^{1,1}_{\mathbb{R}}(X)$. For each $\varepsilon>0$, $\alpha+\varepsilon\omega$ is big, and (i) and (A.4) give

$$\mathrm{vol}(\alpha+\varepsilon\omega) = (\alpha+\varepsilon\omega)\cdot\langle(\alpha+\varepsilon\omega)^{n-1}\rangle \geq \alpha\cdot\langle(\alpha+\varepsilon\omega)^{n-1}\rangle \geq \omega\cdot\langle(\alpha+\varepsilon\omega)^{n-1}\rangle,$$

and hence

$$\mathrm{vol}(\alpha+\varepsilon\omega) \geq \mathrm{vol}(\omega)^{1/n}\,\mathrm{vol}(\alpha+\varepsilon\omega)^{1-1/n}$$

by the Khovanskii-Teissier inequality. This yields $\mathrm{vol}(\alpha+\varepsilon\omega)\geq(\omega^n)>0$ for any $\varepsilon>0$, which proves that α is big by [Bou02]. □

Proof of Theorem A.2. By Theorem A.3, it is enough to show that any big class $\alpha\in H^{1,1}_{\mathbb{R}}(X)$ satisfies $\mathrm{vol}(\alpha)=\alpha\cdot\langle\alpha^{n-1}\rangle$. We do this by adapting the arguments of [BDPP13, §4]. As above, we may choose a sequence of approximate Zariski decompositions $\mu_k^*\alpha = \omega_k + E_k$ where $\mu_k : X_k \to X$ is a projective modification, ω_k is Kähler, E_k is (the class of) an effective $\mathbb{Q}$-divisor, in such a way that $\langle\alpha^n\rangle = \lim(\omega_k^n)$ and $\langle\alpha^{n-1}\rangle = \lim(\mu_k)_*\left(\omega_k^{n-1}\right)$. Property (i) is then equivalent to the asymptotic orthogonality property $\left(\omega_k^{n-1}\cdot E_k\right)\to 0$ (hence the chosen terminology!).

Let H be an ample divisor class on X such that $H-\alpha\in H^{1,1}_{\mathbb{R}}(X)$ is nef. As observed in [BDPP13, §10], the class

$$\mu^*H - E_k = \mu^*(H-\alpha)+\omega_k$$

is nef and rational. For each $t\in[0,1]$, we have

$$\omega_k + tE_k = (\omega_k + t\mu^*H) - t(\mu^*H - E_k)$$

with $\omega_k + t\mu^*H \in H^{1,1}_{\mathbb{R}}(X_k)$ nef, and Theorem A.1 therefore yields

$$\mathrm{vol}(\alpha) \geq \mathrm{vol}(\omega_k + tE_k) \geq (\omega_k + t\mu^*H)^n - nt(\mu^*H - E_k)\cdot(\omega_k + t\mu^*H)^{n-1}$$

$$= (\omega_k)^n + nt(\omega_k^{n-1}\cdot\mu^*H) + \sum_{j=2}^{n} t^j \binom{n}{j} \omega_k^{n-j}\cdot(\mu^*H)^j$$

$$-nt(\mu^*H - E_k)\cdot\omega_k^{n-1} - nt\left(\mu^*(H-\alpha)+\omega_k\right)\cdot\sum_{j=1}^{n-1} t^j\binom{n-1}{j}\omega_k^{n-1-j}\cdot(\mu^*H)^j.$$

Since $H \geq \alpha$ is nef, we have $\mu_k^*H \geq \mu_k^*\alpha \geq \omega_k$, and the monotonicity property of intersection numbers of nef classes implies that

$$0 \leq \omega_k^j\cdot(\mu^*H)^{n-j} \leq (H^n)$$

and

$$0 \leq \mu^*(H-\alpha)\cdot\omega_k^j\cdot(\mu^*H)^{n-1-j} \leq (H^n).$$

We thus get the existence of a uniform constant $C > 0$ such that

$$\mathrm{vol}(\alpha) - \omega_k^n \geq nt(\omega_k^{n-1}\cdot E_k) - Ct^2$$

for all $t \in [0,1]$. Choosing $t = n\frac{(\omega_k^{n-1}\cdot E_k)}{2C}$ gives an estimate

$$(\omega_k^{n-1}\cdot E_k)^2 \leq C'\left(\mathrm{vol}(\alpha) - (\omega_k^n)\right),$$

proving as desired that $(\omega_k^{n-1}\cdot E_k) \to 0$.

□

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