THE VOLUME OF AN ISOLATED SINGULARITY

SEBASTIEN BOUCKSOM, TOMMASO DE FERNEX, and CHARLES FAVRE

Abstract

We introduce a notion of volume of a normal isolated singularity that generalizes Wahl's characteristic number of surface singularities to arbitrary dimensions. We prove a basic monotonicity property of this volume under finite morphisms. We draw several consequences regarding the existence of noninvertible finite endomorphisms fixing an isolated singularity. Using a cone construction, we deduce that the anti-canonical divisor of any smooth projective variety carrying a noninvertible polarized endomorphism is pseudoeffective.

Our techniques build on Shokurov's b-divisors. We define the notions of nef Weil b-divisors and of nef envelopes of b-divisors. We relate the latter to the pullback of Weil divisors introduced by de Fernex and Hacon. Using the subadditivity theorem for multiplier ideals with respect to pairs recently obtained by Takagi, we carry over to the isolated singularity case the intersection theory of nef Weil b-divisors formerly developed by Boucksom, Favre, and Jonsson in the smooth case.

Contents

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0. Introduction

Wahl's [Wa] characterisic number is a topological invariant of the link of a normal surface singularity. Its simple behavior under finite morphisms enables one to characterize surface singularities that carry finite noninvertible endomorphisms. Our main goal is to generalize Wahl's invariant to higher-dimensional isolated normal singularities and to present a few applications to the description of singularities admitting nontrivial finite endomorphisms. Our main result can be stated as follows.

THEOREM A
To any normal isolated singularity $(X,0)$ *there is associated a nonnegative real number* $\mathrm{Vol}(X,0)$ *that we call its volume, satisfying the following properties.*

(i) *For every finite morphism* $\phi\colon (X,0) \to (Y,0)$ *of degree* $e(\phi)$ *we have*

$$\mathrm{Vol}(X,0) \geq e(\phi)\,\mathrm{Vol}(Y,0),$$

and equality holds when ϕ *is étale in codimension* 1.

(ii) *If* $\dim X = 2$, *then* $\mathrm{Vol}(X,0)$ *coincides with Wahl's characteristic number.*

(iii) *If* X *is* $\mathbb{Q}$*-Gorenstein, then* $\mathrm{Vol}(X,0) = 0$ *if and only if* X *has log-canonical (i.e., lc) singularities.*

Our result generalizes in particular the well-known fact that $\mathbb{Q}$-Gorenstein lc singularities are preserved under finite morphisms (see for instance [Kol1, Proposition 3.16]).

Just as in dimension 2, one infers restrictions on isolated singularities admitting finite endomorphisms.

THEOREM B
Suppose that $\phi\colon (X,0) \to (X,0)$ *is a finite noninvertible endomorphism of an isolated singularity. Then* $\mathrm{Vol}(X,0) = 0$.

If X *is* $\mathbb{Q}$*-Gorenstein, then* X *has lc singularities, and it furthermore has Kawamata log-terminal (klt) singularities if* ϕ *is not étale in codimension* 1.

To obtain a more precise classification of singularities carrying finite endomorphisms one would need to get deeper into the structure of singularities with $\mathrm{Vol}(X,0) = 0$. This can be done in dimension 2 (see [Wa], [Fav]), but unfortunately, this task seems very difficult at the moment in arbitrary dimension. To illustrate the previous result, however, we construct several classes of (not necessarily $\mathbb{Q}$-Gorenstein) isolated normal singularities carrying finite endomorphisms (see Sections 6.2–6.3 below). Our examples include quotient singularities, Tsuchihashi's cusp singularities (see [Oda], [Tsu]), toric singularities, and certain simple singularities obtained from cone or deformation constructions.

In dimension 2, the conclusion of Theorem B plays a key role in the classification of projective surfaces admitting noninvertible endomorphisms, which is by now essentially complete (see [FN], [Nak]). In higher dimensions, classifying projective varieties carrying a noninvertible endomorphism has recently attracted quite a lot of attention (see [dqZ] and the references therein), but the general problem remains largely open.

The assumption on the singularity being isolated in Theorem B is too strong to be directly useful in this perspective. Nevertheless we observe that Theorem B has some consequences in the more rigid case of so-called polarized endomorphisms. Recall that an endomorphism $\phi\colon V \to V$ of a projective variety is said to be polarized if there exists an ample line bundle L on V such that $\phi^* L = dL$ in $\operatorname{Pic}(V)$ for some $d \geq 1$ (see [swZ] for a nice survey). By looking at the affine cone over X induced by a large enough multiple of L, we obtain the following.

THEOREM C

If V is a smooth projective variety carrying a noninvertible polarized endomorphism ϕ, then $-K_V$ is pseudoeffective.

Observe that the ramification formula implies $K_V \cdot L^{n-1} \leq 0$. If K_V is pseudoeffective, then $K_V \equiv 0$ and (V, ϕ) is then an endomorphism of an abelian variety up to finite étale cover (see [Fak, Theorem 4.2]). If K_V is not pseudoeffective, then V is uniruled by [BDPP], and our result puts further constraints on the geometry of V.

Throughout the paper, we insist on working with arbitrary non-$\mathbb{Q}$-Gorenstein singularities. This degree of generality is crucial to obtain Theorem C since the cone over V is $\mathbb{Q}$-Gorenstein if and only if $\pm K_V$ is either $\mathbb{Q}$-linearly trivial or ample (see Example 2.31 below).

To understand our construction and the difficulties that one has to overcome to define the volume above, let us recall briefly Wahl's definition for a normal surface singularity $(X, 0)$.

Pick any *log resolution* $\pi : Y \to X$ of $(X, 0)$, that is, a birational morphism which is an isomorphism above $X \setminus \{0\}$, and such that Y is smooth and the scheme-theoretic inverse image $\pi^{-1}(0)$ is a divisor with simple normal crossing support E. Let K_X be a canonical divisor on X, and let K_Y be the induced canonical divisor on Y. Denote by $\pi^* K_X$ Mumford's *numerical pullback* of K_X to Y, which is uniquely determined as a $\mathbb{Q}$-divisor by the conditions $\pi_*(\pi^* K_X) = K_X$ and $\pi^* K_X \cdot C = 0$ for any π-exceptional curve C. The *log-discrepancy* divisor is then defined by the relation $A_{Y/X} := K_Y + E - \pi^* K_X$. Recall that X is (numerically) lc if and only if $A_{Y/X} \geq 0$, while X is (numerically) klt if and only if $A_{Y/X} > 0$ on the whole of E.

Wahl's invariant measures the degree of positivity of the log-discrepancy divisor. The positivity is here relative to the contraction morphism $Y \to X$, and it is thus natural to consider the *relative Zariski decomposition* $A_{Y/X} = P + N$ in the sense of [Sak, p. 408], where N is the smallest effective π-exceptional $\mathbb{Q}$-divisor such that $P = A_{Y/X} - N$ is π-nef. Finally one sets

$$\mathrm{Vol}(X,0) := -P^2 \in \mathbb{Q}_{\geq 0}. \tag{1}$$

Two (related) difficulties arise in generalizing Wahl's construction to higher dimensions: first, one needs to introduce a notion of pullback for Weil divisors; and second, one needs to find a replacement for the relative Zariski decomposition. These problems have already been addressed in [dFH] and in [BFJ1] and [KuMa], respectively. Building on these works our first objective is to explain how these difficulties can be conveniently addressed using Shokurov's language of *b-divisors*. In Sections 1–3, we define and study the notion of a nef Weil b-divisor in the general setting of a normal variety X. This leads to the notion of nef envelopes and relative Zariski decomposition as follows.

Let us recall some terminology. A *Weil b-divisor* W over X is the data of Weil divisors W_π on all birational models $\pi\colon X_\pi \to X$ of X that are compatible under pushforward. A *Cartier b-divisor* C is a Weil b-divisor for which there is a model π such that for every other model π' dominating π the trace $C_{\pi'}$ of C on $X_{\pi'}$ is the pullback of the trace C_π on X_π; any π as above is called a determination of C. All the divisors we consider for the time being have $\mathbb{R}$-coefficients.

Now, suppose we are given a projective morphism $f\colon X \to S$. A Cartier b-divisor C is said to be *nef* (relatively to f) if C_π is nef for one (hence any) determination π of C. Generalizing [BFJ1] and [KuMa], we say that a Weil b-divisor W is nef if and only if there exists a net of nef Cartier b-divisors C_n such that the net $[(C_n)_\pi]$ converges to $[W_\pi]$ in the space $N^1(X_\pi/S)$ of numerical classes over S. This is equivalent to saying that W_π lies in the closed movable cone $\overline{\mathrm{Mov}}(X_\pi/S)$ for all smooth models X_π (cf. Lemma 2.10 below).

In Section 2, we prove that the following definitions make sense (under suitable conditions) and introduce the following two notions of *nef envelopes*.

- The nef envelope $\mathrm{Env}_X(D)$ of a Weil divisor D on X is the largest nef Weil b-divisor Z that is both relatively nef over X and satisfies $Z_X \leq D$.
- The nef envelope $\mathrm{Env}_{\mathfrak{X}}(W)$ of a Weil b-divisor W is the largest nef Weil b-divisor Z that is both relatively nef over X and satisfies $Z \leq W$.

In dimension 2, nef envelopes recover the notions of numerical pullback and relative Zariski decomposition. Specifically, if D is a divisor on a normal surface X, then the trace $\mathrm{Env}_X(D)_\pi$ on a given model X_π coincides with the numerical pullback of D by π, while if D is a divisor on a smooth model X_π over X, then the nef part of D

in its relative Zariski decomposition is given by $\mathrm{Env}_{\mathfrak{X}}(\overline{D})_\pi$, where $\overline{D}$ is the Cartier b-divisor induced by D.

In higher dimensions, $D \mapsto \mathrm{Env}_X(D)$ is nonlinear in general, and $\mathrm{Env}_X(D)_\pi$ coincides up to sign with the pullback $\pi^* D$ defined in [dFH]. However, it is this approach via b-divisors and nef envelopes that brings to light the crucial positivity properties of the pullback of Weil divisors.

We are now in a position to generalize the log-discrepancy divisor and its relative Zariski decomposition. Given a canonical divisor K_X on X, there is a unique canonical divisor K_{X_π}, for each model $\pi\colon X_\pi \to X$, with the property that $\pi_* K_{X_\pi} = K_X$. Thus a choice of K_X determines a *canonical b-divisor* $K_{\mathfrak{X}}$ over X. The *log-discrepancy b-divisor* is then defined as

$$A_{\mathfrak{X}/X} := K_{\mathfrak{X}} + 1_{\mathfrak{X}/X} + \mathrm{Env}_X(-K_X),$$

where the trace of $1_{\mathfrak{X}/X}$ in any model is equal to the reduced exceptional divisor over X. The log-discrepancy b-divisor is exceptional over X and does not depend on the choice of K_X. Its coefficients are given by the (usual) log discrepancies of X when the latter is $\mathbb{Q}$-Gorenstein. The role of the nef part of $A_{\mathfrak{X}/X}$ in its relative Zariski decomposition is in turn played by the nef envelope

$$P := \mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}).$$

To generalize (1), we now face the problem of defining the intersection product of nef b-divisors. This step is nontrivial. The intersection of Cartier b-divisors is defined as their intersection in a common determination. However, it cannot be extended to a multilinear intersection product on the space of *Weil* b-divisors having reasonable continuity properties. As it turns out, it is nevertheless possible to extend it to a multilinear intersection pairing on *nef* Weil b-divisors lying over a point $0 \in X$. This is done following the approach of [BFJ1], in which multiplier ideals appear as a prominent tool.

Assume from now on that $(X, 0)$ is an n-dimensional *isolated* normal singularity. For all (relatively) nef b-divisors $W_1, \dots, W_n$ above zero, we set

$$W_1 \cdot \cdots \cdot W_n := \inf\{C_1 \cdot \cdots \cdot C_n \mid C_j \text{ nef Cartier}, C_j \geq W_j\} \in [-\infty, 0].$$

To develop a reasonable calculus of these intersection numbers, *additivity* in each variable is a desirable property. We obtain this result as a consequence of the fact that any nef envelope of a Cartier b-divisor is the *decreasing* limit of a sequence of nef Cartier b-divisors C_k.

Let us explain how to get this crucial approximation property. The first observation is that the nef envelope of a Cartier b-divisor C is a limit of the graded sequence of ideals $\mathfrak{a}_m := \mathcal{O}_X(mC)$, $m \geq 0$ (see Section 2.1). For any fixed $c > 0$, we use the general notion of (asymptotic) multiplier ideal $\mathcal{J}(X; \mathfrak{a}_\bullet^c)$ introduced in [dFH] for any ambient variety X with normal singularities. As was shown in [dFH] this multiplier ideal can also be computed using *compatible boundaries*: namely, there exist effective $\mathbb{Q}$-boundaries Δ such that $\mathcal{J}(X; \mathfrak{a}_\bullet^c)$ coincides with the standard (asymptotic) multiplier ideal $\mathcal{J}((X, \Delta); \mathfrak{a}_\bullet^c)$ with respect to the pair (X, Δ).

This connection enables us to make use of a recent result of Takagi [Tak2], which extends the usual subadditivity property of multiplier ideals (see [DEL]) to multiplier ideals with respect to a pair (X, Δ), up to an (inevitable) error term involving Δ and the Jacobian ideal of X. The approximation we are looking for then follows by taking the nef Cartier b-divisor C_k associated to $\mathcal{J}(X; \mathfrak{a}_\bullet^k)$.

Now that we have defined the intersection product of nef Weil b-divisors, we can come back to the definition of the volume. We set

$$\mathrm{Vol}(X, 0) := -\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})^n,$$

which is shown to be finite (and nonnegative). Once the volume is defined, the properties stated in Theorem A follow smoothly from transformation laws of envelopes under finite morphisms (see Proposition 2.19).

The volume as defined above relates to other kinds of invariants that were previously defined and are connected to growth rate of pluricanonical forms.

In the 2-dimensional case, we first note that the definition (1) admits an equivalent formulation in terms of the growth rate of a certain quotient of sections. It was indeed shown in [Wa] that if X is a surface, then

$$\dim\bigl(H^0(X \setminus \{0\}, mK_X)/H^0(Y, m(K_Y + E))\bigr) = \frac{m^2}{2}\,\mathrm{Vol}(X, 0) + o(m^2),$$

where the left-hand side is independent of the choice of Y and is equal by definition to the mth *log-plurigenus* $\lambda_m(X, 0)$ in the sense of Morales [Mor], a notion which makes sense in all dimensions.

In line with this point of view M. Fulger [Fulg] has recently considered the following invariant of an isolated singularity $(X, 0)$:

$$\mathrm{Vol}_F(X, 0) := \limsup_m \frac{n!}{m^n} \dim\bigl(H^0(X \setminus \{0\}, mK_X)/H^0(Y, m(K_Y + E))\bigr).$$

It measures by definition the growth rate of $\lambda_m(X, 0)$ or, equivalently, that of Watanabe's L^2*-plurigenera* $\delta_m(X, 0)$ (see [Wat1], [Wat2]), and yields a finite number since

$$\delta_m(X, 0) = \lambda_m(X, 0) + O(m^{n-1}) = O(m^n)$$

(see [Ish], which contains a thorough introduction to these notions, and Section 5.2 below).

The notion of volume considered by Fulger also behaves well under finite morphisms, and the analogue of Theorem A holds true. Moreover, in contrast to our volume, $\mathrm{Vol}_F(X,0)$ is more accessible to explicit computations. On the other hand, our volume $\mathrm{Vol}(X,0)$ relates more closely to lc singularities (see question (b) below).

Fulger [Fulg] explores how the two approaches compare to one another, proving that $\mathrm{Vol}(X,0) \geq \mathrm{Vol}_F(X,0)$ for any isolated normal singularity $(X,0)$. Equality holds when X is $\mathbb{Q}$-Gorenstein, but can fail otherwise (cf. Proposition 5.3 and Example 5.4).

In general these volumes can take irrational values. Urbinati [Urb] constructs examples where the log discrepancy takes irrational values, and Fulger [Fulg] shows that similar examples have irrational volumes $\mathrm{Vol}(X,0)$ and $\mathrm{Vol}_F(X,0)$.

In the 2-dimensional case, we know by the work of Wahl [Wa] that the volume is a topological invariant of the link of the singularity and that its vanishing characterizes log-canonical singularities. Furthermore, Ganter [Gan] has shown that there is a uniform lower bound to the volume of a normal Gorenstein surface singularity with positive volume. An example brought to our attention by Kollár shows that the first property fails in higher dimensions: in general the volume of a normal isolated singularity is not a topological invariant of the singularity (cf. Example 4.23). The following questions remain open.

(a) Does there exist a *positive* lower bound, depending only on the dimension, for the volume of isolated Gorenstein singularities with positive volume?

(b) Is it true that $\mathrm{Vol}(X,0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X,Δ) is log-canonical? (the converse being easily shown).

It is to be noted that (b) fails with $\mathrm{Vol}_F(X,0)$ in place of $\mathrm{Vol}(X,0)$ (cf. Example 5.4).

The plan of our paper is the following. In the first four sections, we work over a normal algebraic variety. Section 1 contains basics on b-divisors. The notion of envelopes is analyzed in detail in Section 2. In this section we also formalize a measure of the failure of a Weil divisor to be Cartier in terms of certain *defect ideals*, which are related to the notion of compatible boundary. In Section 3 we turn to the definition of the log-discrepancy b-divisor and of multiplier ideals. The key result of this section is the subadditivity theorem (Theorem 3.17) that we deduce from Takagi's work.

The rest of the paper deals with normal isolated singularities. We define the volume of such a singularity and prove Theorem A(i), (iii) in Section 4. In Section 5 we complete the proof of Theorem A and compare our notion with the approaches via plurigenera and Fulger's work. Finally Section 6 focuses on endomorphisms and contains a proof of Theorems B and C.

1. Shokurov's b-divisors

In this section, X denotes a normal variety defined over an algebraically closed field of characteristic zero, and we set $n := \dim X$. The goal of this section is to gather general properties of Shokurov's b-divisors over X, for which [Isk] and [Cor] constitute general references. Proposition 1.14 seems to be new.

1.1. The Riemann–Zariski space

The set of all proper birational morphisms $\pi : X_\pi \to X$ modulo isomorphism is (partially) ordered by $\pi' \geq \pi$ if and only if π' factors through π, and the order is inductive (i.e., any two proper birational morphisms to X can be dominated by a third one). For short, we will refer to X_π, or π, as a *model* over X. The *Riemann–Zariski space* of X is defined as the projective limit

$$\mathfrak{X} = \varprojlim_\pi X_\pi,$$

taken in the category of locally ringed topological spaces, each X_π being viewed as a scheme with its Zariski topology. (Note that $\mathfrak{X}$ itself is not a scheme anymore.)

As a topological space, $\mathfrak{X}$ may alternatively be viewed as the set of all valuation subrings $V \subset k(X)$ with nonempty center on X, endowed with the Krull–Zariski topology. Indeed, given a Krull valuation V, the center $c_\pi(V)$ of V on X_π is nonempty for each π by the valuative criterion for properness, and the collection of all scheme-theoretic points $c_\pi(V)$ defines a point in $c(V)$ in $\mathfrak{X}$. By [ZS, Theorem 41, p. 122] the mapping $V \mapsto c(V)$ so defined is a homeomorphism.

1.2. Divisors on the Riemann–Zariski space

Following Shokurov we define the group of *Weil b-divisors* over X (where b stands for birational) as

$$\mathrm{Div}(\mathfrak{X}) := \varprojlim_\pi \mathrm{Div}(X_\pi),$$

where $\mathrm{Div}(X_\pi)$ denotes the group of Weil divisors of X_π and the limit is taken with respect to the pushforward maps $\mathrm{Div}(X_{\pi'}) \to \mathrm{Div}(X_\pi)$, which are defined whenever $\pi' \geq \pi$. It can alternatively be thought of as the group of Weil divisors on the Riemann–Zariski space $\mathfrak{X}$ (hence the notation).

The group of *Cartier b-divisors* over X is in turn defined as

$$\mathrm{CDiv}(\mathfrak{X}) := \varinjlim_\pi \mathrm{CDiv}(X_\pi)$$

with $\mathrm{CDiv}(X_\pi)$ denoting the group of Cartier divisors of X_π. Here the limit is taken with respect to the pullback maps $\mathrm{CDiv}(X_\pi) \to \mathrm{CDiv}(X_{\pi'})$, which are defined whenever $\pi' \geq \pi$. One can easily check that

$$\mathrm{CDiv}(\mathfrak{X}) = H^0(\mathfrak{X}, \mathcal{M}^*_{\mathfrak{X}}/\mathcal{O}^*_{\mathfrak{X}})$$

is indeed the group of Cartier divisors of the locally ringed space $\mathfrak{X}$.

There is an injection $\mathrm{CDiv}(\mathfrak{X}) \hookrightarrow \mathrm{Div}(\mathfrak{X})$ determined by the cycle maps on birational models X_π.

An element of $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}) := \mathrm{Div}(\mathfrak{X}) \otimes \mathbb{R}$ (resp., $\mathrm{CDiv}_{\mathbb{R}}(\mathfrak{X}) := \mathrm{CDiv}(\mathfrak{X}) \otimes \mathbb{R}$) will be called an $\mathbb{R}$-Weil b-divisor (resp., $\mathbb{R}$-Cartier b-divisor), and similarly with $\mathbb{Q}$ in place of $\mathbb{R}$. The space $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$ is naturally isomorphic to the projective limit of the spaces $\mathrm{Div}_{\mathbb{R}}(X_\pi)$, and $\mathrm{CDiv}_{\mathbb{R}}(\mathfrak{X})$ is naturally isomorphic to the direct limit of the spaces $\mathrm{CDiv}_{\mathbb{R}}(X_\pi)$.

Let us now interpret these definitions in more concrete terms. A Weil divisor W on $\mathfrak{X}$ consists of a family of Weil divisors $W_\pi \in \mathrm{Div}(X_\pi)$ that are compatible under pushforward, that is, such that $W_\pi = \mu_* W_{\pi'}$ whenever π' factors through a morphism $\mu\colon X_{\pi'} \to X_\pi$. We say that W_π (also denoted by W_{X_π}) is the *trace* (or *incarnation* as in [BFJ1]) of W on the model X_π. By contrast, a Cartier divisor C on $\mathfrak{X}$ is determined by its trace on a high enough model; that is, there exists π such that $C_{\pi'} = \mu^* C_\pi$ for every $\pi' \geq \pi$, where $\mu\colon X_{\pi'} \to X_\pi$ is the induced morphism. We say that C is *determined on* X_π (or *by* π).

Weil b-divisors can also be interpreted as certain functions on the set of *divisorial valuations* of X. Recall first that a divisorial valuation of X is a rank 1 valuation of transcendence degree $\dim X - 1$ of the function field $k(X)$, whose center on X is nonempty. By a classical result of Zariski (see, e.g., [KoMo, Lemma 2.45]) the divisorial valuations on X are exactly those of the form $\nu = t \operatorname{ord}_E$, where $t \in \mathbb{R}^*_+$ and E is a prime divisor on some birational model X_π over X.

Given an $\mathbb{R}$-Weil b-divisor W over X we can then define $(t \operatorname{ord}_E)(W)$ as t times the coefficient of E in W_π.

LEMMA 1.1

Setting $g_W(\nu) := \nu(W)$ yields an identification $W \mapsto g_W$ between $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$ *and the space of all real-valued* 1*-homogeneous functions g on the set of divisorial valuations of X satisfying the following finiteness property: the set of prime divisors $E \subset X$ (or equivalently on X_π for any given π) such that $g(\operatorname{ord}_E) \neq 0$ is finite.*

The topology of pointwise convergence therefore induces a *topology of coefficient-wise* convergence on $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$, for which $\lim_j W_j = W$ if and only if $\lim_j \operatorname{ord}_E(W_j) = \operatorname{ord}_E(W)$ for each prime divisor E over X.

1.3. Examples of b-divisors

We introduce the main types of b-divisors that we shall consider.

Example 1.2
The choice of a nonzero rational form ω of top degree on X induces a *canonical b-divisor* $K_{\mathfrak{X}}$ whose trace on X_π is equal to the canonical divisor determined by ω on X_π.

Example 1.3
A Cartier divisor D on a given model X_π induces a Cartier b-divisor $\overline{D}$, its *pullback* to $\mathfrak{X}$. It is simply defined by pulling back D to all models dominating X_π and then by pushing forward on all other models. By definition all Cartier b-divisors are actually obtained this way.

Example 1.4
Given a coherent fractional ideal sheaf $\mathfrak{a}$ on X we denote by $Z(\mathfrak{a})$ the Cartier b-divisor determined on the normalized blowup X_π of X along $\mathfrak{a}$ by

$$\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}\big(Z(\mathfrak{a})_\pi\big).$$

In particular we have $Z(f)_\pi = -\pi^* \operatorname{div}(f)$ when f is a rational function on X. Note that with this convention $Z(\mathfrak{a})$ is antieffective when $\mathfrak{a}$ is an actual ideal sheaf.

For any Weil b-divisor we write $Z \geq 0$ if Z_π is an effective divisor for every π. We record the following easy properties.

LEMMA 1.5
Let $\mathfrak{a}, \mathfrak{b}$ be two coherent fractional ideal sheaves on X:
- $Z(\mathfrak{a}) \leq Z(\mathfrak{b})$ *whenever* $\mathfrak{a} \subset \mathfrak{b}$*;*
- $Z(\mathfrak{a} \cdot \mathfrak{b}) = Z(\mathfrak{a}) + Z(\mathfrak{b})$*;*
- $Z(\mathfrak{a} + \mathfrak{b}) = \max\{Z(\mathfrak{a}), Z(\mathfrak{b})\}$*, where the maximum is defined coefficient-wise;*
- $Z(\mathfrak{a}) = Z(\mathfrak{b})$ *if and only if the integral closures of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are equal.*

Remark 1.6
Given an ideal sheaf $\mathfrak{a}$ and a positive number $s > 0$ we set $Z(\mathfrak{a}^s) := sZ(\mathfrak{a})$. Then, by definition, we have $Z(\mathfrak{a}^s) = Z(\mathfrak{b}^t)$ if and only if the $\mathbb{R}$-*ideals* $\mathfrak{a}^s$ and $\mathfrak{b}^t$ are *valuatively* equivalent in the sense of Kawakita [Kaw].

Definition 1.7
Let W be an $\mathbb{R}$-Weil b-divisor over X. We denote by $\mathcal{O}_X(W)$ the fractional ideal sheaf of X whose sections on an open set $U \subset X$ are the rational functions f such that $Z(f) \leq W$ over U.

We emphasize that the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(W)$ is *not* coherent in general, since we are imposing infinitely many (even uncountably many) conditions on f (see [Isk]). Note that $\pi_*\mathcal{O}_{X_\pi}(W_\pi) \subset \tau_*\mathcal{O}_{X_\tau}(W_\tau)$ whenever $\pi \geq \tau$ and

$$\mathcal{O}_X(W) = \bigcap_\pi \pi_*\mathcal{O}_{X_\pi}(W_\pi).$$

However, if C is an $\mathbb{R}$-Cartier b-divisor, then we have $\mathcal{O}_X(C) = \pi_*\mathcal{O}_{X_\pi}(C_\pi)$ for each determination π of C, and in particular $\mathcal{O}_X(C)$ is coherent in that case.

Cartier b-divisors associated with coherent fractional ideal sheaves can be characterized as follows.

LEMMA 1.8
A Cartier b-divisor $C \in \mathrm{CDiv}(\mathfrak{X})$ is of the form $Z(\mathfrak{a})$ for some coherent fractional ideal sheaf $\mathfrak{a}$ on X if and only if C is relatively globally generated over X.

In particular the Cartier divisors $Z(\mathfrak{a})$ with $\mathfrak{a}$ ranging over all coherent (fractional) ideal sheaves of X generate $\mathrm{CDiv}(\mathfrak{X})$ as a group.

Here we say that C is relatively globally generated over X if and only if so is C_π for one (hence any) determination π of C.

Proof
Let C be a Cartier b-divisor determined by π. To say that C is relatively globally generated over X means by definition that the evaluation map

$$\pi^*\pi_*\mathcal{O}_{X_\pi}(C_\pi) \to \mathcal{O}_{X_\pi}(C_\pi)$$

is surjective. If this is the case we thus see that $C = Z(\mathfrak{a})$ with $\mathfrak{a} := \pi_*\mathcal{O}_{X_\pi}(C_\pi) = \mathcal{O}_X(C)$, while the converse direction is equally clear. The second assertion now follows from the fact that any Cartier divisor on a given model X_π can be written as a difference of two π-very ample (hence π-globally generated) Cartier divisors. $\square$

1.4. Numerical classes of b-divisors
Let $X \to S$ be a projective morphism. Recall that the space of codimension 1 relative numerical classes $N^1(X/S)$ is the vector space of $\mathbb{R}$-Cartier divisors modulo those divisors D for which $D \cdot C = 0$ for every irreducible curve C that is mapped to a point in S. One can put together these spaces and define the space of 1-*codimensional numerical classes* of $\mathfrak{X}$ over S by

$$N^1(\mathfrak{X}/S) := \varinjlim_\pi N^1(X_\pi/S),$$

where the maps are given by pulling back. We define in turn the space of $(n-1)$-*dimensional numerical classes* of $\mathfrak{X}$ over S by

$$N_{n-1}(\mathfrak{X}/S) := \varprojlim_\pi N^1(X_\pi/S),$$

where the maps are given by pushing forward and π now runs over all *smooth* (or at least $\mathbb{Q}$-factorial) birational models of X—so that the pushforward map $N^1(X_{\pi'}/S) \to N^1(X_\pi/S)$ is well defined for $\pi' \geq \pi$.

Each $N^1(X_\pi/S)$ is a finite-dimensional $\mathbb{R}$-vector space, and we endow $N^1(\mathfrak{X}/S)$ and $N_{n-1}(\mathfrak{X}/S)$ with their natural inductive and projective limit topologies, respectively.

LEMMA 1.9
The cycle maps induce a natural continuous injection $N^1(\mathfrak{X}/S) \to N_{n-1}(\mathfrak{X}/S)$ with dense image.

Proof
Just as in the case of Cartier and Weil b-divisors described in Section 1.2, any class β in $N^1(X_\pi/S)$ can be identified to the class in $N_{n-1}(\mathfrak{X}/S)$ determined by pulling back β on all higher models. We thus have natural continuous maps $N^1(X_\pi/S) \to N_{n-1}(\mathfrak{X}/S)$ which induce a continuous injective map $N^1(\mathfrak{X}/S) \to N_{n-1}(\mathfrak{X}/S)$. It follows by the definition of the projective limit topology that this map has dense image, since for any class $\alpha \in N_{n-1}(\mathfrak{X}/S)$ the net determined by its traces $\alpha_\pi \in N^1(X_\pi/S)$, viewed as elements of $N_{n-1}(\mathfrak{X}/S)$ as described before, converges to α. □

There are also natural surjections $\mathrm{CDiv}_\mathbb{R}(\mathfrak{X}) \to N^1(\mathfrak{X}/S)$ and $\mathrm{Div}_\mathbb{R}(\mathfrak{X}) \to N_{n-1}(\mathfrak{X}/S)$, but one should be careful that the latter map is *not* continuous with respect to coefficient-wise convergence in general.

Example 1.10
Consider an infinite sequence C_j of (-1)-curves on $X = \mathbb{P}^2$ blown up at 9 points. We then have $C_j \to 0$ in terms of coefficients, but the numerical classes $[C_j] \in N^1(X)$ do not tend to zero since $C_j^2 = -1$ for each j.

LEMMA 1.11
Let $\pi : X_\pi \to X$ be a birational model of X, and let $\alpha \in N^1(X_\pi/X)$. Then there exists at most one π-exceptional $\mathbb{R}$-Cartier divisor D on X_π whose numerical class is equal to α.

Proof
Let D be a π-exceptional and π-numerically trivial $\mathbb{R}$-Cartier divisor. We are to show that $D = 0$. Upon pulling back D to a higher birational model, we may assume

that π is the normalized blowup of X along a subscheme of codimension at least two. If we denote by E_j the π-exceptional divisors, we then have on the one hand $D = \sum_j d_j E_j$, and on the other hand there exist positive integers a_j such that $F := \sum_j a_j E_j$ is π-antiample. Now set $t := \max_j d_j/a_j$. If we assume by contradiction that $D \neq 0$, then upon possibly replacing D by $-D$ we may assume that $t > 0$. Now $tF - D$ is effective and there exists j such that E_j is not contained in its support. If $C \subset E_j$ is a general curve in a fiber of π, we then have $(tF - D) \cdot C \geq 0$ since C is not contained in the support of the effective divisor $tF - D$, which contradicts the fact that $D - tF$ is π-ample. □

Even assuming that X_π is smooth, it is not true in general that any class $\alpha \in N^1(X_\pi/X)$ can be represented by a π-exceptional $\mathbb{R}$-divisor (since π might for instance be small, i.e., without any π-exceptional divisor). It is, however, true when X is $\mathbb{Q}$-factorial, and for any normal X when $\dim X = 2$ thanks to Mumford's numerical pullback.

Using these remarks we may now prove the following simple lemma, which enables us to circumvent the discontinuity of the quotient map $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}) \to N_{n-1}(\mathfrak{X}/S)$.

LEMMA 1.12

(a) *Let W_j be a sequence (or net) of $\mathbb{R}$-Weil b-divisors which converges to an $\mathbb{R}$-Weil b-divisor W coefficient-wise. If there exists a fixed finite-dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $W_{j,X} \in V$ for all j, then $[W_j] \to [W]$ in $N_{n-1}(\mathfrak{X}/S)$.*

(b) *Conversely, let $\alpha_j \to \alpha$ be a convergent sequence (or net) in $N_{n-1}(\mathfrak{X}/S)$. Then there exist representatives $W_j, W \in \mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$ of α_j and α, respectively, and a finite-dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that*
   - *$W_j \to W$ coefficient-wise;*
   - *$W_{j,X} \in V$ for all j.*

   If $\alpha_j \in N^1(\mathfrak{X}/S)$, then W_j can be chosen to be $\mathbb{R}$-Cartier.

Proof

For each smooth model π the existence of V yields a finite-dimensional space V_π of $\mathbb{R}$-divisors on X_π such that $W_{j,\pi} \in V_\pi$ for all j. The natural linear map $V_\pi \to N^1(X_\pi/S)$ is of course continuous since both spaces are finite-dimensional, and it follows that $[W_{j,\pi}] \to [W_\pi]$ in $N^1(X_\pi/S)$ for each smooth model. Since smooth models are cofinal in the family of all models we conclude as desired that $[W_j] \to [W]$ in $N_{n-1}(\mathfrak{X}/S)$.

We now consider the converse. Let X_π be a fixed smooth model of X. For each j, $\alpha_j - \overline{\alpha}_{j,\pi}$ (resp., $\alpha - \overline{\alpha}_\pi$) is exceptional over X_π. By the above remarks it is thus *uniquely* represented by an $\mathbb{R}$-Weil b-divisor Z_j (resp., Z) that is exceptional over X_π. Since $(\alpha_j - \overline{\alpha}_{j,\pi})_{\pi'}$ converges to $(\alpha - \overline{\alpha}_\pi)_{\pi'}$ in $N^1(X_{\pi'}/X_\pi)$ for each $\pi' \geq \pi$ it follows by uniqueness of Z_j that $Z_j \to Z$ coefficient-wise.

On the other hand, since $N^1(X_\pi/S)$ is finite-dimensional there exists a finite-dimensional $\mathbb{R}$-vector space V of $\mathbb{R}$-divisors on X_π such that $V \to N^1(X_\pi/X)$ is surjective. This map is therefore open, and we may thus find representatives $C_j \in V$ of $\alpha_{j,\pi}$ converging to a representative $C \in V$ of α_π. Setting $W_j := Z_j + \overline{C}_j$ concludes the proof. ☐

1.5. Functoriality

If $\phi\colon X \to Y$ is any morphism between two normal varieties, then it is immediate to see that pulling back induces a homomorphism $\phi^*\colon \operatorname{CDiv}(\mathfrak{Y}) \to \operatorname{CDiv}(\mathfrak{X})$ in a functorial way.

Assume furthermore that $\phi\colon X \to Y$ is proper, surjective, and generically finite. In this case pushing forward induces a homomorphism

$$\phi_*\colon \operatorname{Div}(\mathfrak{X}) \to \operatorname{Div}(\mathfrak{Y}),$$

and the homomorphism $\phi^*\colon \operatorname{CDiv}(\mathfrak{Y}) \to \operatorname{CDiv}(\mathfrak{X})$ extends in a natural way to a homomorphism

$$\phi^*\colon \operatorname{Div}(\mathfrak{Y}) \to \operatorname{Div}(\mathfrak{X}).$$

Before going through the constructions of these homomorphisms, we recall the following property.

LEMMA 1.13

Let $\phi\colon X \to Y$ be a proper, surjective, and generically finite morphism of normal varieties. Every divisorial valuation ν on X induces, by restriction via the field extension $\phi^*\colon \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$, a divisorial valuation $\phi_*\nu$ on Y that is defined by

$$(\phi_*\nu)(f) := \nu(f \circ \phi).$$

The correspondence $\nu \mapsto \phi_*\nu$ defines a surjective map with finite fibers from the set of divisorial valuations on X to the set of divisorial valuations on Y.

Proof

If ν is a divisorial valuation on X, then $\phi_*\nu$ is a divisorial valuation on Y since the restriction of the valuation ring of ν to $\mathbb{C}(Y)$ has transcendence degree $\dim Y - 1$ by [ZS, VI.6, Corollary 1]. The assertion is that, if ν' is a divisorial valuation on Y, then there exists a nonzero finite number of divisorial valuations $\nu_1, \dots, \nu_r$ on X that restrict to ν'. Geometrically, if $\nu' = t \operatorname{ord}_F$ where F is a prime divisor on some model

Y' over Y and $t > 0$, then the valuations ν_i are constructed by picking the model X' over X such that ϕ lifts to a well-defined morphism $\phi' \colon X' \to Y'$. If $E_1, \dots, E_r$ are the irreducible components of $(\phi')^* F$ such that $\phi'(E_i) = F$, then the associated valuations ord_{E_i} restrict to a multiple of ord_F on $\mathbb{C}(Y)$. Up to rescaling, these are the only divisorial valuations restricting to ord_F since any divisorial valuation on X with nondivisorial center in X' restricts to a divisorial valuation on Y with nondivisorial center in Y'. $\square$

We then define $\phi_* \colon \operatorname{Div}(\mathfrak{X}) \to \operatorname{Div}(\mathfrak{Y})$ and $\phi^* \colon \operatorname{Div}(\mathfrak{Y}) \to \operatorname{Div}(\mathfrak{X})$ in the following way. If $W \in \operatorname{Div}(\mathfrak{X})$, then $\phi_* W$ is characterized by the condition that

$$\operatorname{ord}_F(\phi_* W) = \sum_i \operatorname{ord}_F\big((\phi')_* E_i\big) \cdot \operatorname{ord}_{E_i}(W)$$

for any prime divisor F over Y. Here we are using the notation as in the proof of Lemma 1.13, so that F is a divisor on a model Y' over Y, X' is a model over X such that the map $\phi' \colon X' \to X$ induced by ϕ is a morphism, and the E_i are the irreducible components of $(\phi')^* F$ dominating F. It follows by the lemma that the sum is finite. Note also that on any model Y' the coefficient $\operatorname{ord}_F(\phi_* W)$ can be nonzero only for finitely many prime divisors F on a model X', so that $\phi_* W$ does define a Weil b-divisor over Y.

Regarding the pullback, if $W \in \operatorname{Div}(\mathfrak{Y})$, then $\phi^* W$ is characterized by the condition that

$$\operatorname{ord}_E(\phi^* W) = (\phi_* \operatorname{ord}_E)(W)$$

for every prime divisor E over X. This is indeed a Weil b-divisor since each prime divisor E on X such that $(\phi_* \operatorname{ord}_E)(W) \neq 0$ is either mapped to a prime divisor F on Y such that $\operatorname{ord}_F(W) \neq 0$ or is contracted by ϕ, so that the set of all such prime divisors E appearing on any model X' over X is finite by Lemma 1.13.

PROPOSITION 1.14

Let $\phi \colon X \to Y$ be a proper, surjective, generically finite morphism. Then $\phi_* \times \operatorname{CDiv}(\mathfrak{X}) \subset \operatorname{CDiv}(\mathfrak{Y})$.

Proof

The assertion is obvious when ϕ is birational because we are just shifting models in that case. Using the Stein factorization of ϕ we may thus assume that ϕ is finite (and still proper and surjective). By Lemma 1.8 it is then enough to show that for every coherent fractional ideal sheaf $\mathfrak{a}$ on X there exists a coherent fractional ideal sheaf $\mathfrak{b}$ on Y such that $\phi_* Z(\mathfrak{a}) = Z(\mathfrak{b})$. In fact we claim that

$$\phi_* Z(\mathfrak{a}) = Z\big(N_{X/Y}(\mathfrak{a})\big), \tag{2}$$

where $N_{X/Y}(\mathfrak{a})$ denotes the image of $\mathfrak{a}$ under the *norm homomorphism* (cf. [Gro, définition 21.5.5]).

More precisely, pick an affine chart $U \subset Y$. Since the restriction $\phi^{-1}(U) \to U$ is finite, $\phi^{-1}(U)$ is affine and $\mathfrak{a}$ is thus generated by its global sections g on $\phi^{-1}(U)$. For each such g its norm is defined by setting

$$N_{X/Y}(g)(x) = \prod_{\phi(y)=x} g(y)$$

for every smooth point $x \in U$ over which ϕ is étale and by extending it to a regular function on U by normality. We then define $N_{X/Y}(\mathfrak{a})(U)$ as the $\mathcal{O}_U$-module generated by all $N_{X/Y}(g)$ with g as above.

Let us now prove (2). Pick a prime divisor F on a model Y' over Y, and choose a birational model X' over X such that ϕ lifts to a morphism $\phi' \colon X' \to Y'$. Note that ϕ' is proper and generically finite. Let $E_1, \dots, E_r$ be the prime divisors of X' dominating F, so that $(\phi')_* E_i = c_i F$ for some positive integer c_i. Then we have

$$\operatorname{ord}_F\big(\phi_* Z(\mathfrak{a})\big) = \sum_i c_i \operatorname{ord}_{E_i}\big(Z(\mathfrak{a})\big) = -\sum_i c_i \operatorname{ord}_{E_i}(\mathfrak{a})$$

by definition of ϕ_*. On the other hand, let $V \subset Y'$ be an affine chart containing a point of F. The ideal sheaf $N_{X/Y}(\mathfrak{a}) \cdot \mathcal{O}_{Y'}$ is generated, over V, by the functions $N_{X'/Y'}(g)$ where g ranges over all global sections of $\mathfrak{a} \cdot \mathcal{O}_{X'}$ on $(\phi')^{-1}(V)$. We have

$$\operatorname{ord}_F\big(N_{X'/Y'}(g)\big) = \sum c_i \operatorname{ord}_{E_i}(g),$$

and hence

$$\begin{aligned} \operatorname{ord}_F\big(N_{X/Y}(\mathfrak{a})\big) &= \min\big\{\operatorname{ord}_F\big(N_{X'/Y'}(g)\big), g \in H^0\big((\phi')^{-1}(V), \mathfrak{a} \cdot \mathcal{O}_{X'}\big)\big\} \\ &= \min\Big\{\sum c_i \operatorname{ord}_{E_i}(f), f \in \mathfrak{a}\Big\}, \end{aligned}$$

which proves the claim since we have $\operatorname{ord}_{E_i}(f) = \operatorname{ord}_{E_i}(\mathfrak{a})$ for each i if $f \in \mathfrak{a}$ is a general element. □

PROPOSITION 1.15
Suppose that $\phi \colon X \to Y$ is a proper, surjective, generically finite morphism of normal varieties, and let $e(\phi) \in \mathbb{N}^*$ be its degree. Then we have

$$\phi_* \phi^* W = e(\phi) W$$

for every $W \in \operatorname{Div}(\mathfrak{Y})$.

Proof

Let F be an arbitrary prime divisor over Y, and let $E_1, \dots, E_r$ be the prime divisors over X such that ord_{E_i} restricts to a multiple of ord_F. Let $X' \to X$ and $Y' \to Y$ be models so that each E_i is on X' and E is on Y'. As before, we can assume that ϕ lifts to a morphism $\phi' \colon X' \to Y'$. Let $c_i = \mathrm{ord}_F((\phi')_* E_i)$. By definition of ϕ_* and ϕ^*, we have

$$\begin{aligned}\mathrm{ord}_F(\phi_*\phi^* W) &= \sum_i c_i \,\mathrm{ord}_{E_i}(\phi^* W) \\ &= \sum_i c_i \,\mathrm{ord}_{E_i}(\phi^* F)\,\mathrm{ord}_F(W) = e(\phi')\,\mathrm{ord}_F(W),\end{aligned}$$

where the last equality follows by projection formula. One concludes by observing that $e(\phi') = e(\phi)$. □

2. Nef envelopes

In this section X still denotes an arbitrary normal variety (over an algebraically closed field of characteristic zero). We reinterpret the pullback construction of [dFH] as a *nef envelope*, which shows in particular that it coincides with Mumford's numerical pullback on surfaces. Section 2.5 introduces the *defect ideal* of a Weil divisor, measuring its failure to be Cartier, and a precise description of the defect ideal is obtained.

2.1. Graded sequences and nef envelopes

Recall that $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 0}$ is a *graded sequence of fractional ideal sheaves* if $\mathfrak{a}_0 = \mathcal{O}_X$, each $\mathfrak{a}_m$ is a coherent fractional ideal sheaf of X, and $\mathfrak{a}_k \cdot \mathfrak{a}_m \subset \mathfrak{a}_{k+m}$ for every k, m (see [Laz, Section 2.4]). We say that $\mathfrak{a}_\bullet$ has *linearly bounded denominators* if there exists a (fixed) Weil divisor D on X such that $\mathcal{O}_X(mD) \cdot \mathfrak{a}_m \subset \mathcal{O}_X$ for all m.

Let us first attach an $\mathbb{R}$-Weil b-divisor to any graded sequence of ideal sheaves with linearly bounded denominators.

PROPOSITION 2.1

Suppose that $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 0}$ is a graded sequence of fractional ideals sheaves $\mathfrak{a}_m$ with linearly bounded denominators. Then we have

$$\frac{1}{l} Z(\mathfrak{a}_l) \leq \frac{1}{m} Z(\mathfrak{a}_m)$$

for every m divisible by l, and the sequence $(1/m)Z(\mathfrak{a}_m)$ converges coefficient-wise to an $\mathbb{R}$-Weil b-divisor. We write

$$Z(\mathfrak{a}_\bullet) := \lim_m \frac{1}{m} Z(\mathfrak{a}_m).$$

Proof
All this follows from the superadditivity property

$$Z(\mathfrak{a}_m) + Z(\mathfrak{a}_n) \leq Z(\mathfrak{a}_{m+n})$$

since the condition that $\mathfrak{a}_\bullet$ has linearly bounded denominators guarantees that the sequence $(1/m)\operatorname{ord}_E Z(\mathfrak{a}_m)$ is bounded below for each prime divisor E over X and even identically zero for all but finitely many prime divisors E on X. $\square$

LEMMA 2.2
Let $\mathfrak{a}_\bullet$ be a graded sequence of fractional ideal sheaves on X with linearly bounded denominators. Then we have $Z(\mathfrak{a}_\bullet) = (1/m_0)Z(\mathfrak{a}_{m_0})$ for some m_0 if and only if the graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0} \overline{\mathfrak{a}_m}$ of integral closures is finitely generated.

Proof
Since $Z(\mathfrak{a}_m)$ depends only on $\overline{\mathfrak{a}_m}$ (cf. Lemma 1.5), we may assume to begin with that every $\mathfrak{a}_m$ is integrally closed. Assume first that the graded algebra is finitely generated, so that there exists $m_0 \in \mathbb{N}$ such that $\mathfrak{a}_{km_0} = \mathfrak{a}_{m_0}^k$ for all $k \in \mathbb{N}$. Then $Z(\mathfrak{a}_{km_0}) = kZ(\mathfrak{a}_{m_0})$; hence, $Z(\mathfrak{a}_\bullet) = (1/m_0)Z(\mathfrak{a}_{m_0})$. Conversely, assume that $Z(\mathfrak{a}_\bullet) = (1/m_0)Z(\mathfrak{a}_{m_0})$ for a given m_0. By Proposition 2.1 it follows that $Z(\mathfrak{a}_{km_0}) = kZ(\mathfrak{a}_{m_0})$ for all k. Let π be the normalized blowup of X along $\mathfrak{a}_{m_0}$. We then have

$$\mathfrak{a}_{km_0} = \overline{\mathfrak{a}_{km_0}} = \pi_*\mathcal{O}_{X_\pi}\big(kZ(\mathfrak{a}_{m_0})_\pi\big)$$

for all k (cf. [Laz, Proposition 9.6.6]). Since the graded algebra of (relative) global sections of multiples of any (relatively) globally generated line bundle is finitely generated, the fact that $Z(\mathfrak{a}_{m_0})_\pi$ is π-globally generated implies that the $\mathcal{O}_X$-algebra $\bigoplus_k \mathfrak{a}_{km_0}$ is finitely generated and hence so is its finite integral extension $\bigoplus_m \mathfrak{a}_m$. $\square$

Definition 2.3
Let D be an $\mathbb{R}$-Weil divisor on X_π for a given π. The *nef envelope* $\operatorname{Env}_\pi(D)$ of D is defined as the $\mathbb{R}$-Weil b-divisor associated with the graded sequence $\pi_*\mathcal{O}_{X_\pi}(mD)$, $m \geq 0$. When π is the identity we write Env_X for Env_π.

We shall see how this definition relates to relative Zariski decomposition and numerical pullback in the surface case (see Theorem 2.22). A nontrivial toric example is worked out in Example 2.23.

Remark 2.4
If D is an $\mathbb{R}$-Weil divisor on X, then $-\operatorname{Env}_X(-D)_\pi$ coincides by definition with π^*D in the sense of [dFH, Definition 2.9].

Remark 2.5
We introduce later in Section 2.3 a notion of nef envelopes over $\mathfrak{X}$ of a b-divisor W (under some condition on W). The relation between the two notions of envelopes is explained in Remark 2.17.

PROPOSITION 2.6
Let D, D' be two $\mathbb{R}$-Weil divisors on a model X_π. Then we have
- $\mathrm{Env}_\pi(D + D') \geq \mathrm{Env}_\pi(D) + \mathrm{Env}_\pi(D')$*;*
- $\mathrm{Env}_\pi(tD) = t\,\mathrm{Env}_\pi(D)$ *for each* $t \in \mathbb{R}_+$.

Proof
For each $m \geq 0$ we have

$$\big(\pi_*\mathcal{O}_{X_\pi}(mD)\big) \cdot \big(\pi_*\mathcal{O}_{X_\pi}(mD')\big) \subset \pi_*\mathcal{O}_{X_\pi}\big(m(D + D')\big),$$

whence the first point.

To prove the second point we may assume that D is effective (since we may add to D the pullback of an appropriate Cartier divisor of X to make it effective). Now observe that $\mathrm{Env}_\pi(mD) = m\,\mathrm{Env}_\pi(D)$ for each positive integer m since $\mathrm{Env}_\pi(D) = \lim_k (1/k) Z(\pi_*\mathcal{O}_{X_\pi}(kD))$; hence, $\mathrm{Env}_\pi(tD) = t\,\mathrm{Env}_\pi(D)$ for each $t \in \mathbb{Q}_+^*$. On the other hand, $D \mapsto \mathrm{Env}_\pi(D)$ is obviously nondecreasing, so if we pick $t \in \mathbb{R}_+^*$ and approximate it from below and from above by rational numbers s_j, t_j, we get

$$s_j\,\mathrm{Env}_\pi(D) = \mathrm{Env}_\pi(s_j D) \leq \mathrm{Env}_\pi(tD) \leq \mathrm{Env}_\pi(t_j D) = t_j\,\mathrm{Env}_\pi(D)$$

and hence the result. □

Linearity of nef envelopes fails in general. The obstruction to linearity will be studied in greater detail in Section 2.5 (see also Example 2.23 and [dFH]).

COROLLARY 2.7
For every finite-dimensional vector space V of $\mathbb{R}$-Weil divisors on X_π and every divisorial valuation ν, the map $D \mapsto \nu(\mathrm{Env}_\pi(D))$ is continuous on V.

Proof
Proposition 2.6 implies that $D \mapsto \nu(\mathrm{Env}_\pi(D))$ is a concave function on V, and the result follows. □

PROPOSITION 2.8
For every $\mathbb{R}$-Weil divisor D on X the trace $(\mathrm{Env}_X(D))_X$ of $\mathrm{Env}_X(D)$ on X coincides with D.

Proof

If D is a Weil divisor on X, then we have $Z(\mathcal{O}_X(D))_X = D$. Indeed this means that $\operatorname{ord}_E \mathcal{O}_X(D) = -\operatorname{ord}_E D$ for each prime divisor E of X, which holds true since X, being normal, is regular at the generic point of E.

As a consequence we get $D = (\operatorname{Env}_X(D))_X$ when D is a $\mathbb{Q}$-Weil divisor on X, and the general case follows by density, using Corollary 2.7. □

2.2. Variational characterization of nef envelopes

Let $X \to S$ be a projective morphism. In the usual theory of b-divisors one says that an $\mathbb{R}$-Cartier b-divisor C is relatively nef over S (or S-nef for short) if C_π is S-nef for one (hence any) determination π of C. Following [BFJ1] and [KuMa] we extend this definition to arbitrary $\mathbb{R}$-Weil b-divisors.

Definition 2.9

Let $X \to S$ be a projective morphism. We define $\operatorname{Nef}(\mathfrak{X}/S) \subset N_{n-1}(\mathfrak{X}/S)$ as the closed convex cone generated by all S-nef classes $\beta \in N^1(\mathfrak{X}/S)$, that is, all classes of S-nef $\mathbb{R}$-Cartier b-divisors.

Since the usual notion of nefness is preserved by pullback, it is immediate to check that S-nef classes in the sense of the above definition are also preserved by pullback. On the other hand, nefness is in general not preserved under pushforward when $\dim X > 2$, and the traces W_π of an S-nef $\mathbb{R}$-Weil b-divisor are therefore *not* S-nef in general.

Given a projective morphism $Y \to S$, the *S-movable cone* $\overline{\operatorname{Mov}}(Y/S) \subset N^1(Y/S)$ is the closed convex cone $\overline{\operatorname{Mov}}(Y/S)$ generated by the numerical classes of all Cartier divisors D on Y whose S-base locus has codimension at least two: recall that the S-base locus of a Cartier divisor D on Y is the cosupport of the ideal sheaf obtained as the image of the natural evaluation map $f^* f_* \mathcal{O}_Y(D) \otimes \mathcal{O}_Y(-D) \to \mathcal{O}_Y$.

We now have the following alternative description of nef b-divisors.

LEMMA 2.10

Let $X \to S$ be a projective morphism. Then we have

$$\operatorname{Nef}(\mathfrak{X}/S) = \operatorname*{proj\,lim}_{\pi} \overline{\operatorname{Mov}}(X_\pi/S),$$

where the limit is taken over all smooth (or $\mathbb{Q}$-factorial) models X_π. In other words, an $\mathbb{R}$-Weil b-divisor W is S-nef if and only if W_π is S-movable on each smooth (or $\mathbb{Q}$-factorial) model X_π. In particular the restriction of (the class of) W_π to any prime divisor of X_π is S-pseudoeffective.

Proof
Let $\alpha \in N_{n-1}(\mathfrak{X}/S)$. Since the latter is endowed with the inverse limit topology the sets

$$V_{\pi,U} := \{\beta \in N_{n-1}(\mathfrak{X}/S), \beta_\pi \in U\},$$

where π ranges over all smooth models of X and $U \subset N^1(X_\pi/S)$ ranges over all conical open neighborhoods of α_π, form a neighborhood basis of α.

We infer by definition that α is S-nef if and only if for every π and U there exists an S-nef class $\beta \in N^1(\mathfrak{X}/S)$ such that $\beta_\pi \in U$. On the other hand, since U is conical it is immediate to see that β may be assumed to be the class of an S-globally generated Cartier b-divisor, and the result follows. □

The next result is a limiting case of Lemma 1.8.

LEMMA 2.11
Let $\mathfrak{a}_\bullet$ be a graded linearly bounded denominator. Then the $\mathbb{R}$-Weil b-divisor $Z(\mathfrak{a}_\bullet)$ is X-nef.

Proof
Since $\mathfrak{a}_\bullet$ has linearly bounded denominators it is in particular clear that there exists a finite-dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $Z(\mathfrak{a}_m) \in V$ for all m. By Lemma 1.12 it thus follows that $[(1/m)Z(\mathfrak{a}_m)]$ converges to $[Z(\mathfrak{a}_\bullet)]$ in $N_{n-1}(\mathfrak{X}/X)$. But each $Z(\mathfrak{a}_m)$ is X-globally generated by Lemma 1.8, and we thus conclude that $Z(\mathfrak{a}_\bullet)$ is X-nef. □

PROPOSITION 2.12 (Negativity lemma)
Let W be an X-nef $\mathbb{R}$-Weil b-divisor over X. Then for each π we have $W \le \mathrm{Env}_\pi(W_\pi)$.

The following argument provides in particular an alternative proof of the well-known *negativity lemma* [KoMo, Lemma 3.39].

Proof
Let X_π be a fixed model of X.

Step 1. Let C be an X-globally generated Cartier b-divisor, determined on some model X_τ that may be assumed to dominate X_π. As in the proof of Lemma 1.8 we have $C = Z(\mathcal{O}_X(C))$ since C is X-globally generated, and we infer that $C \le \mathrm{Env}_\pi(C_\pi)$. Indeed $\tau \ge \pi$ implies

$$\mathcal{O}_X(C) = \tau_* \mathcal{O}_{X_\tau}(C_\tau) \subset \pi_* \mathcal{O}_{X_\pi}(C_\pi);$$

hence,

$$C = Z\big(\mathcal{O}_X(C)\big) \le Z\big(\pi_* \mathcal{O}_{X_\pi}(C_\pi)\big) \le \mathrm{Env}_\pi(C_\pi)$$

by Proposition 2.1.

Step 2. Let C be an X-nef $\mathbb{R}$-Cartier b-divisor, determined on a model X_τ that may again be assumed to be projective over X and to dominate X_π. The class of C_τ in $N^1(X_\tau/X)$ is X-nef and hence belongs to the closed convex cone spanned by the classes of X-very ample divisors of X_τ. As in Lemma 1.12(ii), we may then find a sequence of X-very ample Cartier divisors A_j on X_τ and a sequence $t_j \in \mathbb{R}_+^*$ such that $t_j A_j \to C_\tau$ coefficient-wise, while staying in a fixed finite-dimensional vector space of $\mathbb{R}$-divisors on X_τ. By Step 1 and Proposition 2.6 we have $t_j \overline{A_j} \le \mathrm{Env}_\pi(t_j (\overline{A_j})_\pi)$ for each j. By Corollary 2.7 we infer

$$\nu(C) = \lim_j t_j \nu(\overline{A_j}) \le \nu\big(\mathrm{Env}_\pi(t_j \overline{A_j})\big) = \nu\big(\mathrm{Env}_\pi(C_\pi)\big)$$

for each divisorial valuation ν, hence $C \le \mathrm{Env}_\pi(C_\pi)$. This step recovers in particular the usual statement of the negativity lemma.

Step 3. Let W be an arbitrary X-nef $\mathbb{R}$-Weil b-divisor. By Lemma 1.12 there exists a net W_j of X-nef $\mathbb{R}$-Cartier divisors such that $W_j \to W$ coefficient-wise and $W_{j,X}$ stays in a fixed finite-dimensional space of $\mathbb{R}$-Weil divisors on X. The result now follows by another application of Corollary 2.7. □

As a consequence we get the following variational characterization of nef envelopes.

COROLLARY 2.13

If D is an $\mathbb{R}$-Weil divisor on X_π, then $\mathrm{Env}_\pi(D)$ *is the largest X-nef $\mathbb{R}$-Weil b-divisor W such that $W_\pi \le D$. In particular we have*

- $\mathrm{Env}_\pi(D) = \overline{D}$ *if D is $\mathbb{R}$-Cartier and X-nef.*
- *The b-divisor* $\mathrm{Env}_\pi(D)$ *is $\mathbb{R}$-Cartier, determined by a given $\tau \ge \pi$, if and only if the trace of* $\mathrm{Env}_\pi(D)$ *on X_τ is $\mathbb{R}$-Cartier and X-nef.*

Proof

The $\mathbb{R}$-Weil b-divisor $\mathrm{Env}_\pi(D)$ is X-nef by Lemma 2.11. We also clearly have $(1/m)Z(\pi_* \mathcal{O}_{X_\pi}(mD))_\pi \le D$, and hence $\mathrm{Env}(D)_\pi \le D$ in the limit. Conversely if Z is an X-nef $\mathbb{R}$-Weil b-divisor such that $Z_\pi \le D$, then $Z \le \mathrm{Env}_\pi(Z_\pi) \le \mathrm{Env}_\pi(D)$ by the negativity lemma. □

As an illustration we now prove the following.

PROPOSITION 2.14

Assume that X has klt singularities in the sense that there exists an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is klt (cf. [dFH]). Then $\mathrm{Env}_X(D)$ *is an $\mathbb{R}$-Cartier b-divisor for every $\mathbb{R}$-Weil divisor D on X. When D has $\mathbb{Q}$-coefficients we even have* $\mathrm{Env}_X(D) = (1/m)Z(\mathcal{O}_X(mD))$ *for some m.*

The result easily follows from [Kol2, Exercise 109], but we provide some details for the convenience of the reader.

Note that the analogous result for $\mathrm{Env}_\pi(D)$, D being a Weil divisor on a higher model X_π, fails even when X is smooth (cf. [Cut], [Kür] for an explicit example).

Proof

Since (X, Δ) is klt it follows from [BCHM, Corollary 1.4.3] that there exists a $\mathbb{Q}$-*factorialization* $\pi\colon X_\pi \to X$, that is, a small birational morphism π such that X_π is $\mathbb{Q}$-factorial. Denote by $\hat{\Delta}_\pi$ and $\hat{D}_\pi$ the strict transforms on X_π of Δ and D, respectively. Since π is small we have $\pi^*(K_X + \Delta) = K_{X_\pi} + \hat{\Delta}_\pi$, which shows that $(X_\pi, \hat{\Delta}_\pi)$ is klt, hence so is $(X_\pi, \hat{\Delta}_\pi + \varepsilon \hat{D}_\pi)$ for $0 < \varepsilon \ll 1$. By applying [BCHM, Corollary 1.4.3] to $\varepsilon \hat{D}_\pi$, which is π-numerically equivalent to $K_{X_\pi} + \hat{\Delta} + \varepsilon \hat{D}$ as well as π-big (since π is birational) we infer the existence of a new $\mathbb{Q}$-factorialization $\tau\colon X_\tau \to X$ such that the strict transform $\hat{D}_\tau$ of D on X_τ is furthermore X-nef. Since τ is small it is easily seen that $\tau_* \mathcal{O}_{X_\tau}(m\hat{D}_\tau) = \mathcal{O}_X(mD)$ for all m; hence, $\mathrm{Env}_\tau(\hat{D}_\tau) = \mathrm{Env}_X(D)$, and it follows by Corollary 2.13 that $\mathrm{Env}_X(D)$ is the $\mathbb{R}$-Cartier b-divisor determined by $\hat{D}_\tau$.

When D has rational coefficients the base point free theorem shows that $\hat{D}_\tau$ is X-globally generated, so that

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mD) = \bigoplus_{m \geq 0} \tau_* \mathcal{O}_{X_\tau}(m\hat{D}_\tau)$$

is finitely generated over $\mathcal{O}_X$. We thus have $\mathrm{Env}_X(D) = (1/m)Z(\mathcal{O}_X(mD))$ for some m. □

2.3. *Nef envelopes of Weil b-divisors*

The next result is a variant in the relative case of [BFJ1, Proposition 2.13] and [KuMa, Theorem D].

PROPOSITION 2.15

Let W be an $\mathbb{R}$-Weil b-divisor. If the set of X-nef $\mathbb{R}$-Weil b-divisors Z such that $Z \leq W$ is nonempty, then it admits a largest element.

Definition 2.16
We say that *the nef envelope of W is well defined* if the assumption of the proposition holds. We then denote the largest element in question by $\mathrm{Env}_{\mathfrak{X}}(W)$ and call it the *nef envelope* of W.

Proof of Proposition 2.15
Every Z as in the proposition satisfies $Z \le \mathrm{Env}_\pi(W_\pi)$ for all π by Corollary 2.13, which also implies that $\pi \mapsto \mathrm{Env}_\pi(W_\pi)$ is nonincreasing; that is,

$$\mathrm{Env}_{\pi'}(W_{\pi'}) \le \mathrm{Env}_\pi(W_\pi)$$

whenever $\pi' \ge \pi$. If there exists at least one Z as above, then it follows that $\mathrm{Env}_{\mathfrak{X}}(W) := \lim_\pi \mathrm{Env}_\pi(W_\pi)$ is well defined as a b-divisor and satisfies $\mathrm{Env}_{\mathfrak{X}}(W) \ge Z$ for every such Z. There remains to show that $\mathrm{Env}_{\mathfrak{X}}(W)$ is X-nef and satisfies $\mathrm{Env}_{\mathfrak{X}}(W) \le W$. But the existence of Z guarantees the existence of a finite-dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $\mathrm{Env}_\pi(W_\pi)_X \in V$ for all π. Since $\mathrm{Env}_\pi(W_\pi)$ converges to $\mathrm{Env}_{\mathfrak{X}}(W)$ coefficient-wise, we conclude as before by Lemma 1.12 that $\mathrm{Env}_{\mathfrak{X}}(W)$ is X-nef, whereas $\mathrm{Env}_{\mathfrak{X}}(W) \le W$ follows from $\mathrm{Env}_\pi(W_\pi)_\tau \le W_\tau$ for $\tau \le \pi$ by letting $\pi \to \infty$. $\square$

Remark 2.17
Note that the proof gives

$$\mathrm{Env}_{\mathfrak{X}}(W) = \inf_\pi \mathrm{Env}_\pi(W_\pi).$$

If W is an $\mathbb{R}$-Cartier b-divisor, then we have

$$\mathrm{Env}_{\mathfrak{X}}(W) = \mathrm{Env}_\pi(W_\pi)$$

for each determination π.

PROPOSITION 2.18
Let $(W_i)_{i\in I}$ be a net of b-divisors decreasing to W such that $\mathrm{Env}_{\mathfrak{X}}(W)$ *is well defined. Then* $\mathrm{Env}_{\mathfrak{X}}(W_i)$ *is well defined for every i, and the net decreases to* $\mathrm{Env}_{\mathfrak{X}}(W)$.

Proof
By assumption, $\mathrm{Env}_{\mathfrak{X}}(W)$ is well defined, so that there exists an X-nef $\mathbb{R}$-Weil b-divisor $Z \le W$. Since $W_i \ge W$ for all i, the envelopes $\mathrm{Env}_{\mathfrak{X}}(W_i)$ are also well defined, and form a net that decreases to a b-divisor $Z' \ge \mathrm{Env}_{\mathfrak{X}}(W)$. Pick any π. Since $W_{i,\pi} \to W_\pi$, we have $\mathrm{Env}_{\mathfrak{X}}(W_i) \le \mathrm{Env}_\pi(W_{i,\pi}) \to \mathrm{Env}_\pi(W_\pi)$. Letting $i \to \infty$, we get $Z' \le \mathrm{Env}_\pi(W_\pi)$. We conclude using the preceding remark. $\square$

PROPOSITION 2.19
Suppose that $\phi\colon X \to Y$ *is a finite dominant morphism of normal varieties. Let* W *be any* $\mathbb{R}$*-Weil* b*-divisor over* Y *whose nef envelope* $\mathrm{Env}_{\mathfrak{Y}}(W)$ *is well defined. Then* $\mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ *is also well defined and we have*

$$\mathrm{Env}_{\mathfrak{X}}(\phi^* W) = \phi^* \mathrm{Env}_{\mathfrak{Y}}(W).$$

We similarly have

$$\mathrm{Env}_X(\phi^* D) = \phi^* \mathrm{Env}_Y(D)$$

for every $\mathbb{R}$*-Weil divisor* D *on* Y*.*

Proof
Since $\mathrm{Env}_{\mathfrak{Y}}(W)$ is Y-nef, its pullback $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$ is Y-nef as well and hence also X-nef. Since we have $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W) \le \phi^* W$ this shows that $\mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ is well defined and satisfies $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W) \le \mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ by Proposition 2.15.

Conversely, Lemma 2.20 below shows that $\phi_* \mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ is Y-nef. Since $\phi_* \times \mathrm{Env}_{\mathfrak{X}}(\phi^* W) \le \phi_*\phi^* W = e(\phi) W$ by Proposition 1.15 it follows that

$$\phi_* \mathrm{Env}_{\mathfrak{X}}(\phi^* W) \le e(\phi) \mathrm{Env}_{\mathfrak{Y}}(W) = \phi_*\phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$$

by Proposition 1.15 again, and we conclude by applying Lemma 2.21 below to $Z := \mathrm{Env}_{\mathfrak{X}}(\phi^* W) - \phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$. □

LEMMA 2.20
Let $\phi\colon X \to Y$ *be a finite dominant morphism between normal varieties, and let* W *be an* X*-nef* $\mathbb{R}$*-Weil* b*-divisor over* X*. Then* $\phi_* W$ *is* Y*-nef.*

Proof
By assumption, the class of W in $N_{n-1}(\mathfrak{X}/X)$ is X-nef and hence can be written as the limit of a net of X-nef classes of $N^1(\mathfrak{X}/X)$. By Lemma 1.12(b) there exists a net W_j of X-nef $\mathbb{R}$-Cartier b-divisors such that $W_j \to W$ coefficient-wise and $W_{j,X}$ stays in a fixed finite-dimensional vector of $\mathbb{R}$-Weil divisors on X. It follows that the divisors $(\phi_* W_j)_Y$ also stay in a fixed finite-dimensional vector space of $\mathbb{R}$-Weil divisors on Y. Using the definition of ϕ_* on Weil b-divisors, it is immediate to see that $\phi_* W_j \to \phi_* W$ coefficient-wise. Using Lemma 1.12(a) it thus follows that $[\phi_* W_j] \to [\phi_* W]$ in $N_{n-1}(\mathfrak{Y}/Y)$, and we are reduced to the case where W is $\mathbb{R}$-Cartier.

Now let π be a determination of W. By Corollary 2.13 we have in particular $W = \mathrm{Env}_\pi(W_\pi)$, so that the fractional ideals $\mathfrak{a}_m := \pi_* \mathcal{O}(m W_\pi)$ satisfy $W = \lim (1/m) Z(\mathfrak{a}_m)$ coefficientwise, and it is clear that the $Z(\mathfrak{a}_m)_X$ stay in a fixed finite-dimensional vector space by monotonicity. We are now reduced to the case where

$W = Z(\mathfrak{a})$ for some fractional ideal, in which case we have $\phi_* Z(\mathfrak{a}) = Z(N_{X/Y}(\mathfrak{a}))$ by (the proof of) Proposition 1.14. We conclude that $\phi_* Z(\mathfrak{a})$ is Y-globally generated, hence in particular Y-nef, by Lemma 1.8. □

LEMMA 2.21
Let $\phi\colon X \to Y$ be a proper, surjective, generically finite morphism. Suppose that $Z \geq 0$ is an $\mathbb{R}$-Weil b-divisor over X. Then $\phi_* Z = 0$ only if $Z = 0$.

Proof
Suppose that there is a prime divisor E lying in some model X' over X such that $\operatorname{ord}_E Z > 0$. Since ϕ is generically finite, we can choose a model Y' over Y such that E maps to a prime divisor F on Y' via the rational map $\phi'\colon X' \dashrightarrow Y'$ obtained by lifting ϕ. Then $\operatorname{ord}_F(\phi_* Z) \geq \operatorname{ord}_E Z > 0$; hence, $\phi_* Z$ cannot be zero. □

2.4. The case of surfaces and toric varieties

THEOREM 2.22
Let X be a normal surface, and let $\pi: X_\pi \to X$ be a smooth (or at least $\mathbb{Q}$-factorial) model.

(i) *If D is an $\mathbb{R}$-divisor on X_π, then the b-divisor $\operatorname{Env}_\pi(D)$ is $\mathbb{R}$-Cartier, determined on X_π, and*

$$D = \operatorname{Env}_\pi(D)_\pi + \big(D - \operatorname{Env}_\pi(D)_\pi\big)$$

coincides with the relative Zariski decomposition of D with respect to $\pi : X_\pi \to X$.

(ii) *If D is an $\mathbb{R}$-Weil divisor on X, then $\operatorname{Env}_X(D) = \overline{\pi^* D}$, where $\pi^* D$ is the numerical pullback of D in the sense of Mumford.*

Recall that the *numerical pullback* of D is defined as the unique $\mathbb{R}$-divisor D' on X_π such that $\pi_* D' = D$ and $D' \cdot E = 0$ for all π-exceptional divisors E.

Proof
Let us prove (i). The first assertion follows from Corollary 2.13, since each movable class is nef when $\dim X = 2$.

The divisor $P := \operatorname{Env}_\pi(D)_\pi$ is an X-nef $\mathbb{R}$-divisor on X_π such that $D \geq P$ and $P \geq Q$ for every X-nef divisor Q on X_π such that $D \geq Q$, by Corollary 2.13 again. Write $N := P - D$. Then we have $P \cdot E = 0$ for any prime divisor E in the support of N, since otherwise $P + sE$ is X-nef and at least D for $s \ll 1$. This is one of the characterizations of the (relative) Zariski decomposition (see [Sak, p. 408]). This concludes the proof of (i).

Let us now prove (ii). Let π^*D be the numerical pullback of D to X_π. Since π^*D is π-nef it follows that $C := \overline{\pi^*D}$ is X-nef and satisfies $C_X = D$; hence, $C \leq \mathrm{Env}_X(D)$ by Corollary 2.13. Conversely set $D' := \mathrm{Env}_X(D)_\pi$. We claim that $D' = \pi^*D$. Taking this for granted for the moment we then get $\mathrm{Env}_X(D) \leq C$ by the negativity lemma, and the result follows.

Since we have $\pi_*D' = D$ by Proposition 2.8, the claim will follow if we show that $D' \cdot E = 0$ for each π-exceptional prime divisor E on X_π. This is a consequence of the variational characterization of $\mathrm{Env}_X(D)$. Indeed note that $D' \cdot E \geq 0$ since D' is π-nef by Lemma 2.10. If we assume by contradiction that $D' \cdot E > 0$, then $D' + \varepsilon E$ is still π-nef for $0 < \varepsilon \ll 1$ and $C := \overline{D' + \varepsilon E}$ is then an X-nef b-divisor with $C_X = D$. It follows that $C \leq \mathrm{Env}_X(D)$ by Corollary 2.13; hence, $D' + \varepsilon E \leq D'$, a contradiction. □

Let us now decribe the case of toric varieties. We refer to [Fult], [Oda], and [CLS] for basics on toric varieties. Let N be a free abelian group of rank n, and suppose that we are given two rational polyhedral fans Δ, Δ' in N such that $\Delta \subset \Delta'$. For the sake of simplicity we assume that Δ and Δ' have the same support S. Denote by $X(\Delta)$ and $X(\Delta')$ the corresponding toric varieties. Since Δ is a subset of Δ', we have an induced birational map $\pi : X(\Delta') \to X(\Delta)$.

Let D be an $\mathbb{R}$-Weil toric divisor on $X(\Delta)$. It is given by a real-valued function h_D on the set of primitive vectors $\Delta(1)$ generating the 1-dimensional faces of Δ, and D is $\mathbb{R}$-Cartier if and only if h_D extends to a continuous function on S that is linear on each face. In that case D is π-nef if and only if h_D is convex on the union S_0 of all faces of Δ' that contain a ray in $\Delta'(1) \setminus \Delta(1)$. By Corollary 2.13 it follows that the function attached to $\mathrm{Env}_\pi(D)_\pi$ is the supremum of all 1-homogeneous functions on the convex set S such that $g \leq h_D$ on $\Delta(1)$ and g is convex on the subset S_0.

Example 2.23
Take Δ in $\mathbb{R}^3$ the fan having a single 3-dimensional cone generated by the four rays $(1,0,0), (0,1,0), (0,0,1), (1,1,-1)$. Then $X(\Delta)$ is an affine variety having an isolated singularity at the origin and is locally isomorphic to a quadratic cone there.

Let Δ' be the regular fan having $(1,0,0), (0,1,0), (0,0,1), (1,1,-1), (1,1,0)$ as vertices. The natural map $X(\Delta') \to X(\Delta)$ is a proper birational map which gives a (nonminimal) desingularization of $X(\Delta)$. Denote by E_v the divisor associated to the corresponding ray $v \in \mathbb{R}^3$ either in $X(\Delta)$ or $X(\Delta')$.

Now take $D_1 = E_{100} + E_{010} + E_{001}$ and $D_2 = E_{100} + E_{001} + E_{11-1}$. Then $D_1 + D_2$ is a Cartier divisor on $X(\Delta)$ whose support function is given by $2x_1 + x_2 + 2x_3$ in the standard coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$. Hence $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1 + D_2) = 3$. On the other hand, for any convex function g having value 1 at $(0,0,1)$ and zero at

$(1,1,-1)$, we have $g(1,1,0) \leq 1$; hence, $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1) \leq 1$. The same argument shows that $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_2) \leq 1$; hence, $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1) + \mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_2) < \mathrm{ord}_{E_{110}}(\mathrm{Env}_X(D_1 + D_2))$.

2.5. *Defect ideals*

Definition 2.24
The *defect ideal* of an $\mathbb{R}$-Weil divisor D on X is defined as

$$\mathfrak{d}(D) := \mathcal{O}_X(D) \cdot \mathcal{O}_X(-D).$$

Note that $\mathfrak{d}(D) \subset \mathcal{O}_X(D - D) = \mathcal{O}_X$ is an ideal sheaf. The following proposition summarizes immediate properties of defect ideals.

PROPOSITION 2.25
Let D, D' be $\mathbb{R}$-Weil divisors on X. Then we have
(i) $\mathfrak{d}(D + C) = \mathfrak{d}(D)$ *for every Cartier divisor C,*
(ii)

$$\mathfrak{d}(D) \cdot \mathcal{O}_X(D + D') \subset \mathcal{O}_X(D) \cdot \mathcal{O}_X(D') \subset \mathcal{O}_X(D + D'),$$

(iii)

$$\phi^{-1}\mathfrak{d}_X(D) \cdot \mathcal{O}_Y(\phi^* D) \subset \phi^{-1}\mathcal{O}_X(D) \cdot \mathcal{O}_Y \subset \mathcal{O}_Y(\phi^* D)$$

for every finite dominant morphism $\phi\colon Y \to X$.
(iv) *The sequence*

$$\mathfrak{d}_\bullet(D) := \big(\mathfrak{d}(mD)\big)_{m \geq 0}$$

is a graded sequence of ideals, and

$$Z\big(\mathfrak{d}_\bullet(D)\big) = \mathrm{Env}_X(D) + \mathrm{Env}_X(-D).$$

Definition 2.26
We say that an $\mathbb{R}$-Weil divisor D on X is *numerically Cartier* if $\mathrm{Env}_X(-D) = -\mathrm{Env}_X(D)$. In the special case where $D = K_X$ we say that X is *numerically Gorenstein* if K_X is numerically Cartier.

Remark 2.27
If D is a $\mathbb{Q}$-Weil divisor, then the property of being numerically Cartier can be equivalently checked using valuations, so that D is numerically Cartier if and only if, given a positive integer k such that kD is an integral divisor, for every divisorial valuation ν the sequence $\nu(\mathcal{O}_X(mkD)) - \nu(\mathcal{O}_X(-mkD))$ is in $o(m)$.

By Proposition 2.6 it is straightforward to see that numerically Cartier divisors form an $\mathbb{R}$-vector space. We also have the following.

LEMMA 2.28
Let D be an $\mathbb{R}$-Weil divisor on X. Then D is numerically Cartier if and only if

$$\mathrm{Env}_X(D + D') = \mathrm{Env}_X(D) + \mathrm{Env}_X(D')$$

for every $\mathbb{R}$-Weil divisor D' on X.

Proof
Assume that D is numerically Cartier, so that $\mathrm{Env}_X(-D) = -\mathrm{Env}_X(D)$. Then we have on the one hand $\mathrm{Env}_X(D + D') \geq \mathrm{Env}_X(D) + \mathrm{Env}_X(D')$ and on the other hand $\mathrm{Env}_X(-D) + \mathrm{Env}_X(D + D') \leq \mathrm{Env}_X(D')$, and additivity follows. The converse is equally easy and left to the reader. □

Example 2.29 (Surfaces)
Since Mumford's pullback of Weil divisors on surfaces is linear, it follows from Theorem 2.22 that all $\mathbb{R}$-Weil divisors on a normal surface X are numerically Cartier.

Example 2.30 (Toric varieties)
If D is a toric $\mathbb{R}$-Weil divisor on a toric variety X, then it follows from the discussion from the last section that D is numerically Cartier if and only if D is already $\mathbb{R}$-Cartier.

Example 2.31 (Cone singularities)
Let (V, L) be a smooth projective variety endowed with an ample line bundle L. Recall that the affine cone over (V, L) is the algebraic variety defined by

$$X = C(V, L) := \mathrm{Spec}\Big(\bigoplus_{m \geq 0} H^0(V, mL)\Big).$$

If L is sufficiently positive, then X has an isolated normal singularity at its vertex $0 \in X$ and is obtained by blowing down the zero section $E \simeq V$ in the total space Y of the dual bundle L^*. We denote by $\pi : Y \to X$ the contraction map, which is isomorphic to the blowup of X at zero. Every divisor D on V induces a Weil divisor $C(D)$ on X, and the map $D \mapsto C(D)$ induces an isomorphism $\mathrm{Pic}(V)/\mathbb{Z}L \simeq \mathrm{Cl}(X)$ onto the divisor class group of X.

LEMMA 2.32
Let (V, L) be a smooth polarized variety, and let D be an $\mathbb{R}$-Weil divisor on V. Assume that L is sufficiently positive, so that $C(V, L)$ is normal.

(1) *$C(D)$ is $\mathbb{R}$-Cartier if and only if D and L are $\mathbb{R}$-linearly proportional in* $\mathrm{Pic}(X) \otimes \mathbb{R}$.

(2) *$C(D)$ is numerically Cartier if and only if D and L are numerically proportional in $N^1(V)$.*

Proof

Property (1) follows from the description of the divisor class group of $X = C(V)$ recalled above. Let us prove (2). Let $\pi : Y \to X$ be the blowup of X at its vertex zero. The restriction to $E \simeq V$ of the strict transform $C(D)'$ is linearly equivalent to D. If D is numerically Cartier, then the restriction to E of $\mathrm{Env}_X(-C(D))_Y = -\mathrm{Env}_X(C(D))_Y$ is both pseudoeffective and anti-pseudoeffective by Lemma 2.10, so $\mathrm{Env}_X(C(D))_Y$ is numerically equivalent to zero in $N^1(Y/X)$. But $\mathrm{Env}_X(C(D))_Y - C(D)'$ is π-exceptional, hence proportional to E, and we conclude as desired that $D \equiv C(D)'|_E$ is proportional to $L \equiv -E|E$ in $N^1(V)$.

Conversely assume that $D \equiv aL$ are proportional in $N^1(V)$. Then $C(D)'$ and E are proportional in $N^1(Y/X)$; hence, there exists $t \in \mathbb{R}$ such that $\mathrm{Env}_X(C(D))_Y \equiv -tE$ in $N^1(Y/X)$. Since $-E$ is X-ample and the numerical class of $\mathrm{Env}_X(C(D))_Y$ is in the X-movable cone, it follows that $t \geq 0$, which implies that $\mathrm{Env}_X(C(D))_Y$ is X-nef. This in turn shows as in the proof of Theorem 2.22 that the b-divisor $\mathrm{Env}_X(C(D))$ is $\mathbb{R}$-Cartier, determined on Y by $C(D)' - aE$. If we replace D by $-D$, then we get that $\mathrm{Env}_X(C(D))$ is determined on Y by $C(-D)' + aE = -(C(D)' - aE)$; that is, $\mathrm{Env}_X(-C(D)) = -\mathrm{Env}_X(C(D))$ holds as desired. □

We now give a more precise description of defect ideals, which is basically an elaboration of [dFH, Theorem 5.4]. As a matter of terminology we introduce the following.

Definition 2.33

We say that a determination π of an $\mathbb{R}$-Cartier b-divisor C is a *log resolution* of C if X_π is smooth, the exceptional locus $\mathrm{Exc}(\pi)$ has codimension one, and $\mathrm{Exc}(\pi) + C_\pi$ has simple normal crossing (SNC) support.

Another $\mathbb{R}$-Cartier b-divisor C' is then said to be *transverse* to π and C if π is also a log resolution of $C + C'$ and C'_π has no common component with $\mathrm{Exc}(\pi) + C_\pi$.

Every $\mathbb{R}$-Cartier b-divisor admits a log resolution by Hironaka's theorem.

PROPOSITION 2.34

Let D be a Weil divisor on X, and assume that X is quasi-projective. Then we have

$$\mathfrak{d}(D) = \sum_E \mathcal{O}_X(-E),$$

where the sum is taken over the set of all prime divisors E of X such that $D - E$ is Cartier (and this set is in particular nonempty).

Given a Cartier b-divisor C and a joint log resolution π of C and $\mathcal{O}_X(D)$, the sum can be further restricted to those E such that $Z(\mathcal{O}_X(E))$ is transverse to π and C.

Proof
Observe first that

$$\mathcal{O}_X(-E) \subset \mathcal{O}_X(-E) \cdot \mathcal{O}_X(E) = \mathfrak{d}(E) = \mathfrak{d}(D)$$

for all effective Weil divisors E such that $D - E$ is Cartier.

Since X is quasi-projective there exists a line bundle L on X such that $L \otimes \mathcal{O}_X(D)$ is generated by a finite-dimensional vector space of global sections V, which we view as rational sections of L. For each $s \in V$ set $E_s := D + \operatorname{div}(s)$, which is an effective Weil divisor congruent to D modulo Cartier divisors.

We claim that there exists a (nonempty) Zariski open subset U of V such that

$$\mathfrak{d}(D) = \sum_{s \in U} \mathcal{O}_X(-E_s) \tag{3}$$

and

- E_s is a prime divisor on X,
- $Z(\mathcal{O}_X(E_s))$ is transverse to π and C,

for each $s \in U$, which concludes the proof of Proposition 2.34.

Since π dominates the blowup of $\mathcal{O}_X(D)$ it is easily seen that the effective divisors

$$M_s := Z\big(\mathcal{O}_X(E_s)\big)_\pi = Z\big(\mathcal{O}_X(D)\big)_\pi + \pi^* \operatorname{div}(s)$$

move in a base point free linear system on X_π as s moves in V. We may thus find a nonempty Zariski open subset U of V such that for each $s \in U$ we have

- M_s has no common component with $\operatorname{Exc}(\pi) + C_\pi$,
- M_s is smooth and irreducible,
- $M_s + \operatorname{Exc}(\pi) + C_\pi$ has SNC support,

where the last two points follow from Bertini's theorem. Since $\pi_* M_s = Z(\mathcal{O}_X(D))_X + \operatorname{div}(s) = E_s$ by Proposition 2.8, we see in particular that E_s is a prime divisor for each $s \in U$ and $Z(\mathcal{O}_X(E_s))$ is transverse to π and C. There remains to show (3). Observe that

$$s \cdot \mathcal{O}_X(-D) \subset L \otimes \mathcal{O}_X\big(-\operatorname{div}(s)\big) \cdot \mathcal{O}_X(-D) = L \otimes \mathcal{O}_X(-E_s)$$

for each $s \in V$. Since V generates $L \otimes \mathcal{O}_X(D)$ and U is open in V we obtain

$$L \otimes \mathfrak{d}(D) = L \otimes \mathcal{O}_X(D) \cdot \mathcal{O}_X(-D)$$
$$= \sum_{s \in U} s \cdot \mathcal{O}_X(-D) \subset L \otimes \sum_{s \in U} \mathcal{O}_X(-E_s),$$

and the result follows since L is invertible. □

3. Multiplier ideals and approximation

In this section, X still denotes a normal variety. Our main goal here is to show how to obtain from Takagi's subadditivity theorem for multiplier ideals of pairs a similar statement for the general multiplier ideals defined in [dFH]. This result in turn enables us to approximate nef envelopes of Cartier divisors from above by nef Cartier divisors, in the spirit of [BFJ1].

3.1. Log discrepancies

We say that an $\mathbb{R}$-Weil divisor Δ on X is an $\mathbb{R}$-*boundary* (resp., a $\mathbb{Q}$-boundary, resp., an m-boundary) if $K_X + \Delta$ is $\mathbb{R}$-Cartier (resp., $K_X + \Delta$ is $\mathbb{Q}$-Cartier, resp., $m(K_X + \Delta)$ is Cartier).

Let ω be a rational top-degree form on X, and consider the associated canonical b-divisor $K_{\mathfrak{X}}$. Given an $\mathbb{R}$-boundary Δ on X we define the *relative canonical b-divisor* of (X, Δ) by

$$K_{\mathfrak{X}/(X,\Delta)} = K_{\mathfrak{X}} - \overline{K_X + \Delta},$$

which is independent of the choice of ω. If E is a prime divisor above X, then $\operatorname{ord}_E K_{\mathfrak{X}/(X,\Delta)}$ is nothing but the *discrepancy* of the pair (X, Δ) along E. Following [dFH] we introduce on the other hand the following.

Definition 3.1

The *m-limiting relative canonical b-divisor* is defined by

$$K_{m,\mathfrak{X}/X} := K_{\mathfrak{X}} + \frac{1}{m} Z\big(\mathcal{O}_X(-mK_X)\big),$$

and the *relative canonical b-divisor* is

$$K_{\mathfrak{X}/X} = K_{\mathfrak{X}} + \operatorname{Env}_X(-K_X).$$

They are both independent of the choice of ω and are exceptional over X by Proposition 2.8. Note that $K_{m,\mathfrak{X}/X} \to K_{\mathfrak{X}/X}$ coefficient-wise as $m \to \infty$.

Recall that the *log discrepancy* of a pair (X, Δ) along a prime divisor E above X is defined by adding 1 to the discrepancy. Let us reformulate this by introducing the *pseudo b-divisor* $1_{\mathfrak{X}}$, that is, the homogeneous function on the set of divisorial valuations of X such that

$$(t\,\mathrm{ord}_E)(1_{\mathfrak{X}}) = t$$

for each divisorial valuation $t\,\mathrm{ord}_E$, so that $\mathrm{ord}_E(K_{\mathfrak{X}/(X,\Delta)} + 1_{\mathfrak{X}})$ is now equal to the log discrepancy of (X, Δ) along E. We also consider the reduced exceptional b-divisor $1_{\mathfrak{X}/X}$, which takes value 1 on the prime divisors that are exceptional over X and value zero on the prime divisors contained in X.

The following well-known properties show that $K_{\mathfrak{X}} + 1_{\mathfrak{X}}$ is better behaved than $K_{\mathfrak{X}}$.

LEMMA 3.2
Assume that X is smooth, and let E be a reduced SNC divisor on X. Then we have $K_{\mathfrak{X}} + 1_{\mathfrak{X}} \geq \overline{K_X + E}$.

This result is [Kol1, Lemma 3.11], whose proof we reproduce for the convenience of the reader.

Proof
Let F be a smooth irreducible divisor in some model $\pi\colon X_\pi \to X$. We may choose local coordinates $(x_1, \dots, x_n)$ near the generic point of $\pi(F)$ such that the local equation of E writes $x_c \cdots x_p = 0$ for some $p = 0, \dots, n$, and we let z be a local equation of F at its generic point. We then have $\pi^* x_i = z^{b_i} u_i$, where u_i is a unit at the generic point of F and $b_i \in \mathbb{N}$ vanishes for $i > p$. It follows that $\pi^* dx_i = b_i z^{b_i - 1} u_i \, dz + z^{b_i} \, du_i$, and hence

$$\begin{aligned}\mathrm{ord}_F(K_{\mathfrak{X}} - \pi^* K_X) &= \mathrm{ord}_F(K_{X_\pi/X}) = \mathrm{ord}_F\big(\pi^*(dx_1 \wedge \cdots \wedge dx_n)\big)\\ &\geq -1 + \sum_i b_i = -1 + \mathrm{ord}_F \overline{E}.\end{aligned}$$

□

LEMMA 3.3
Let $\phi\colon X \to Y$ be a generically finite dominant morphism between normal varieties. Let ω_Y be a rational top-degree form on Y, let ω_X be its pullback to X, and let $K_{\mathfrak{Y}}$, $K_{\mathfrak{X}}$ be the associated canonical b-divisors. Then we have

$$K_{\mathfrak{X}} + 1_{\mathfrak{X}} = \phi^*(K_{\mathfrak{Y}} + 1_{\mathfrak{Y}}).$$

Proof
Let F be a prime divisor on a smooth model Y' over Y, and pick a smooth model X' over X such that ϕ lifts to a morphism $\phi'\colon X' \to Y'$. The model X' can be constructed by taking a desingularization of the graph of the rational map $X \dashrightarrow Y'$. Let E be a prime divisor on X' with $\phi'(E) = F$. We then have $\phi_* \mathrm{ord}_E = b\,\mathrm{ord}_F$ with $b :=$

$\mathrm{ord}_E(\phi'^* F)$. The same computation as above shows that the ramification order of ϕ' at the generic point of E is equal to $b-1$, so that we have

$$\mathrm{ord}_E\big(K_{X'} - (\phi')^* K_{Y'}\big) = b - 1.$$

It follows that

$$\mathrm{ord}_E(K_{X'}) = b\,\mathrm{ord}_F(K_{Y'}) + b - 1;$$

that is,

$$\mathrm{ord}_E(K_{\mathfrak{X}} + 1_{\mathfrak{X}}) = (b\,\mathrm{ord}_F)(K_{\mathfrak{Y}} + 1_{\mathfrak{Y}}),$$

as was to be shown. □

Definition 3.4
The *m-limiting log-discrepancy b-divisor* $A_{m,\mathfrak{X}/X}$ and the *log-discrepancy b-divisor* $A_{\mathfrak{X}/X}$ are the Weil b-divisors defined by

$$A_{m,\mathfrak{X}/X} := K_{m,\mathfrak{X}/X} + 1_{\mathfrak{X}/X}$$

and

$$A_{\mathfrak{X}/X} := K_{\mathfrak{X}/X} + 1_{\mathfrak{X}/X}.$$

Note that $\lim_{m\to\infty} A_{m,\mathfrak{X}/X} = A_{\mathfrak{X}/X}$ coefficient-wise.

If $\phi\colon X \to Y$ is a finite dominant morphism, recall that the *ramification divisor* R_ϕ is the effective Weil divisor on X such that

$$K_X = \phi^* K_Y + R_\phi,$$

where K_Y and K_X are defined by ω_Y and $\phi^*\omega_Y$, respectively, the divisor R_ϕ being again independent of the choice of ω_Y.

COROLLARY 3.5
Let $\phi : X \to Y$ be a finite dominant morphism between normal varieties. Then we have

$$0 \le \mathrm{Env}_X(R_\phi) \le \phi^* A_{\mathfrak{Y}/Y} - A_{\mathfrak{X}/X} \le -\mathrm{Env}_X(-R_\phi)$$

and the second (resp., third) inequality is an equality when X (resp., Y) is numerically Gorenstein.

Proof
Since ϕ is finite, we have

$$\begin{aligned}\phi^* A_{\mathfrak{Y}/Y} - A_{\mathfrak{X}/X} &= \phi^*(K_{\mathfrak{Y}/Y} + 1_{\mathfrak{Y}}) - (K_{\mathfrak{X}/X} + 1_{\mathfrak{X}}) \\ &= \phi^* \operatorname{Env}_Y(-K_Y) - \operatorname{Env}_X(-K_X) \\ &= \operatorname{Env}_X(-\phi^* K_Y) - \operatorname{Env}_X(-K_X)\end{aligned}$$

by Lemma 3.3 and Proposition 2.19. Now we have on the one hand

$$\operatorname{Env}_X(-\phi^* K_Y) = \operatorname{Env}_X(-K_X + R_\phi) \geq \operatorname{Env}_X(-K_X) + \operatorname{Env}_X(R_\phi),$$

and this is an equality when X is numerically Gorenstein by Lemma 2.28. On the other hand,

$$\operatorname{Env}_X(-K_X) = \operatorname{Env}_X(-\phi^* K_Y - R_\phi) \geq \operatorname{Env}_X(-\phi^* K_Y) + \operatorname{Env}_X(-R_\phi),$$

which is an equality if Y is numerically Gorenstein by Proposition 2.19 and Lemma 2.28. The result follows, noting that $\operatorname{Env}(R_\phi) \geq 0$ since $R_\phi \geq 0$. ☐

3.2. *Multiplier ideals*

The following definition is a straightforward extension of the usual notion of multiplier ideals with respect to a pair.

Definition 3.6
Let Δ be an effective $\mathbb{R}$-boundary on X, and let C be an $\mathbb{R}$-Cartier b-divisor. We define the *multiplier ideal sheaf* of C with respect to (X, Δ) as the fractional ideal sheaf

$$\mathcal{J}\big((X,\Delta); C\big) := \mathcal{O}_X(\lceil K_{\mathfrak{X}/(X,\Delta)} + C \rceil).$$

We have in particular

$$\mathcal{J}\big((X,\Delta); C\big) \subset \mathcal{O}_X(\lceil C_X - \Delta_X \rceil),$$

which shows that the (fractional) multiplier ideal is an actual ideal as soon as $C_X \leq 0$. By Lemma 3.2 we have

$$\mathcal{J}\big((X,\Delta); C\big) = \pi_* \mathcal{O}_{X_\pi}\big(\lceil K_{X_\pi} - \pi^*(K_X + \Delta) + C_\pi \rceil\big)$$

for each joint log-resolution π of (X, Δ) and C. This shows in particular that $\mathcal{J}((X,\Delta); C)$ is coherent, and in the case $C = Z(\mathfrak{a}^c)$ for a coherent ideal sheaf $\mathfrak{a}$ and $c > 0$ we recover

$$\mathcal{J}\big((X,\Delta); Z(\mathfrak{a}^c)\big) = \mathcal{J}\big((X,\Delta); \mathfrak{a}^c\big),$$

where the right-hand side is defined in [Laz, Definition 9.3.56].

We similarly introduce the following straightforward generalization of the notion of multiplier ideal defined in [dFH].

Definition 3.7
Let C be an $\mathbb{R}$-Cartier b-divisor over X.

- For each positive integer m the *m-limiting multiplier ideal sheaf* of C is the fractional ideal sheaf
$$\mathcal{J}_m(C) := \mathcal{O}_X(\lceil K_{m,\mathfrak{X}/X} + C\rceil).$$
- The *multiplier ideal sheaf* $\mathcal{J}(C)$ is the unique maximal element in the family of fractional ideal sheaves $\mathcal{J}_m(C)$, $m \geq 1$.

Here again Lemma 3.2 implies that

$$\mathcal{J}_m(C) = \pi_*\mathcal{O}_{X_\pi}\Big(\Big\lceil K_{X_\pi} + \frac{1}{m} Z\big(\mathcal{O}_X(-mK_X)\big)_\pi + C_\pi\Big\rceil\Big)$$

for each joint log-resolution π of $\mathcal{O}_X(-mK_X)$ and C, which shows in particular that $\mathcal{J}_m(C)$ is coherent. We also have

$$\mathcal{J}_m(C) \subset \mathcal{O}_X(\lceil C_X\rceil),$$

which implies the existence of a unique maximal element in the set of fractional ideals $\{\mathcal{J}_m(C), m \geq 1\}$, by using as usual

$$\frac{1}{lm} Z\big(\mathcal{O}_X(-lmK_X)\big) \geq \max\Big(\frac{1}{m} Z\big(\mathcal{O}_X(-mK_X)\big), \frac{1}{l} Z\big(\mathcal{O}_X(-lK_X)\big)\Big).$$

As in [dFH] we now relate the above notions of multiplier ideals, obtaining in particular a more precise version of [dFH, Theorem 5.4].

THEOREM 3.8
Assume that X is quasi-projective, let C be an $\mathbb{R}$-Cartier b-divisor, and let $m \geq 2$. Then we have

$$\mathfrak{d}(mK_X) = \sum_{\Delta} \mathcal{O}_X(-m\Delta),$$

where Δ ranges over the set of all effective m-boundaries such that

$$\mathcal{J}_m(C) = \mathcal{J}\big((X,\Delta); C\big)$$

(so that this set is in particular nonempty).

Proof

Let π be a joint log resolution of $\mathfrak{a}$ and $\mathcal{O}_X(-mK_X)$. By Proposition 2.34 applied to $-mK_X$ we have

$$\mathfrak{d}(mK_X) = \sum_E \mathcal{O}_X(-E),$$

where E ranges over all prime divisors such that $mK_X + E$ is Cartier and $Z(\mathcal{O}_X(E))$ is transverse to π and C. There remains to set $\Delta := (1/m)E$ and to observe that $\lfloor \Delta \rfloor = 0$, so that $\mathcal{J}_m(C) = \mathcal{J}((X,\Delta);C)$ by Lemma 3.9 below. □

LEMMA 3.9

Let C be an $\mathbb{R}$-Cartier b-divisor, let π be a joint log resolution of C and $\mathcal{O}_X(-mK_X)$, and let Δ be an effective m-boundary.

- *We have*

$$\mathcal{J}\big((X,\Delta);C\big) \subset \mathcal{J}_m(C).$$

- *If $\lfloor \Delta \rfloor = 0$ and $Z(\mathcal{O}_X(m\Delta))$ is transverse to π and C, then*

$$\mathcal{J}\big((X,\Delta);C\big) = \mathcal{J}_m(C).$$

Proof

Since $m(K_X + \Delta)$ is Cartier we have

$$\mathcal{O}_X\big(-mK_X\big) = \mathcal{O}_X(m\Delta)\cdot \mathcal{O}_X\big(-m(K_X+\Delta)\big);$$

hence,

$$\frac{1}{m} Z\big(\mathcal{O}_X(-mK_X)\big) = \frac{1}{m} Z\big(\mathcal{O}_X(m\Delta)\big) - \overline{K_X + \Delta}, \tag{4}$$

and the first point follows because $Z(\mathcal{O}_X(m\Delta)) \geq 0$.

Assume now that $\lfloor \Delta \rfloor = 0$ and that $Z(\mathcal{O}_X(m\Delta))$ is transverse to π and C. By (4) we have

$$\begin{aligned}\lceil K_{X_\pi} - \pi^*(K_X+\Delta) + C_\pi \rceil = {} & \left\lceil K_{X_\pi} + \frac{1}{m} Z\big(\mathcal{O}_X(-mK_X)\big)_\pi + C_\pi \right\rceil \\ & - \left\lfloor \frac{1}{m} Z\big(\mathcal{O}_X(m\Delta)\big)_\pi \right\rfloor.\end{aligned}$$

Indeed, by the transversality assumption $(1/m)Z(\mathcal{O}_X(m\Delta)_\pi$ has no common component with C_π and no common component with $K_{X_\pi} + (1/m)Z(\mathcal{O}_X(-mK_X))_\pi$, the latter being π-exceptional by Proposition 2.8. But by transversality we also have $(1/m)Z(\mathcal{O}_X(m\Delta))_\pi = \widehat{\Delta}_\pi$, the strict transform of Δ on X_π, and the result follows since $\lfloor \widehat{\Delta}_\pi \rfloor = 0$. □

As a consequence we get the following extension of [dFH, Corollary 5.5] to b-divisors.

COROLLARY 3.10

Let X be a normal quasi-projective variety, and let $\mathbf{C}$ be an $\mathbb{R}$-Cartier b-divisor.

- *The m-limiting multiplier ideal $\mathcal{J}_m(\mathbf{C})$ is the largest element of the set of multiplier ideals $\mathcal{J}((X,\Delta);\mathbf{C})$ where Δ ranges over all effective m-boundaries on X.*
- *The multiplier ideal $\mathcal{J}(\mathbf{C})$ is the largest element of the set of multiplier ideals $\mathcal{J}((X,\Delta);\mathbf{C})$ where Δ ranges over all effective $\mathbb{Q}$-boundaries on X.*

We will need the following variant of Lemma 3.9.

COROLLARY 3.11

With the same assumption as in Lemma 3.9, if $m \geq 3$, then we can find an effective m-compatible boundary Δ such that

$$\mathcal{J}\Big((X,\Delta);\mathbf{C}+\frac{1}{m}\mathbf{Z}\big(\mathcal{O}_X(-m\Delta)\big)\Big)=\mathcal{J}_m\Big(\mathbf{C}+\frac{1}{m}\mathbf{Z}\big(\mathfrak{d}(mK_X)\big)\Big).$$

Proof

The problem is local, so we can assume that X is affine. Let π be as in the statement of Lemma 3.9. If $f\in\mathfrak{d}(mK_X)$ is a general element, then $\operatorname{ord}_F(f)=\operatorname{ord}_F(\mathfrak{d}(mK_X))$ for every π-exceptional prime divisor F. By Theorem 3.8 and its proof, we can find an effective m-boundary of the form $\Delta=(1/m)E$ where E is a prime divisor, such that $f\in\mathcal{O}_X(-m\Delta)\subset\mathfrak{d}(mK_X)$ and

$$\mathcal{J}\Big((X,\Delta);\mathbf{C}+\frac{1}{m}\mathbf{Z}\big(\mathfrak{d}(mK_X)\big)\Big)=\mathcal{J}_m\Big(\mathbf{C}+\frac{1}{m}\mathbf{Z}\big(\mathfrak{d}(mK_X)\big)\Big).$$

Note that $\operatorname{ord}_F(\mathcal{O}_X(-m\Delta))=\operatorname{ord}_F(\mathfrak{d}(mK_X))$ for every π-exceptional prime divisor F. Thus, bearing in mind that $\mathbf{Z}(\mathfrak{d}(mK_X))$ is exceptional as X is regular in codimension 1, we have

$$\mathbf{Z}\big(\mathcal{O}_X(-mD)\big)_\pi=\mathbf{Z}\big(\mathfrak{d}(mK_X)\big)_\pi-m\widehat{\Delta}_\pi.$$

Since $\widehat{\Delta}_\pi$ does not share any component with $\mathbf{C}_\pi$, and $\lfloor 2\widehat{\Delta}_\pi\rfloor=0$, we see that

$$\begin{aligned}&\Big\lceil K_{X_\pi}-\pi^*(K_X+\Delta)+\mathbf{C}_\pi+\frac{1}{m}\mathbf{Z}\big(\mathcal{O}_X(-m\Delta)\big)\Big\rceil\\&\quad=\Big\lceil K_{X_\pi}-\pi^*(K_X+\Delta)+\mathbf{C}_\pi+\frac{1}{m}\mathbf{Z}\big(\mathfrak{d}(mK_X)\big)\Big\rceil,\end{aligned}$$

which gives

$$\mathcal{J}\Big((X,\Delta); C + \frac{1}{m} Z\big(\mathcal{O}_X(-m\Delta)\big)\Big) = \mathcal{J}\Big((X,\Delta); C + \frac{1}{m} Z\big(\mathfrak{d}(mK_X)\big)\Big).$$

This completes the proof of the corollary. □

Asymptotic multiplier ideals can also be generalized to this setting. For short, we say that a sequence of $\mathbb{R}$-Cartier b-divisors $Z_\bullet = (Z_m)_{m\ge 1}$ is a *bounded graded sequence* if there is an $\mathbb{R}$-Cartier b-divisor B such that $B \ge (1/km)Z_{km} \ge \max\{(1/k)Z_k, (1/m)Z_m)$ for all $m, k \ge 0$. The following definition relies on the Noetherian property.

Definition 3.12

Let Δ be an effective $\mathbb{R}$-boundary on X, let C be an $\mathbb{R}$-Cartier b-divisor, and let $Z_\bullet = (Z_m)_{m\ge 1}$ be a bounded graded sequence of $\mathbb{R}$-Cartier b-divisors.

- The *asymptotic multiplier ideal sheaf* $\mathcal{J}((X,\Delta); C + Z_\bullet)$ with respect to (X,Δ) is the unique maximal element in the family of multiplier ideal sheaves $\mathcal{J}((X,\Delta); C + (1/k)Z_k)$, $k \ge 1$.
- The *asymptotic multiplier ideal sheaf* $\mathcal{J}(C + Z_\bullet)$ is the unique maximal element in the family of multiplier ideal sheaves $\mathcal{J}(C + (1/k)Z_k)$, $k \ge 1$.

LEMMA 3.13

We have $\mathcal{J}(C + Z_\bullet) = \mathcal{J}_m(C + (1/m)Z_m)$ for every sufficiently divisible m.

Proof

We have $\mathcal{J}(C + Z_\bullet) = \mathcal{J}(C + (1/p)Z_p)$ for every sufficiently divisible p. If we fix any such p, then we have $\mathcal{J}(C + (1/p)Z_p) = \mathcal{J}_m(C + (1/p)Z_p)$ for every sufficiently divisible m. In particular, if we pick m to be a multiple of p, then we have

$$\begin{aligned}\mathcal{J}(C + Z_\bullet) = \mathcal{J}\Big(C + \frac{1}{p} Z_p\Big) = \mathcal{J}_m\Big(C + \frac{1}{p} Z_p\Big) \subset \mathcal{J}_m\Big(C + \frac{1}{m} Z_m\Big) \\ \subset \mathcal{J}\Big(C + \frac{1}{m} Z_m\Big) \subset \mathcal{J}(C + Z_\bullet).\end{aligned}$$

The lemma follows. □

In the case $C = cZ(\mathfrak{a})$ for some $c \ge 0$ and some nonzero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$, and $Z_k = dZ(\mathfrak{b}_k)$ for some $d \ge 0$ and some graded sequence of ideal sheaves $\mathfrak{b}_\bullet = (\mathfrak{b}_m)_{m\ge 0}$, then we also use the notation

$$\mathcal{J}\big((X,\Delta); \mathfrak{a}^c \cdot \mathfrak{b}_\bullet^d\big), \qquad \mathcal{J}_m(\mathfrak{a}^c \cdot \mathfrak{b}_\bullet^d), \qquad \mathcal{J}(\mathfrak{a}^c \cdot \mathfrak{b}_\bullet^d),$$

to denote $\mathcal{J}((X,\Delta); C + Z_\bullet)$, $\mathcal{J}_m(C + Z_\bullet)$, and $\mathcal{J}(C + Z_\bullet)$, respectively.

PROPOSITION 3.14
For every nonzero ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$*, we have* $\mathfrak{a} \cdot \mathcal{J}(\mathcal{O}_X) \subset \mathcal{J}(\mathfrak{a})$.

Proof
Let $f = gh$, with $g \in \mathfrak{a}$ and $h \in \mathcal{J}(\mathcal{O}_X)$. Then $Z(f) = Z(g) + Z(h) \leq Z(g) + K_{m,X_m/X}$ for every $m \geq 1$, which implies the statement. □

3.3. Subadditivity and approximation
Recall that the Jacobian ideal sheaf $\mathrm{Jac}_X \subset \mathcal{O}_X$ of X is defined as the nth Fitting ideal $\mathrm{Fitt}^n(\Omega^1_X)$ with $n = \dim X$.

Takagi obtained in [Tak2] the following general subadditivity result for multiplier ideals with respect to a pair.

THEOREM 3.15 ([Tak2])
Let X *be a normal variety, and let* Δ *be an effective* $\mathbb{Q}$*-Weil divisor such that* $m(K_X + \Delta)$ *is Cartier for some integer* $m > 0$*. If* $\mathfrak{a}, \mathfrak{b}$ *are two nonzero coherent ideal sheaves on* X *and* $c, d \geq 0$*, then we have*

$$\mathrm{Jac}_X \cdot \mathcal{J}\big((X,\Delta); \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-m\Delta)^{1/m}\big) \subset \mathcal{J}\big((X,\Delta); \mathfrak{a}^c\big) \cdot \mathcal{J}\big((X,\Delta); \mathfrak{b}^d\big).$$

Note that when X is smooth and $\Delta = 0$ the statement reduces to the original subadditivity theorem of [DEL]. Takagi gives two independent proofs of this result. The first one is based on positive characteristic techniques and relies on the corresponding statement for test ideals. The other one builds on the work of Eisenstein [Eis] and relies on Hironaka's desingularization theorem.

We now show how to deduce from Takagi's result a subadditivity theorem for multiplier ideals in the sense of [dFH].

THEOREM 3.16 (Subadditivity)
Let X *be a normal variety. If* $\mathfrak{a}, \mathfrak{b}$ *are two nonzero coherent ideal sheaves on* X *and* $c, d \geq 0$*, then we have*

$$\mathrm{Jac}_X \cdot \mathcal{J}\big(\mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathfrak{d}_\bullet(K_X)\big) \subset \mathcal{J}(\mathfrak{a}^c) \cdot \mathcal{J}(\mathfrak{b}^d).$$

The results in [Tak1] and [Sch], combined, suggest the possibility that the correction term $\mathfrak{d}_\bullet(K_X)$ in the left-hand side might be unnecessary.

Proof
By Lemma 3.13 we have

$$\mathcal{J}\big(\mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathfrak{d}_\bullet(K_X)\big) = \mathcal{J}_m\big(\mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathfrak{d}(mK_X)^{1/m}\big)$$

for every sufficiently divisible m. Fix any such m; we can assume that $m \geq 3$. By Corollary 3.11, we can find an effective m-compatible boundary Δ such that

$$\mathcal{J}_m\big(\mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathfrak{d}(mK_X)^{1/m}\big) = \mathcal{J}\big((X,\Delta); \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-m\Delta)^{1/m}\big).$$

Now we apply Theorem 3.15 to get the inclusion

$$\mathcal{J}\big((X,\Delta); \mathfrak{a}^c \cdot \mathfrak{b}^d \cdot \mathcal{O}_X(-m\Delta)^{1/m}\big) \subset \mathcal{J}\big((X,\Delta); \mathfrak{a}^c\big) \cdot \mathcal{J}\big((X,\Delta); \mathfrak{b}^d\big).$$

We conclude by observing that $\mathcal{J}((X,\Delta); \mathfrak{a}^c) \subset \mathcal{J}_m(\mathfrak{a}^c) \subset \mathcal{J}(\mathfrak{a}^c)$, and the similar statement for $\mathfrak{b}^d$ holds at any rate, by Lemma 3.9. □

THEOREM 3.17

Let X be a normal variety, and let $\mathfrak{a}_\bullet$ be a graded sequence of ideal sheaves on X. Then we have

$$Z(\mathrm{Jac}_X) + Z\big(\mathfrak{d}_\bullet(K_X)\big) \leq Z\big(\mathcal{J}(\mathfrak{a}_\bullet)\big) - Z(\mathfrak{a}_\bullet) \leq A_{\mathfrak{X}/X}.$$

In particular $(1/k)Z(\mathcal{J}(\mathfrak{a}_\bullet^k)) \to Z(\mathfrak{a}_\bullet)$ coefficient-wise as $k \to \infty$, uniformly with respect to $\mathfrak{a}_\bullet$.

This result is an extension to the singular case of [BFJ1, Proposition 3.18], which was in turn a direct elaboration of the main result of [ELS].

Proof

For each $k \geq 1$ we have

$$Z\big(\mathcal{J}(\mathfrak{a}_k^{1/k})\big) \leq \frac{1}{k} Z(\mathfrak{a}_k) + A_{\mathfrak{X}/X}$$

by definition of multiplier ideals, and the right-hand inequality follows.

Regarding the other inequality, let for short $\mathfrak{d}_\bullet = (\mathfrak{d}_m)_{m \geq 0} := \mathfrak{d}_\bullet(K_X)$. A recursive application of Theorem 3.16 yields

$$\mathrm{Jac}_X^{k-1} \cdot \mathcal{J}(\mathfrak{a}_k \cdot \mathfrak{d}_\bullet^{k-1}) \subset \mathcal{J}(\mathfrak{a}_k^{1/k})^k.$$

On the other hand, by Proposition 3.14 and the definition of asymptotic multiplier ideal, we have

$$\mathfrak{a}_k \cdot \mathfrak{d}_{k-1} \cdot \mathcal{J}(\mathcal{O}_X) \subset \mathcal{J}(\mathfrak{a}_k \cdot \mathfrak{d}_{k-1}) \subset \mathcal{J}(\mathfrak{a}_k \cdot \mathfrak{d}_\bullet^{k-1}).$$

In terms of b-divisors, this gives

$$(k-1)Z(\mathrm{Jac}_X) + Z(\mathfrak{a}_k) + Z(\mathfrak{d}_{k-1}) + Z\big(\mathcal{J}(\mathcal{O}_X)\big) \leq kZ\big(\mathcal{J}(\mathfrak{a}_k^{1/k})\big).$$

We conclude by dividing by k and letting $k \to \infty$. □

4. Normal isolated singularities

From now on X has an *isolated* normal singularity at a given point $0 \in X$, and $\mathfrak{m} \subset \mathcal{O}_X$ denotes the maximal ideal of zero. We first show how to extend to this setting the intersection theory of nef b-divisors introduced in the smooth case in [BFJ1]. The main ingredient to do so is the approximation theorem from the previous section. We next define the *volume* of $(X, 0)$ as the self-intersection of the nef envelope of the log-canonical b-divisor.

4.1. b-divisors over zero

Observe that every Weil b-divisor W over X decomposes in a unique way as a sum

$$W = W^0 + W^{X \setminus 0},$$

where all irreducible components of W^0 have center zero, and none of $W^{X \setminus 0}$ have center zero. If $W = W^0$, then we say that W *lies over zero*, and we denote by

$$\mathrm{Div}(\mathfrak{X}, 0) \subset \mathrm{Div}(\mathfrak{X})$$

the subspace of all Weil b-divisors over $0 \in X$. An element of $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}, 0)$ is the same thing as a real-valued homogeneous function on the set of divisorial valuations on X centered at zero.

Example 4.1

For every coherent ideal sheaf $\mathfrak{a}$ on X we have

$$Z(\mathfrak{a})^0 = \lim_{k \to \infty} Z(\mathfrak{a} + \mathfrak{m}^k).$$

On the other hand, we say that a Cartier b-divisor $C \in \mathrm{CDiv}(\mathfrak{X})$ is *determined* over zero if it admits a determination π which is an *isomorphism away from zero*, and we say that C is a *Cartier b-divisor over zero* if C furthermore lies over zero. We denote by $\mathrm{CDiv}(\mathfrak{X}, 0)$ the space of Cartier b-divisors over zero. There is an inclusion

$$\mathrm{CDiv}(\mathfrak{X}, 0) \subset \mathrm{CDiv}(\mathfrak{X}) \cap \mathrm{Div}(\mathfrak{X}, 0),$$

but this is in general not an equality. The following example was kindly suggested to us by Fulger.

Example 4.2

Consider $(X, 0) = (\mathbb{C}^3, 0)$. Let $f : Y \to X$ be the morphism given by first taking the blowup $f_1 : Y_1 \to X$ along a line L passing through zero, and then taking the blowup $f_2 : Y \to Y_1$ at a point p on the fiber of f_1 over zero. Let E be the exceptional divisor of f_1, and let D be the exceptional divisor of f_2. Note that D lies over zero. We claim that the Cartier b-divisor $\overline{D}$ cannot be determined over zero. If that were the case, then there would exist a model $X' \to X$ that is an isomorphism outside zero, and a divisor D' on X' such that $\overline{D} = \overline{D'}$ as b-divisors over X. To show that this is impossible, consider two sections of the $\mathbb{P}^1$-bundle $E \to L$ induced by f_1, the second one passing through p but not the first, and let C_0 and C_1 be their respective proper transforms on Y, so that $D \cdot C_i = i$. If L' is the proper transform of L on X', then projection formula yields $D \cdot C_i = D' \cdot L'$, and thus $D \cdot C_0 = D \cdot C_1$. This gives a contradiction.

Remark 4.3
The previous example can be understood torically. Consider in general $(X, 0) = (\mathbb{C}^n, 0)$. It is a toric variety defined by the regular fan Δ_0 in $\mathbb{R}^n$ having the canonical basis as vertices. Any proper birational toric modification $\pi\colon X(\Delta) \to \mathbb{C}^n$ is determined by a refinement Δ of Δ_0. We assume $X(\Delta)$ to be smooth. Denote by $V(\sigma)$ the torus invariant subvariety of $X(\Delta)$ associated to a face σ of Δ. For any vertex v of Δ, let $D(v)$ be the Cartier b-divisor determined in $X(\Delta)$ by the divisor $V(\mathbb{R}_+ v)$. Observe that for any face σ of Δ, we have $\pi(V(\sigma)) = 0$ if and only if σ is included in the open cone $(\mathbb{R}^*)^n_+$. Whence $D(v)$ lies over zero if and only if $v \in (\mathbb{R}^*)^n_+$. And $D(v)$ is determined over zero if and only if any face of Δ containing v is included in $(\mathbb{R}^*)^n_+$.

Example 4.4
Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal. Then $Z(\mathfrak{a})$ is determined over zero as soon as $\mathfrak{a}$ is locally principal outside zero since the normalized blow-up of X along $\mathfrak{a}$ is then an isomorphism away from zero. If $\mathfrak{a}$ is furthermore $\mathfrak{m}$-primary, then $Z(\mathfrak{a})$ is a Cartier b-divisor over zero.

Definition 4.5
We shall say than an $\mathbb{R}$-Weil b-divisor W over zero is *bounded below* if there exists $c > 0$ such that $W \geq c Z(\mathfrak{m})$.

Recall that $Z(\mathfrak{m}) \leq 0$, so that the condition means that the function $\nu \mapsto \nu(W)/\nu(\mathfrak{m})$ is bounded below on the set of divisorial valuations centered at zero.

PROPOSITION 4.6
$(A_{\mathfrak{X}/X})^0$ is bounded below.

Proof
Since $Z(\mathcal{O}_X(-K_X)) \leq \mathrm{Env}_X(K_X)$ by the definition of nef envelope, it follows that $A_{\mathfrak{X}/X} \geq A_{1,\mathfrak{X}/X}$, and hence it suffices to check that $(A_{1,\mathfrak{X}/X})^0$ is bounded below. Let

π be a resolution of the singularity of X, chosen to be an isomorphism away from zero. For each divisorial valuation ν centered at zero we have

$$\nu(A_{1,\mathfrak{X}/X}) = \nu\big((K_{\mathfrak{X}} + 1_{\mathfrak{X}}) - \overline{K_{X_\pi}}\big) + \nu\big(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\big).$$

The first term in the right-hand side is nonnegative since it is equal to the log discrepancy of the smooth variety X_π along ν. On the other hand the Cartier b-divisor $\big(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\big)$ is determined over zero since $\mathcal{O}_X(-K_X)$ is locally principal outside zero by assumption (cf. Example 4.4) and it also lies over zero by Proposition 2.8. We thus see that

$$\big(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\big) \in \mathrm{CDiv}(\mathfrak{X}, 0)$$

and we conclude by Lemma 4.7 below. □

LEMMA 4.7
Every $C \in \mathrm{CDiv}(\mathfrak{X}, 0)$ is bounded below.

Proof
Let π be a determination of C which is an isomorphism away from zero. The result follows directly from the fact that $Z(\mathfrak{m})_\pi$ contains every π-exceptional prime divisor E in its support (since ord_E is centered at zero). □

4.2. Nef b-divisors over zero
We see that an $\mathbb{R}$-Weil b-divisor over zero is *nef* if its class in $N^1(\mathfrak{X}/X)$ is X-nef. If W is an $\mathbb{R}$-Weil b-divisor over zero that is bounded below, then $\mathrm{Env}_{\mathfrak{X}}(W)$ is well defined, nef, and it lies over zero.

By a result of Izumi [Izu] for every two divisorial valuations ν, ν' on X centered at zero, there is a constant $c = c(\nu, \nu') > 0$ such that

$$c^{-1}\nu(f) \le \nu'(f) \le c\nu(f)$$

for every $f \in \mathcal{O}_X$. This result extends to nef b-divisors by approximation.

THEOREM 4.8
Given two divisorial valuations ν, ν' centered at zero there exists $c > 0$ such that

$$c\nu(W) \le \nu'(W) \le c^{-1}\nu(W)$$

for every X-nef $\mathbb{R}$-Weil b-divisor W such that $W \le 0$ (which amounts to $W_X \le 0$ by the negativity lemma).

Proof
Since $\mathrm{Env}_\pi(W_\pi)$ decreases coefficient-wise to W as $\pi \to \infty$ by Proposition 2.15, it is enough to treat the case, where $W = \mathrm{Env}_{\mathfrak{X}}(C)$ for some $\mathbb{R}$-Cartier b-divisor $C \leq 0$. But we then have

$$W = \lim_{m\to\infty} \frac{1}{m} Z\big(\mathcal{O}_X(mC)\big)$$

with $\mathcal{O}_X(mC) \subset \mathcal{O}_X$, so we are reduced to the case of an ideal, for which the result directly follows from Izumi's theorem. □

COROLLARY 4.9
For each X-nef $\mathbb{R}$-Weil b-divisor W such that $W \leq 0$ and $W^0 \neq 0$ there exists $\varepsilon > 0$ such that

$$W \leq \varepsilon Z(\mathfrak{m}).$$

Proof
Since $W^0 \neq 0$ there exists a divisorial valuation ν_0 centered at zero such that $\nu_0(W) < 0$, and it follows that $\nu(W) < 0$ for *all* divisorial valuations centered at zero by Theorem 4.8.

Now let π be the normalized blowup of $\mathfrak{m}$. Since W_π contains each π-exceptional prime in its support there exists $\varepsilon > 0$ such that $W_\pi \leq \varepsilon Z(\mathfrak{m})_\pi$ and the result follows by the negativity lemma. □

For nef envelopes of Weil divisors with integer coefficients this result can be made uniform as follows.

THEOREM 4.10
There exists $\varepsilon > 0$ depending only on X such that

$$\mathrm{Env}_X(-D) \leq \varepsilon Z(\mathfrak{m})$$

for all effective Weil divisors (with integer coefficients) D on X containing zero.

Proof
By Hironaka's resolution of singularities we may choose a smooth birational model X_π which dominates the blow-up of $\mathfrak{m}$ and is isomorphic to X away from zero, and such that there exists a π-ample and π-exceptional Cartier divisor A on X_π. If we denote by $E_1, \dots, E_r$ the π-exceptional prime divisors, then $A = -\sum_j a_j E_j$ with $a_j \geq 1$ by the negativity lemma.

By the negativity lemma the desired result means that there exists $\varepsilon > 0$ such that for each effective Weil divisor D through 0 on X we have

$$\mathrm{Env}_X(-D)_\pi \leq \varepsilon Z(\mathfrak{m})_\pi .$$

If we set $c_j(D) := -\operatorname{ord}_{E_j} \mathrm{Env}_X(-D)$, then in view of Theorem 4.8 this amounts to proving the existence of $\varepsilon > 0$ such that

$$\max_{1\leq j\leq r} c_j(D) \geq \varepsilon$$

for each D. Note that

$$\sum_j c_j(D)E_j = -\mathrm{Env}_X(-D)_\pi - \widehat{D}_\pi \tag{5}$$

by Proposition 2.8. Now we have on the one hand

$$\begin{aligned} -A^{n-1}\cdot \mathrm{Env}_X(-D)_\pi &= \sum a_j E_j \cdot A^{n-2}\cdot \mathrm{Env}_X(-D)_\pi \\ &= \sum_j a_j (A|_{E_j})^{n-2}\cdot \big(\mathrm{Env}_X(-D)_\pi|_{E_j}\big) \geq 0 \end{aligned}$$

since $A|_{E_j}$ is ample and $\mathrm{Env}_X(-D)_\pi|_{E_j}$ is pseudoeffective by Lemma 2.10. On the other hand,

$$-A^{n-1}\cdot \widehat{D}_\pi = \sum_j a_j (A|_{E_j})^{n-2}\cdot (\widehat{D}_\pi|_{E_j}) \geq 1$$

since $\widehat{D}_\pi|_{E_j}$ is an effective Cartier divisor on E_j, and is nonzero for at least one j. We thus get $\sum_j c_j(D)(E_j\cdot A^{n-1}) \geq 1$ from (5), and we infer that

$$\max_j c_j(D) \geq \varepsilon := 1/\max_j (E_j\cdot A^{n-1}). \qquad \square$$

We conclude this section by the following crucial consequence of Theorem 3.17.

THEOREM 4.11
Let $C \in \mathrm{CDiv}(\mathfrak{X},0)$, and set $W := \mathrm{Env}_{\mathfrak{X}}(C)$. Then there exists a sequence of $\mathfrak{m}$-primary ideals $\mathfrak{b}_k$ and a sequence of positive rational numbers $c_k \to 0$ such that

- *$c_k Z(\mathfrak{b}_k) \geq W$ for all k;*
- *$\lim_{k\to\infty} c_k Z(\mathfrak{b}_k) = W$ coefficient-wise.*

Proof
Consider the graded sequence of $\mathfrak{m}$-primary ideals $\mathfrak{a}_m := \mathcal{O}_X(mW) = \mathcal{O}_X(mC)$, and set $\mathfrak{b}_k := \mathcal{J}(\mathfrak{a}_\bullet^k)$. By Theorem 3.17 we have in particular

$$Z(\mathfrak{b}_k) \geq kW + Z\big(\mathfrak{d}(K_X)\big) + Z(\mathrm{Jac}_X)$$

and $(1/k)Z(\mathfrak{b}_k) \to W$ coefficient-wise. Since $0 \in X$ is an isolated singularity we see that both $\mathfrak{d}(K_X)$ and Jac_X are $\mathfrak{m}$-primary ideals and Lemma 4.7 yields $c > 0$ such that

$$Z\big(\mathfrak{d}(K_X)\big) + Z(\mathrm{Jac}_X) \geq cZ(\mathfrak{m}).$$

On the other hand, there exists $\varepsilon > 0$ such that $W \leq \varepsilon Z(\mathfrak{m})$ by Corollary 4.9, and we conclude that there exists $c > 0$ such that

$$Z(\mathfrak{b}_k) \geq kW + cW$$

for all k. There remains to set $c_k := 1/(k + c)$. □

4.3. Intersection numbers of nef b-divisors

We indicate in this subsection how to extend to the singular case the local intersection theory of nef b-divisors introduced in [BFJ1, Section 4] in the smooth case. The main point is to replace the approximation result [BFJ1, Proposition 3.13] by Theorem 4.11.

Let $C_1, \dots, C_n$ be $\mathbb{R}$-Cartier b-divisors over zero. Pick a common determination π which is an isomorphism away from zero, and set

$$C_1 \cdot \dots \cdot C_n := C_{1,\pi} \cdot \dots \cdot C_{n,\pi}.$$

The right-hand side is well defined since $C_{1,\pi}$ has compact support, and it does not depend on the choice of π by the projection formula since $C_{i,\pi'} = \mu^* C_{i,\pi}$ for any higher model $\mu \colon X_{\pi'} \to X_\pi$.

The following property is a direct consequence of the definition of $Z(\mathfrak{a}_i)$ and the formula displayed in [Laz, p. 92].

PROPOSITION 4.12

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset \mathcal{O}_X$ be $\mathfrak{m}$-primary ideals. Then

$$-Z(\mathfrak{a}_1) \cdot \dots \cdot Z(\mathfrak{a}_n) = e(\mathfrak{a}_1, \dots, \mathfrak{a}_n),$$

where $e(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ denotes the mixed multiplicity (see, e.g., [Laz, p. 91] for a definition).

The intersection numbers of *nef* $\mathbb{R}$-Cartier b-divisors $C_1, \dots, C_n$, $C'_1, \dots, C'_n$ over zero satisfy the monotonicity property

$$C_1 \cdot \dots \cdot C_n \leq C'_1 \cdot \dots \cdot C'_n$$

if $C_i \leq C'_i$ for each i.

Definition 4.13
If $W_1, \dots, W_n$ are arbitrary nef $\mathbb{R}$-Weil b-divisors over zero we set

$$W_1 \cdot \dots \cdot W_n := \inf_{C_i \geq W_i} (C_1 \cdot \dots \cdot C_n) \in [-\infty, +\infty[,$$

where the infimum is taken over all nef $\mathbb{R}$-Cartier b-divisors C_i over zero such that $C_i \geq W_i$ for each i.

Note that $(W_1 \cdot \dots \cdot W_n)$ is finite when all W_i are bounded below. This is for instance the case if each W_i is the nef envelope of a Cartier b-divisor by Lemma 4.7.

The next theorem summarizes the main properties of the intersection product. The nontrivial part of the assertion is additivity, which requires the approximation theorem.

THEOREM 4.14
The intersection product $(W_1, \dots, W_n) \mapsto W_1 \cdot \dots \cdot W_n$ of nef $\mathbb{R}$-Weil b-divisors over zero is symmetric, upper semicontinuous, and continuous along monotonic families (for the topology of coefficient-wise convergence).

It is also homogeneous, additive, and nondecreasing in each variable. Furthermore, $W_1 \cdot \dots \cdot W_n < 0$ if $W_i \neq 0$ for each i.

Proof
We follow the same lines as [BFJ1, Proposition 4.4]. Symmetry, homogeneity, and monotonicity are clear. If $W_i \neq 0$ for all i, then there exists $\varepsilon > 0$ such that $W_i \leq \varepsilon Z(\mathfrak{m})$ for all i by Corollary 4.9; hence

$$W_1 \cdot \dots \cdot W_n \leq \varepsilon^n Z(\mathfrak{m})^n = -\varepsilon^n e(\mathfrak{m}) < 0,$$

where $e(\mathfrak{m})$ is the Samuel multiplicity of $\mathfrak{m}$.

Let us prove the semicontinuity. Suppose that $W_i \neq 0$ for all i, and pick $t \in \mathbb{R}$ such that $W_1 \cdot \dots \cdot W_n < t$. By definition there exist nef $\mathbb{R}$-Cartier b-divisors C_i over zero such that $W_i \leq C_i$ and $C_1 \cdot \dots \cdot C_n < t$. Replacing each C_i by $(1-\varepsilon)C_i$ we may assume $C_i \neq W_i$ while still preserving the previous conditions. Now consider the set U_i of all nef b-divisors W_i' such that $W_i' \leq C_i$. This is a neighborhood of W_i in the topology of coefficient-wise convergence, and $(W_1' \cdot \dots \cdot W_n') < t$ for all $W_i' \in U_i$. This proves the upper semicontinuity.

As a consequence we get the following continuity property: for all families $W_{j,k}$ such that

- $W_{j,k} \geq W_j$ for all j, k and
- $\lim_k W_{j,k} = W_j$ for all j,

we have $\lim_k W_{1,k} \cdot \dots \cdot W_{n,k} = W_1 \cdot \dots \cdot W_n$. Indeed $W_{1,k} \cdot \dots \cdot W_{n,k} \geq W_1 \cdot \dots \cdot W_n$ holds by monotonicity, and the claim follows by upper semicontinuity.

We now turn to additivity. Assume first that W', $W_1, W_2, \dots, W_n$ are nef envelopes of Cartier b-divisors over zero. By Theorem 4.11 there exist two sequences C'_k and $C_{j,k}$ of nef Cartier divisors above zero such that $C_{j,k} \geq W_j$ and $C_{j,k} \to W_j$ as $k \to \infty$, and similarly for C'_k and W'. Since $C_{1,k} + C'_k \geq W_1 + W'$ also converges to $W_1 + W'$ the above remark yields

$$(C_{1,k} + C'_k) \cdot C_{2,k} \cdot \dots \cdot C_{n,k} \to (W_1 + W') \cdot W_2 \cdot \dots \cdot W_n.$$

On the other hand, we have

$$(C_{1,k} + C'_k) \cdot C_{2,k} \cdot \dots \cdot C_{n,k} = (C_{1,k} \cdot C_{2,k} \cdot \dots \cdot C_{n,k}) + (C'_k \cdot C_{2,k} \cdot \dots \cdot C_{n,k}),$$

where

$$(C_{1,k} \cdot C_{2,k} \cdot \dots \cdot C_{n,k}) \to (W_1 \cdot W_2 \cdot \dots \cdot W_n)$$

and

$$(C_{1,k} \cdot C_{2,k} \cdot \dots \cdot C_{n,k}) \to (W' \cdot W_2 \cdot \dots \cdot W_n),$$

so we get additivity for nef envelopes.

In the general case, let $W', W_1, W_2, \dots, W_n$ be arbitrary nef b-divisors over zero. We then have $\mathrm{Env}_\pi(W_{j,\pi}) \geq W_j$, and $\mathrm{Env}_\pi(W_{j,\pi})$ is a nonincreasing net converging to W_j by Remark 2.17. The additivity then follows from the previous case and the continuity along decreasing nets.

Finally, the continuity along nondecreasing sequences is the content of Theorem A.1, which is proven in the appendix and will appear in a more general setting in [BFJ2]. □

The expected local Khovanskii–Teissier inequality holds.

THEOREM 4.15
For all nef $\mathbb{R}$-Weil b-divisors $W_1, \dots, W_n$ over zero we have

$$|W_1 \cdot \dots \cdot W_n| \leq |W_1^n|^{1/n} \cdots |W_n^n|^{1/n}. \tag{6}$$

In particular we have

$$|(W_1 + W_2)^n|^{1/n} \leq |W_1^n|^{1/n} + |W_2^n|^{1/n}.$$

Proof
Arguing as in the proof of Theorem 4.14 we may use Theorem 4.11 to reduce to the case where $W_i = Z(\mathfrak{a}_i)$ for some $\mathfrak{m}$-primary ideals $\mathfrak{a}_i$. In that case the result follows from Proposition 4.12 and the local Khovanskii–Teissier inequality (cf. [Laz, Theorem 1.6.7(iii)]). □

PROPOSITION 4.16
Suppose that $\phi\colon (X,0) \to (Y,0)$ *is a finite map of degree* $e(\phi)$*. Then for all nef* $\mathbb{R}$*-Weil b-divisors* $W_1,\dots,W_n$ *over* $0 \in Y$ *we have*

$$(\phi^* W_1)\cdot \cdots \cdot (\phi^* W_n) = e(\phi) W_1 \cdot \cdots \cdot W_n. \tag{7}$$

Proof
Arguing as in the proof of Theorem 4.14 by successive approximation relying on Theorem 4.11, we reduce to the case where each W_j is $\mathbb{R}$-Cartier over zero. Let $\pi\colon Y' \to Y$ be a common determination of the W_j which is an isomorphism away from zero. Since $\phi^{-1}(0) = 0$ there exists a birational morphism $\mu\colon X' \to X$ which is an isomorphism away from zero such that ϕ lifts as a morphism $\phi'\colon X' \to Y'$, whose degree is still equal to $e(\phi)$, and the result follows. □

Remark 4.17
For every graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}$-primary ideals we have

$$-Z(\mathfrak{a}_\bullet)^n = \lim_{k\to\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_X/\mathfrak{a}_k)}{k^n/n!}.$$

Indeed, it was shown by Lazarsfeld and Mustaţă [LM, Theorem 3.8] that the right-hand-side limit exists and coincides with $\lim_{k\to\infty} e(\mathfrak{a}_k)/k^n$ (which corresponds to a local version of the Fujita approximation theorem). On the other hand, $Z(\mathfrak{a}_\bullet)$ is the nondecreasing limit of $(1/k!)Z(\mathfrak{a}_{k!})$; hence, $Z(\mathfrak{a}_\bullet)^n = \lim_{k\to\infty} Z(\mathfrak{a}_k)^n/k^n$ by using the continuity of intersection numbers along the nondecreasing sequence, and the claim follows in view of Proposition 4.12.

4.4. The volume of an isolated singularity
By Proposition 4.6 the log-discrepancy divisor $A_{\mathfrak{X}/X}$ is always bounded below. Its nef envelope $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})$ is therefore well defined and bounded below as well, and we may introduce the following.

Definition 4.18
The *volume* of a normal isolated singularity $(X,0)$ is defined as

$$\mathrm{Vol}(X,0) := -\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})^n.$$

We have the following characterization of singularities with zero volume.

PROPOSITION 4.19
$\mathrm{Vol}(X,0) = 0$ *if and only if* $A_{\mathfrak{X}/X} \geq 0$. *When* X *is* $\mathbb{Q}$*-Gorenstein,* $\mathrm{Vol}(X,0) = 0$ *if and only if it has log-canonical singularities.*

Proof
By Theorem 4.14 we have $\mathrm{Vol}(X,0) = 0$ if and only if $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) = 0$, which is equivalent to $A_{\mathfrak{X}/X} \geq 0$ since every X-nef b-divisor over zero is antieffective by the negativity lemma.

When X is $\mathbb{Q}$-Gorenstein, then $A_{\mathfrak{X}/X} = A_{m,\mathfrak{X}/X}$ for any integer m such that mK_X is Cartier. We conclude recalling that X is log-canonical if the trace of the log-discrepancy divisor $A_{m,\mathfrak{X}/X}$ in one (or equivalently any) log resolution of X is effective. □

The volume satisfies the following basic monotonicity property.

THEOREM 4.20
Let $\phi : (X,0) \to (Y,0)$ *be a finite morphism between normal isolated singularities. Then we have*

$$\mathrm{Vol}(X,0) \geq e(\phi)\,\mathrm{Vol}(Y,0),$$

with equality if ϕ *is étale in codimension* 1.

Proof
We have $A_{\mathfrak{X}/X} \leq \phi^* A_{\mathfrak{Y}/Y}$ by Corollary 3.5, and equality holds if and only if $R_\phi = 0$, that is, if and only if ϕ is étale in codimension 1. The result follows immediately by using Theorem 2.19 and Proposition 4.16. □

4.5. The volume of a cone singularity
In the case of a cone singularity, the volume relates to the positivity of the anticanonical divisor of the exceptional divisor in the following way.

PROPOSITION 4.21
Let $0 \in X$ *be the affine cone over a polarized smooth variety* (V,L) *as in Example 2.31. We assume in particular that* X *is normal.*
(a) *If* $|-mK_V|$ *contains a smooth element for some* $m \geq 1$, *then* $\mathrm{Vol}(X,0) = 0$.
(b) *Conversely, if* $\mathrm{Vol}(X,0) = 0$, *then* $-K_V$ *is pseudoeffective.*

Proof
Denote by $\pi: X_\pi \to X$ the blowup at zero, with exceptional divisor $E \simeq V$. If $D \in |-mK_V|$ is a smooth element, then we consider the pair (X, Δ), where Δ is the cone over D divided by m. Note that π gives a log resolution of (X, Δ) and $K_{X_\pi} + E - \pi^*(K_X + \Delta)$ has order one along E, by adjunction. Therefore (X, Δ) is log-canonical, and hence $A_{m,\mathfrak{X}/X} \geq 0$. This implies that $A_{\mathfrak{X}/X} \geq 0$, and thus $\mathrm{Vol}(X, 0) = 0$ by Proposition 4.19.

Conversely, assume that $\mathrm{Vol}(X, 0) = 0$. We then have $a = \mathrm{ord}_E(A_{\mathfrak{X}/X}) \geq 0$ by Proposition 4.19 and

$$K_{X_\pi} + E + \mathrm{Env}_X(-K_X)_\pi = aE$$

since E is the only π-exceptional divisor. Now $\mathrm{Env}_X(-K_X)_\pi$ restricts to a pseudo-effective class in $N^1(E)$ by Lemma 2.10. The pseudoeffectivity of $-K_E$ follows by adjunction (one can also see that $-K_E$ is big if the *generalized log-discrepancy* a is positive). $\square$

In [Kol3, Chapter 2, Example 55], Kollár gives an example of a family of singular threefolds where the central fiber admits a boundary which makes it into a log-canonical pair while the nearby fibers do not. The same kind of example can be used to show that the volume defined above is not a topological invariant of the *link* of the singularity in general, in contrast with the 2-dimensional case. We are grateful to János Kollár for bringing this example to our attention.

Recall first that a link M of an isolated singularity $0 \in X$ is a compact real-analytic hypersurface of $X \setminus \{0\}$ with the property that X is homeomorphic to the (real) cone over M. It can be constructed as follows (cf., e.g., [Loo, Section 2A]). Let $r: X \to \mathbb{R}_+$ be a real-analytic function defined in a neighborhood of zero such that $r^{-1}(0) = \{0\}$ (for instance, the restriction to X of $\|z\|^2$ in a local analytic embedding in $\mathbb{C}^N$). Upon shrinking X we may assume that r has no criticial point on $X \setminus \{0\}$, and M can then be taken to be any level set $r^{-1}(\varepsilon)$ for $0 < \varepsilon \ll 1$.

If $0 \in X$ is the affine cone over a polarized variety (V, L), then its link M is diffeomorphic to the (unit) circle bundle of any Hermitian metric on L^*. Indeed, we may take the function r to be given by $r(v) = \sum_j |\langle s_j, v^m \rangle|^{2/m}$, where (s_j) is a basis of sections of mL for $m \gg 1$. As a consequence, the links of the cone singularities X_t induced by any smooth family of polarized varieties $(V_t, L_t)_{t \in T}$ are all diffeomorphic—as follows by applying the Ehresmann–Feldbau theorem to the family of circle bundles with respect to a Hermitian metric on L over the total space of the family V_t.

We use the following result.

LEMMA 4.22
Let S_r be the blowup of $\mathbb{P}^2$ at r very general points. Then $-K_{S_r}$ is not pseudoeffective if (and only if) $r \geq 10$.

This fact is certainly well known to experts, but we provide a proof for the convenience of the reader.

Proof
By semicontinuity it is enough to show that $-K_{S_r}$ is not pseudoeffective for the blowup S_r of $\mathbb{P}^2$ at some family of $r \geq 10$ points. We may also reduce to the case $r = 10$ since the anticanonical bundle only becomes less effective when we keep blowing up points.

First, by [Sak, Lemma 3.1], for any rational surface S we have $-K_S$ pseudoeffective if and only if $h^0(-mK_S) > 0$ for some positive integer m. The short proof goes as follows. The nontrivial case is when $-K_S$ is pseudoeffective but not big. Let $-K_S = P + N$ be the Zariski decomposition, which satisfies $P^2 = P \cdot K_S = 0$. By Riemann–Roch it follows that $\chi(mP) = \chi(\mathcal{O}_S) = 1$ for any m such that mP is Cartier. But $h^2(mP) = h^0(K_S - mP) = 0$ because K_S is not pseudoeffective; hence, $h^0(mP) \geq \chi(mP)$, and the result follows.

Second, let S_9 be the blowup of $\mathbb{P}^2$ at 9 very general points p_i of a given smooth cubic curve C with inflection point p. We then have $h^0(-mK_{S_9}) = 1$ for all positive integers m; otherwise we would get $H^0(\mathcal{O}_C(3m)(-m\sum_i p_i)) \neq 0$ by restriction to the strict transform of C, and $9p - \sum_i p_i$ would be m-torsion in $\operatorname{Pic}^0(C) \simeq C$. In other words, we see that mC is the only degree $3m$ curve in $\mathbb{P}^2$ passing through each p_i with multiplicity at least m. If we let p_{10} be any point outside C it follows of course that no degree $3m$ curve passes through $p_1, \ldots, p_{10}$ with multiplicity at least m. But this means that the blowup S_{10} of $\mathbb{P}^2$ at $p_1, \ldots, p_{10}$ has $h^0(-mK_{S_{10}}) = 0$ for all m, so that $-K_{S_{10}}$ is not pseudoeffective by Sakai's lemma. $\square$

We are now in a position to state our example.

Example 4.23
Let T be the parameter space of all sets of r distinct points $\Sigma_t \subset \mathbb{P}^2$, and for each $t \in T$ let V_t be the blowup of $\mathbb{P}^2$ at Σ_t. Let L be a polarization of the smooth projective family $(V_t)_{t \in T}$, and let X_t be the associated family of cone singularities, whose links are all diffeomorphic according to the above discussion. After possibly replacing L by a multiple, we can assume that each X_t is normal.

If for a given $t \in T$ the points Σ_t all lie on a smooth cubic curve, then the anticanonical system $|-K_{V_t}|$ contains the strict transform of that curve, and we thus have

$\mathrm{Vol}(X_t, 0) = 0$ for such values of t by Proposition 4.21. On the other hand, Proposition 4.21 and Lemma 4.22 show that $\mathrm{Vol}(X_t, 0) > 0$ for $t \in T$ very general.

5. Comparison with other invariants of isolated singularities

5.1. Wahl's characteristic number

As recalled in the introduction, Wahl [Wa] defined the *characteristic number* of a normal surface singularity $(X, 0)$ as $-P^2$ of the nef part P in the Zariski decomposition of $K_{X_\pi} + E$, where $\pi : X_\pi \to X$ is any log resolution of $(X, 0)$ and E is the reduced exceptional divisor of π. The following result proves that the volume defined above extends Wahl's invariant to all isolated normal singularities.

PROPOSITION 5.1
If $(X, 0)$ *is a normal surface singularity, then* $\mathrm{Vol}(X, 0)$ *coincides with Wahl's characteristic number.*

Proof
Let $\pi : X_\pi \to X$ be the log resolution of $(X, 0)$, and let E be its reduced exceptional divisor. By Theorem 2.22 we see that $\mathrm{Env}_\pi(A_{X_\pi/X})$ coincides with the nef part of $K_{X_\pi} + E - \pi^* K_X$. Since the latter is π-numerically equivalent to $K_{X_\pi} + E$ it follows that $\mathrm{Env}_\pi(A_{X_\pi/X})$ is π-numerically equivalent to the nef part P of $K_{X_\pi} + E$, so that

$$-P^2 = -\mathrm{Env}_\pi(A_{X_\pi/X})^2.$$

On the other hand, we claim that $\mathrm{Env}_\pi(A_{X_\pi/X}) = \mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})$, which concludes the proof. Indeed on the one hand we have

$$\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) \leq \mathrm{Env}_\pi(A_{X_\pi/X})$$

as for any Weil b-divisor. On the other hand, Lemma 3.2 implies that

$$K_{\mathfrak{X}} + 1_{\mathfrak{X}} \geq \overline{K_{X_\pi} + E}$$

over zero; hence, $A_{\mathfrak{X}/X} \geq \overline{A_{X_\pi/X}}$, and we infer $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) \geq \mathrm{Env}_\pi(A_{X_\pi/X})$ as desired. □

Proof of Theorem A
The definition of the volume is given in Section 4.4. Theorem A(i) is precisely Theorem 4.20. Statement (ii) is Proposition 5.1. Statement (iii) is Proposition 4.19. □

5.2. Plurigenera and Fulger's volume

Let $0 \in X$ be (a germ of) an isolated singularity, and let $\pi : X_\pi \to X$ be a log resolution with reduced exceptional SNC divisor E. One may then consider the following plurigenera (see [Ish] for a review):

- Knöller's plurigenera (see [Knö]), defined by

$$\gamma_m(X,0) := \dim H^0(X_\pi \setminus E, mK_{X_\pi})/H^0(X_\pi, mK_{X_\pi});$$

- Watanabe's L^2-plurigenera (see [Wat1]), defined by

$$\delta_m(X,0) := \dim H^0(X_\pi \setminus E, mK_{X_\pi})/H^0\big(X_\pi, mK_{X_\pi} + (m-1)E\big);$$

- Morales's log-plurigenera (see [Mor, Definition 0.5.4]), defined by

$$\lambda_m(X,0) := \dim H^0(X_\pi \setminus E, mK_{X_\pi})/H^0\big(X_\pi, m(K_{X_\pi} + E)\big).$$

These numbers do not depend on the choice of log resolution. They satisfy

$$\lambda_m(X,0) \le \delta_m(X,0) \le \gamma_m(X,0) = O(m^n),$$

and one may use them to define various notions of the Kodaira dimension of an isolated singularity.

In a recent work, Fulger [Fulg] has explored in more detail the growth of these numbers. His framework is the following. Given a Cartier divisor D on X_π, consider the local cohomological dimension

$$h^1_{\{0\}}(D) = \dim H^0(X_\pi \setminus E, D)/H^0(X_\pi, D) = \dim \mathcal{O}_X(\pi_* D)/\mathcal{O}_X(D).$$

Observe that $\gamma_m(X,0) = h^1_{\{0\}}(mK_{X_\pi})$ and $\lambda_m(X,0) = h^1_{\{0\}}(m(K_{X_\pi} + E))$. Fulger proves that $h^1_{\{0\}}(mD) = O(m^n)$ and defines the *local volume* of D by setting

$$\mathrm{vol}_{\mathrm{loc}}(D) := \limsup_{m\to\infty} \frac{n!}{m^n} h^1_{\{0\}}(mD).$$

When the Cartier divisor D lies over zero one has the following.

PROPOSITION 5.2

Suppose that D is a Cartier divisor in X_π lying over zero. Then

$$\mathrm{vol}_{\mathrm{loc}}(D) = -\operatorname{Env}_{\mathfrak{X}}(\overline{D})^n.$$

Proof

We may assume $D \le 0$. The envelope of D is the b-divisor associated to the graded sequence of $\mathfrak{m}$-primary ideals $\mathcal{O}_X(-mD)$. The result follows from Remark 4.17. $\square$

Fulger [Fulg] then introduces an alternative notion of volume of an isolated singularity by setting

$$\mathrm{Vol}_F(X,0) := \mathrm{vol}_{\mathrm{loc}}(K_{X_\pi} + E).$$

PROPOSITION 5.3
We have $\operatorname{Vol}(X,0) = \operatorname{Vol}_F(X,0)$ *if* X *is* $\mathbb{Q}$*-Gorenstein.*

Proof
For any integer m such that mK_X is Cartier, one has $A_{\mathfrak{X}/X} = A_{m,\mathfrak{X}/X}$. Pick any log-resolution $\pi: X_\pi \to X$. Then Lemma 3.2 applied to X_π shows that $\overline{A_{X_\pi/X}} \le A_{\mathfrak{X}/X}$. In particular, these b-divisors share the same envelope. We conclude by Proposition 5.2 above. □

In general, Fulger proves that there is always an inequality

$$\operatorname{Vol}(X,0) \ge \operatorname{Vol}_F(X,0).$$

We know by Wahl [Wa] that in dimension 2 these volumes always coincide. In higher dimension these two invariants may, however, differ, as shown by the following example.

Example 5.4
Let V be any smooth projective variety such that neither K_V nor $-K_V$ is pseudoeffective, for instance $V = C \times \mathbb{P}^1$, where C is a curve of genus at least 2. Pick any ample line bundle L on V such that the the affine cone $0 \in X$ over (V,L) is normal. We claim that

$$\operatorname{Vol}(X,0) > 0 = \operatorname{Vol}_F(X,0).$$

Indeed, Proposition 4.21 and the fact that $-K_V$ is not pseudoeffective show that $\operatorname{Vol}(X,0) > 0$. On the other hand, the fact that K_V is not pseudoeffective implies that $\delta_m(X,0) = 0$ for all m and hence $\operatorname{Vol}_F(X,0) = 0$. To see this, let $\pi: X_\pi \to X$ be the blowup of zero, with exceptional divisor $E \simeq V$. Since L is ample, $mK_V - (p-m)L$ is not pseudoeffective for any $p \ge m$; hence,

$$H^0\big(E, mK_E + (p-m)E|_E\big) \simeq H^0\big(V, mK_V - (p-m)L\big) = 0.$$

Now $(K_{X_\pi} + E)|_E = K_E$ by adjunction, and the restriction morphism

$$\begin{aligned} H^0(X_\pi, mK_{X_\pi} + pE)/H^0\big(X_\pi, mK_{X_\pi} + (p-1)E\big) \\ \to H^0\big(E, mK_E + (p-m)E|_E\big) \end{aligned}$$

is injective. We have thus shown $H^0(X_\pi, mK_{X_\pi} + (m-1)E) = H^0(X_\pi, mK_{X_\pi} + pE)$ for all $p \ge m$; hence, $H^0(X_\pi, mK_{X_\pi} + (m-1)E) = H^0(X_\pi \setminus E, mK_{X_\pi})$, that is, $\delta_m(X,0) = 0$.

6. Endomorphisms

We apply the previous analysis to the study of normal isolated singularities admitting endomorphisms.

6.1. *Proofs of Theorems B and C*

We start by proving the following result.

THEOREM 6.1

Assume that X is numerically Gorenstein, and let $\phi : (X,0) \to (X,0)$ be a finite endomorphism of degree $e(\phi) \geq 2$ such that $R_\phi \neq 0$. Then there exists $\varepsilon > 0$ such that $A_{\mathfrak{X}/X} \geq -\varepsilon Z(\mathfrak{m})$.

Remark 6.2

When X is $\mathbb{Q}$-Gorenstein or $\dim X = 2$, the condition $A_{\mathfrak{X}/X} \geq -\varepsilon Z(\mathfrak{m})$ for some $\varepsilon > 0$ is equivalent to $A_{m,\mathfrak{X}/X} > 0$ for some m. By Corollary 3.10 the latter condition means in turn that X has klt singularities in the sense that there exists a $\mathbb{Q}$-boundary Δ such that (X, Δ) is klt. It is possible to prove this result unconditionnally; we shall return to this problem in a later work.

Remark 6.3

Tsuchihashi's cusp singularities (see below) show that the assumption $R_\phi \neq 0$ is essential even when K_X is Cartier.

Proof

Since X is numerically Gorenstein, $R_{\phi^k} = K_X - (\phi^k)^* K_X$ is numerically Cartier for each k and Corollary 3.5 yields

$$(\phi^k)^* A_{\mathfrak{X}/X} = A_{\mathfrak{X}/X} + \mathrm{Env}_X(R_{\phi^k}).$$

On the other hand, observe that $R_{\phi^k} = \sum_{j=0}^{k-1} (\phi^j)^* R_\phi$ by the chain rule. Each $(\phi^j)^* R_\phi$ is numerically Cartier as well, so that

$$\mathrm{Env}_X(R_{\phi^k}) = \sum_{j=0}^{k-1} (\phi^j)^* \,\mathrm{Env}_X(R_\phi)$$

by Lemma 2.28 and Proposition 2.19. Using Proposition 4.6 and Theorem 4.10 we thus obtain $c_1, c_2 > 0$ such that

$$(\phi^k)^*(A_{\mathfrak{X}/X}) \geq c_1 Z(\mathfrak{m}) - c_2 \sum_{j=0}^{k-1} (\phi^j)^* Z(\mathfrak{m})$$

for all divisorial valuations ν centered at zero and all k. Since we have $(\phi^j)^*\mathfrak{m} \subset \mathfrak{m}$ it follows that

$$(\phi^k)^* A_{\mathfrak{X}/X} \geq -Z(\mathfrak{m})(kc_2 - c_1).$$

But the action of ϕ^k on divisorial valuations centered at zero is surjective by Lemma 1.13. We furthermore have $\nu((\phi^k)^* A_{\mathfrak{X}/X}) = \nu((\phi^k)^*\mathfrak{m})\nu(A_{\mathfrak{X}/X})$ for each divisorial valuation ν centered at zero, and there exists $c_k > 0$ such that $\nu((\phi^k)^*\mathfrak{m}) \leq c_k \nu(\mathfrak{m})$ for all ν by Lemma 4.7. We thus get $A_{\mathfrak{X}/X} \geq -\varepsilon_k Z(\mathfrak{m})$ with

$$\varepsilon_k := \frac{kc_2 - c_1}{c_k} > 0$$

as soon as $k > c_1/c_2$. □

Proof of Theorem B
If $\phi\colon X \to X$ is a finite endomorphism with $e(\phi) \geq 2$, then Theorem A implies $\mathrm{Vol}(X,0) \geq 2\,\mathrm{Vol}(X,0)$; hence $\mathrm{Vol}(X,0) = 0$. When X is $\mathbb{Q}$-Gorenstein and ϕ is not étale in codimension 1, then X is klt by Theorem 6.1 and Remark 6.2. □

Proof of Theorem C
By assumption, there exists an endomorphism $\phi\colon V \to V$ and an ample line bundle L such that $\phi^* L \simeq dL$ for some $d \geq 2$. The composite map

$$H^0(V, mL) \xrightarrow{\phi^*} H^0(V, m\phi^* L) \simeq H^0(V, dmL)$$

induces an endomorphism of the finitely generated algebra $\bigoplus_{m\geq 0} H^0(V, mL)$ (which does not preserve the grading). Since the spectrum of this algebra is equal to $X = C(V)$, we get an induced endomorphism $C(\phi)$ on $C(V)$. It is clear that $C(\phi)$ is finite, fixes the vertex $0 \in X$, and is not an automorphism. We conclude that $\mathrm{Vol}(X,0) = 0$, which implies that $-K_V$ is pseudoeffective by Proposition 4.21. □

6.2. Simple examples of endomorphisms
A quotient singularity is locally isomorphic to $(\mathbb{C}^n/G, 0)$ where G is a finite group acting linearly on $\mathbb{C}^n$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ be the natural projection. For any holomorphic maps $h_1, \ldots, h_n \colon \mathbb{C}^n/G \to \mathbb{C}$ such that $\bigcap h_i^{-1}(0) = (0)$, the composite map $\pi \circ (h_1, \ldots, h_n)\colon (\mathbb{C}^n/G, 0) \to (\mathbb{C}^n/G, 0)$ is a finite endomorphism of degree ≥ 2 if the singularity is nontrivial. Note also that any toric singularity admits finite endomorphisms of degree ≥ 2 (induced by the multiplication by an integer ≥ 2 on its associated fan).

We saw the above examples of endomorphisms on cone singularities. One can modify this construction to get examples of other kinds of simple singularities.

Consider a smooth projective morphism $f : Z \to C$ to a smooth pointed curve $0 \in C$, and suppose given a noninvertible endomorphism ϕ such that $f \circ \phi = f$. Note that ϕ is automatically finite since the injective endomorphism ϕ^* of $N^1(Z/C)$ has to be bijective.

Assume that $D \subset Z_0$ is a smooth irreducible ample divisor of the fiber Z_0 over zero that does not intersect the ramification locus of ϕ and such that $\phi(D) \subset D$. Denote by $Y \to Z$ the blowup of Z along D. Then ϕ lifts to a rational self-map of Y over C, and the fact that ϕ is étale around D implies that the indeterminacy locus of this rational lift is contained in $\mu^{-1}(\phi^{-1}(D) \setminus D)$ and hence in the strict transform E of Z_0 on Y.

Since the conormal bundle of E in Y is ample, E contracts to a simple singularity $0 \in X$ by [Gra]. (We are therefore dealing with an analytic germ $0 \in X$ in that case.) The above discussion shows that ϕ induces a finite endomorphism of $(X, 0)$, which is furthermore not invertible since ϕ was assumed not to be an automorphism.

Basic examples of this construction include deformations of abelian varieties having a section, with ϕ the multiplication by a positive integer.

6.3. Endomorphisms of cusp singularities

Our basic references are [Oda] and [Tsu]. Let $C \subset \mathbb{R}^n$ be an open convex cone that is strongly convex (i.e., its closure contains no line), and let $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$ be a subgroup leaving C invariant, whose action on $C/\mathbb{R}_+^*$ is properly discontinuous without fixed point, and has compact quotient. Denote by

$$M := \Gamma \backslash C/\mathbb{R}_+^*$$

the corresponding $(n-1)$-dimensional orientable manifold.

Consider the convex envelope Θ of $C \cap \mathbb{Z}^n$. It is proved in [Tsu] that the faces of $\overline{\Theta}$ are convex polytopes contained in C and with integral vertices. Since Θ is Γ-invariant the cones over the faces of Θ therefore give rise to a Γ-invariant rational fan Σ of $\mathbb{R}^n$ with $|\Sigma| = C \cup \{0\}$. This fan is infinite but is finite modulo Γ since M is compact.

The (infinite type) toric variety $X(\Sigma)$ comes with a Γ-action which preserves the toric divisor $D := X(\Sigma) \setminus (\mathbb{C}^*)^n$ as well the inverse image of C by the map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ defined by

$$\mathrm{Log}(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|).$$

The Γ-invariant set $U := \mathrm{Log}^{-1}(C) \cup D$ is open in $X(\Sigma)$, and the action of Γ is properly discontinuous and without fixed point on U. One then shows that the divisor $E := D/\Gamma \subset U/\Gamma =: Y$, which is compact since Σ is a finite fan modulo Γ, admits a strictly pseudoconvex neighborhood in Y, so that it can be contracted to a normal singularity $0 \in X$, which is furthermore isolated since $Y - E$ is smooth. Note that Y, though possibly not smooth along E, has at most rational singularities since U does, being an open subset of a toric variety. The isolated normal singularity $(X,0)$ is called the *cusp singularity* attached to (C,Γ). It is shown in [Tsu] that (C,Γ) is determined up to conjugation in $\mathrm{GL}(n,\mathbb{Z})$ by the (analytic) isomorphism type of the germ $(X,0)$.

LEMMA 6.4
The canonical divisor K_X is Cartier; X is lc but not klt.

Remark 6.5
Cusp singularities are, however, not Cohen–Macaulay in general and hence not Gorenstein.

Proof
The n-form $\Omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ on the torus $(\mathbb{C}^*)^n$ extends to $X(\Sigma)$ with poles of order one along D. It is Γ-invariant since Γ is a subgroup of $\mathrm{SL}(n,\mathbb{Z})$; thus it descends to a meromorphic form on U/Γ with order one poles along D/Γ. We conclude that K_X is zero and that X is lc but not klt since $\pi : (Y,E) \to X$ is crepant and $(X(\Sigma), D)$ is lc but not klt as for any toric variety. □

Now let $A \in \mathrm{GL}(n,\mathbb{R})$ with integer coefficient which preserves C and commutes with Γ (e.g., a homothety). Then Z induces a regular map on U that descends to the quotient Y and preserves the divisors E, and we get a finite endomorphism $\phi : (X,0) \to (X,0)$ whose topological degree is equal to $|\det A|$.

Example 6.6 (Hilbert modular cusp singularities)
Let K be a totally real number field of degree n over $\mathbb{Q}$, and let N be a free $\mathbb{Z}$-submodule of K of rank n (for instance $N = \mathcal{O}_K$). Using the n distinct embeddings of K into $\mathbb{R}$ we get a canonical identification $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$, and we may view N as a lattice in $\mathbb{R}^n$. Now set $C := (\mathbb{R}_+^*)^n \subset N_{\mathbb{R}}$, and consider the group Γ_N^+ of totally positive units of $u \in \mathcal{O}_K^*$ such that $uN = N$, where u is said to be totally positive if its image under any embedding of K in $\mathbb{R}$ is positive. By Dirichlet's unit theorem, Γ_N^+ is isomorphic to $\mathbb{Z}^{n-1}$, and there is a canonical injective homomorphism $\Gamma_N^+ \hookrightarrow \mathrm{SL}(N)$. For any subgroup $\Gamma \subset \Gamma_N^+$ of finite index, the triple (N, C, Γ) then satisfies the requirements of the definition of a cusp singularities. The singularities obtained by this construction are called Hilbert modular cusp singularities.

Appendix. Continuity of intersection products along nondecreasing nets
In this appendix, we fix an isolated normal singularity $0 \in X$ as in Section 4. The following theorem is taken from [BFJ2], where the result will appear in a more general

form. We are very grateful to Mattias Jonsson for allowing us to include a proof here.

THEOREM A.1 (Increasing limits)
For $1 \le r \le n$, *let* $\{W_{r,i}\}_{i\in I}$ *be a net of nef* $\mathbb{R}$*-Weil b-divisors over zero increasing to* W_r. *Assume that there exists some constant* $C > 0$ *such that* $W_{r,i} \ge C Z(\mathfrak{m})$ *for all* r, i. *Then we have*

$$W_{1,i} \cdot \cdots \cdot W_{n,i} \to W_1 \cdot \cdots \cdot W_n.$$

Proof
After rescaling, we may assume that $W_{r,i} \ge Z(\mathfrak{m})$ for all r, i. We prove the statement by induction on $p = 0, \dots, n-1$ under the assumption that $W_{r,i} = W_r$ for all i and all $r > p$.

The case $p = 0$ is trivial, so first suppose $p = 1$. Let $C_2, \dots, C_n$ be nef $\mathbb{R}$-Cartier b-divisors such that $C_r \ge W_r$ for $2 \le r \le n$. It follows from Lemma A.4 that

$$\begin{aligned}
0 &\le W_1 \cdot \cdots \cdot W_n - W_{1,i} \cdot W_2 \cdot \cdots \cdot W_n \\
&= -(W_1 \cdot C_2 \cdot \cdots \cdot C_n - W_1 \cdot W_2 \cdot \cdots \cdot W_n) \\
&\quad + (W_1 - W_{1,i}) \cdot C_2 \cdot \cdots \cdot C_n \\
&\quad + W_{1,i} \cdot C_2 \cdot \cdots \cdot C_n - W_{1,i} \cdot W_2 \cdot \cdots \cdot W_n \\
&\le (W_1 - W_{1,i}) \cdot C_2 \cdot \cdots \cdot C_n + \sum_{r=2}^{n} \big((C_r - W_r) \cdot W_r \cdot \cdots \cdot W_r\big)^{1/(2^{n-1})}.
\end{aligned}$$

Fix $\varepsilon > 0$. We can assume that the b-divisors C_r are chosen such that $0 \le (C_r - W_r) \cdot W_r \cdot \cdots \cdot W_r \le \epsilon$. On the other hand, since C_r are $\mathbb{R}$-Cartier b-divisors and $W_{1,i} \to W_1$, we have $(W_1 - W_{1,i}) \cdot C_2 \cdot \cdots \cdot C_n \le \epsilon$ for i large enough.

Now assume that $1 < p < n$ and that the statement is true for $p - 1$. Write

$$a_i = W_{1,i} \cdot \cdots \cdot W_{p,i} \cdot W_{p+1} \cdot W_n.$$

Clearly a_i is increasing in i, and we must show that $\sup_i a_i = W_1 \cdot \cdots \cdot W_n$. If $j \le i$, then $W_{p,j} \le W_{p,i} \le W_p$, and so

$$W_{1,i} \cdot \cdots \cdot W_{p-1,i} \cdot W_{p,j} \cdot W_{p+1} \cdot \cdots \cdot W_n \le a_i \le W_{1,i} \cdot \cdots \cdot W_{p-1,i} \cdot W_p \cdot W_{p+1} \cdot \cdots \cdot W_n.$$

Taking the supremum over all i, we get by the inductive assumption that

$$W_1 \cdot \cdots \cdot W_{p-1} \cdot W_{p,j} \cdot W_{p+1} \cdot \cdots \cdot W_n \le \sup_i a_i \le W_1 \cdot \cdots \cdot W_n.$$

The inductive assumption implies that the supremum over j of the first term equals $W_1 \cdot \cdots \cdot W_n$. Thus $\sup_i a_i = W_1 \cdot \cdots \cdot W_n$, which completes the proof. $\square$

LEMMA A.2 (Hodge index theorem)
Let $Z_3, \dots, Z_n$ be nef $\mathbb{R}$-Cartier b-divisors over zero. Then

$$(Z, W) := Z \cdot W \cdot Z_3 \cdot \cdots \cdot Z_n$$

defines a bilinear form on the space of Cartier b-divisors over zero that is negative semidefinite.

Proof
By choosing a common determination $Y \to X$, we are reduced to proving this statement for exceptional divisors lying in Y. We may perturb Z_i and assume that they are rational and ample over zero. By intersecting by general elements of multiples of Z_i, we are then reduced to the 2-dimensional case. Since the intersection form on the exceptional components of any birational surface map is negative definite, the result follows. □

LEMMA A.3
If $Z, W, Z_2, \dots, Z_n$ are nef $\mathbb{R}$-Weil b-divisors over zero with $Z(\mathfrak{m}) \le Z \le W \le 0$ and $Z(\mathfrak{m}) \le Z_j \le 0$ for $j \ge 2$, then

$$0 \le (W - Z) \cdot Z_2 \cdot \cdots \cdot Z_n \le \big((W - Z) \cdot Z \cdot \cdots \cdot Z\big)^{1/2^{n-1}}.$$

Proof
We may assume that all the b-divisors involved are $\mathbb{R}$-Cartier. By Lemma A.2, the bilinear form $(Z, W) \mapsto Z \cdot W \cdot Z_3 \cdot \cdots \cdot Z_n$ is negative semidefinite. Hence,

$$\begin{aligned} 0 &\le (W - Z) \cdot Z_2 \cdot \cdots \cdot Z_n \le |Z_2 \cdot Z_2 \cdot Z_3 \cdot \cdots \cdot Z_n)|^{1/2} \\ &\quad \cdot |(W - Z) \cdot (W - Z) \cdot Z_3 \cdot \cdots \cdot Z_n|^{1/2} \\ &\le |(W - Z) \cdot (W - Z) \cdot Z_3 \cdot \cdots \cdot Z_n|^{1/2} \le |(W - Z) \cdot Z \cdot Z_3 \cdot \cdots \cdot Z_n|^{1/2}. \end{aligned}$$

Repeating this procedure $n - 2$ times, we conclude the proof. □

LEMMA A.4
If Z_r, W_r are nef $\mathbb{R}$-Weil b-divisors with $Z(\mathfrak{m}) \le Z_r \le W_r \le 0$ for $1 \le r \le n$, then

$$0 \le W_1 \cdot \cdots \cdot W_n - Z_1 \cdot \cdots \cdot Z_n \le \sum_{r=1}^{n} \big((W_r - Z_r) \cdot Z_r \cdot \cdots \cdot Z_r\big)^{1/2^{n-1}}.$$

Proof
It follows from Lemma A.3 by writing

$$
\begin{aligned}
&W_1 \cdot \dots \cdot W_n - Z_1 \cdot \dots \cdot Z_n \\
&\quad = (W_1 - Z_1) \cdot W_2 \cdot \dots \cdot W_n \\
&\qquad + Z_1 \cdot (W_2 - Z_2) \cdot W_3 \cdot \dots \cdot W_n + \dots + Z_1 \cdot \dots \cdot Z_{n-1} \cdot (Z_n - W_n). \quad \square
\end{aligned}
$$

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Boucksom
CNRS - Institut de Mathématiques de Jussieu, 75252 Paris CEDEX, France; boucksom@math.jussieu.fr

de Fernex
Department of Mathematics, University of Utah, Salt Lake City, Utah 48112-0090, USA; defernex@math.utah.edu

Favre
CNRS - Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau CEDEX, France; favre@math.polytechnique.fr