SINGULAR SEMIPOSITIVE METRICS ON LINE BUNDLES ON VARIETIES OVER TRIVIALLY VALUED FIELDS

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Abstract. Let X be a smooth projective Berkovich space over a trivially or discretely valued field k of residue characteristic zero, and let L be an ample line bundle on X. We develop a theory of plurisubharmonic (or semipositive) metrics on L. In particular we show that the (non-Archimedean) Monge-Ampère operator induces a bijection between plurisubharmonic metrics and Radon probability measures of finite energy. In the discretely valued case, these results refine earlier work obtained in collaboration with C. Favre. In the trivially valued case, the results are new and will in subsequent work be shown to have ramifications for the study of K-stability.

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Introduction

Consider a polarized smooth projective variety (X, L) defined over a field k of characteristic zero. Our main objective in this paper is to define and study the class of *singular semipositive metrics* on the Berkovich analytification of L with respect to the *trivial* abolute value on k. In forthcoming work [BoJ17], we will use these results to study K-stability from a non-Archimedean point of view, continuing the ideas of [BHJ17]. These results will in turn be applied in [BBJ17] to the existence of twisted Kähler-Einstein metrics.

Our results are similar to—but often more precise than—the ones in [BFJ16a, BFJ15, BGJKM16], which dealt with the case of a *discretely* valued ground field of residue characteristic zero. Namely, first we introduce and study the class $\mathrm{PSH}(L)$ of singular semipositive metrics on L. Then we define a subclass $\mathcal{E}^1(L)$ consisting of metrics of finite energy. This class contains the class of *continuous* semipositive metrics, as studied by many authors. We then define a Monge-Ampère operator on this class, and completely describe its image. Generally speaking, the results are inspired by analogous results in the *Archimedean* case, by which we mean complex geometry: more on this below, and see the survey [BFJ16b].

Let us now describe the results in the paper in more detail. In what follows, we write X also for the Berkovich analytication of X with respect to the trivial absolute value on k. Similarly, we will write L for the analytification of the total space of a line bundle (or invertible sheaf) L on X. These conventions are reasonable in view of the GAGA results in [Berk90, §3.5].

Singular semipositive metrics. We use additive notation for line bundles and metrics. A metric (resp. singular metric) on a line bundle L on X is then a function on the complement $L^\times$ of the zero section in L with values in $\mathbf{R}$ (resp. $\mathbf{R} \cup \{-\infty\}$). If ϕ_i is a metric on L_i and $m_i \in \mathbf{Z}$, $i = 1, 2$, then $m_1\phi_1 + m_2\phi_2$ is a metric on $m_1L_1 + m_2L_2$. A metric on $\mathcal{O}_X$ can and will be identified with a function on X. Any section s of L defines a singular metric $\log |s|$, whose value on the image of s is constantly equal to 0. Since k is trivially valued, there is a canonical *trivial metric* ϕ_{triv} on L, using which we can identify metrics by functions: any metric on L can be written as $\phi = \phi_{\mathrm{triv}} + \varphi$, for some function φ on X.

Now let L be an *ample* line bundle on X. Define $\mathrm{PSH}(L)$ as the smallest class of singular metrics on L such that:

(i) $\mathrm{PSH}(L)$ contains all metrics of the form $m^{-1} \log |s|$, where $m \geq 1$ and s is a nonzero section of mL;

(ii) $\mathrm{PSH}(L)$ is closed under maxima, addition of constants, and decreasing limits.

Here the last part of (ii) means that if $(\phi_j)_j$ is a decreasing net in $\mathrm{PSH}(L)$ and $\phi := \lim_j \phi_j \not\equiv -\infty$ on $L^\times$, then $\phi \in \mathrm{PSH}(L)$. The elements of $\mathrm{PSH}(L)$ will be called *semipositive* or *plurisubharmonic* (singular) metrics.

The definition of $\mathrm{PSH}(L)$ makes sense when L is the analytification of an ample line bundle with respect to any multiplicative norm on the ground field k. In particular, it makes sense for the usual (Archimedean) norm on $k = \mathbf{C}$. One can then show that the class $\mathrm{PSH}(L)$ coincides with the class of semipositive singular metrics usually considered in complex geometry.

Back in the trivially valued case, the space $\mathrm{PSH}(L)$ has two fundamental properties mirroring the Archimedean situation. To state them, we need to define the topology on $\mathrm{PSH}(L)$, as well as the analogues of smooth metrics on L. First, the Berkovich space X contains a dense subset X^{qm} of *quasimonomial* points, and the topology on $\mathrm{PSH}(L)$ in defined by pointwise convergence on X^{qm}. Second, define $\mathrm{FS}(L) \subset \mathrm{PSH}(L)$ as the set of *Fubini-Study metrics*, i.e. metrics of the form

$$\phi = \max_{1 \le l \le N} m^{-1}(\log|s_l| + a_l),$$

where $m \in \mathbf{Z}_{>0}$, $a_l \in \mathbf{Z}$ and s_l, $1 \le l \le N$ are sections of mL having no common zero. The class $\mathrm{FS}(L)$ coincides with the class $\mathcal{H}$, defined in [BHJ17] via ample *test configurations*.

Theorem A. *Let L be an ample line bundle on X.*

(a) *The quotient space* $\mathrm{PSH}(L)/\mathbf{R}$ *is compact.*
(b) *Any singular metric in* $\mathrm{PSH}(L)$ *is the limit of a decreasing net in* $\mathrm{FS}(L)$.

In the Archimedean case, the compactness statement in (a) is classical, whereas the regularization statement in (b) is due to Demailly [Dem92]; see also [BłK07].

Theorem A will be deduced from the main results in [BFJ16a], which dealt with the case of a discretely valued ground field. See below for details.

The Monge-Ampère operator and the Calabi-Yau theorem. The second main result involves the Monge-Ampère equation

$$\mathrm{MA}(\phi) = \mu, \tag{MA}$$

where $\phi \in \mathrm{PSH}(L)$ and μ is a Radon probability measure on X. To state it properly, we need some further definitions.

There are by now several ways to define the Monge-Ampère operator on (sufficiently regular) metrics on line bundles, the most flexible of which is the one developed by Chambert-Loir and Ducros [CD12] in their pioneering work on forms and currents on Berkovich spaces. Suffice it to say that if $\phi_1, \dots, \phi_n \in \mathrm{FS}(L)$, then the *mixed Monge-Ampère measure*

$$\mathrm{MA}(\phi_1, \dots, \phi_n) = V^{-1} dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n,$$

where $V = (L^n)$, is an atomic probability measure supported on Shilov points in X. We also write $\mathrm{MA}(\phi) := \mathrm{MA}(\phi, \dots, \phi)$. The *Monge-Ampère energy*[1][2] of a metric $\phi \in \mathrm{FS}(L)$ is defined by

$$E(\phi) := \frac{1}{n+1} \sum_{j=0}^{n} \int (\phi - \phi_{\mathrm{triv}})\, \mathrm{MA}_j(\phi),$$

where $\mathrm{MA}_j(\phi) = \mathrm{MA}(\phi, \dots, \phi, \phi_{\mathrm{triv}} \dots, \phi_{\mathrm{triv}})$ is defined as above, using j copies of ϕ. For a general metric $\phi \in \mathrm{PSH}(L)$ we set

$$E(\phi) := \inf\{E(\psi) \mid \psi \in \mathrm{FS}(L), \psi \ge \phi\},$$

and define $\mathcal{E}^1(L)$ as the space of metrics $\phi \in \mathrm{PSH}(L)$ of *finite energy*, $E(\phi) > -\infty$. This space contains all *continuous* semipositive metrics, as defined and studied by Zhang [Zha95], Gubler [Gub98] and others. In the Archimedean case, the corresponding class was first defined by Guedj and Zeriahi [GZ07], following the fundamental work of Cegrell [Ceg98].

[1]In the Archimedean case, this also goes under the name Aubin-Mabuchi energy.

[2]In [BHJ17], the Monge-Ampère operator MA, the energy functional E, as well as many related functionals were all written with a superscript "NA", but we don't do so here.

The metric ϕ in the left-hand side of (MA) will be in the class $\mathcal{E}^1(L)$. As for the right-hand side, define the *energy* of a Radon probability measure μ (with respect to L) by

$$E^*(\mu) := \sup\{E(\phi) - \int_X (\phi - \phi_{\mathrm{triv}})\mu \mid \phi \in \mathcal{E}^1(L)\},$$

and let $\mathcal{M}^1(X)$ be the space of Radon probability measure of finite energy. For example, any finite atomic measure supported on quasimonomial points has finite energy. In the Archimedean case, the space $\mathcal{M}^1(X)$ was introduced in [Ceg98, GZ07].

Theorem B. *Let L be an ample line bundle on X.*

(a) *There exist a unique extension of the Monge-Ampére operator to $\mathcal{E}^1(L)$ that is continuous under increasing and decreasing limits.*
(b) *The operator in (a) defines a bijection*

$$\mathrm{MA}\colon \mathcal{E}^1(L)/\mathbf{R} \to \mathcal{M}^1(X)$$

between plurisubharmonic metrics of finite energy modulo constants, and Radon probability measures of finite energy.

Theorem B is the analogue of [BBGZ13, Theorem A], itself is a generalization of the celebrated Calabi-Yau theorem [Yau78] and subsequent work by Kołodziej [Koł98]. We note that the injectivity of MA on the space of *continuous* semipositive metrics modulo constants was first proved in [YZ16], whereas a sketch of a proof of the solvability of (MA) for μ a Dirac mass can be found in [KT00].

Strategy. The definitions of the classes $\mathrm{PSH}(L)$, $\mathcal{E}^1(L)$, and the Monge-Ampère operator make sense over any non-Archimedean ground field k, any k-variety X, and any ample line bundle L on X. We partially develop the theory under these assumptions; see also [BGJKM16, GM16, GJKM17] for important work in this direction. However, we were not able to prove Theorems A and B in full generality.

Instead, following [BFJ16a, BFJ15] we first prove these theorems when k is *discretely valued* and of residue characteristic zero. In this case, we can use the existence of snc models (constructed using Hironaka's theorem), as well as vanishing theorems in algebraic geometry. Theorem A is essentially a direct consequence of the main results of [BFJ16a], except that our definition of the class $\mathrm{PSH}(L)$ differs from the one in *loc. cit*, and some work is needed to prove that the two definitions are equivalent. As for Theorem B, the injectivity of the Monge-Ampère operator was proved in [BFJ15] (see also [YZ16]), whereas solutions to (MA) were established in [BFJ15, BGJKM16] for measures μ supported on the dual complex of an snc model of X. For a general measure $\mu \in \mathcal{M}^1(X)$ of finite energy, we apply a regularization process using snc models, inspired by the approach in [GZ07].

We then reduce the case of a trivially valued ground field to the discretely valued case using the non-Archimedean field extension $k \hookrightarrow k((\varpi))$. Indeed, points in the Berkovich space X can be identified with $\mathbf{G}_m$-invariant points in the base change $X_{k((\varpi))}$ or in the product $X \times_k \mathbf{A}^1_k$. In this way, the role of models of (X, L) is played by test configurations for (X, L).

Applications to K-stability. Following the solution to the Yau-Tian-Donaldson conjecture for Fano manifolds [CDS15, Tia15]—the existence of a Kähler–Einstein metric is equivalent to K-stability—much interest has gone into analyzing K-stability. In [BoJ17] we will show how the the Calabi-Yau theorem naturally fits into this analysis.

Let X be a smooth complex Fano variety, that is, X is smooth and projective over $k = \mathbf{C}$, and $L := -K_X$ is ample. As explained in [BHJ17] (and observed much earlier by S.-W. Zhang), K-stability admits a natural non-Archimedean interpretation, using the trivial absolute value on k. Namely, we can define a natural *Mabuchi functional* $M\colon \mathrm{FS}(L) \to \mathbf{R}$ with the property that X is K-stable iff $M(\phi) \geq 0$ for all $\phi \in \mathrm{FS}(L)$, with equality iff $\phi = \phi_{\mathrm{triv}}$.[3] In [BHJ17, Der15] a stronger notion was introduced: X is *uniformly K-stable* if there exists $\delta > 0$ such that $M \geq \delta J$ on $\mathrm{FS}(L)$. Here $J(\phi) = \max(\phi - \phi_{\mathrm{triv}}) - E(\phi) \geq 0$, with equality iff $\phi = \phi_{\mathrm{triv}} + c$, $c \in \mathbf{R}$. It was shown in [BBJ15] that if X has no nontrivial vector fields, then X admits a Kähler-Einstein metric iff X is uniformly K-stable.

The proof in [BBJ15] uses another natural functional, the *Ding functional* $D\colon \mathrm{FS}(L) \to \mathbf{R}$, and we say that X is *uniformly Ding stable* if there exists $\delta > 0$ such that $D \geq \delta J$ on $\mathrm{FS}(L)$. It was shown using the Minimal Model Program in [BBJ15] (see also [Fuj16b]) that X is uniformly Ding stable iff it is uniformly K-stable. Further, X is K-semistable iff it is Ding semistable; this corresponds to the case $\delta = 0$.

The Calabi-Yau theorem gives a different proof of this equivalence, not relying on the Minimal Model Program, but instead inspired by thermodynamics and the Legendre transform, as in [Berm13]. For simplicity we only discuss K-semistability. The Mabuchi functional naturally extends to $\mathcal{E}^1(L)$ and can be written as $M(\phi) = F(\mathrm{MA}(\phi))$, where $F(\mu)$, the *free energy* of the measure $\mu \in \mathcal{M}^1(X)$, is of the form $F(\mu) = L^*(\mu) - E^*(\mu)$; here $E^*(\mu)$ is the energy, whereas $L^*(\mu) = \int_X A\,\mu$ is the integral of μ agains the *log discrepancy* function $A = A_X$ [JM12]. Similarly, the Ding functional D is defined on $\mathcal{E}^1(L)$ and satisfies $D = L - E$, where $E(\phi)$ is the energy, and $L(\phi) = \inf_X(A + \phi - \phi_{\mathrm{triv}})$. As the notation indicates, E^* and L^* are the Legendre transforms of E and L, respectively. Using the Calabi–Yau theorem it now follows that $M \geq 0$ on $\mathcal{E}^1(L)$ iff $F \geq 0$ on $\mathcal{M}^1(X)$ iff $D \geq 0$ on $\mathcal{E}^1(L)$.

Fujita [Fuj16b] and C. Li [Li15b] (see also [FO16, BlJ17]) have recently given a valuative criterion for K-semistability and uniform K-stability; here we only discuss the former. There exists a natural invariant $S(x) \geq 0$ defined for Shilov (or quasimonomial) points x on X such that X is K-semistable iff $A(x) \geq S(x)$ for all x. Now, the Dirac mass δ_x is a probability measure of finite energy, and $F(\delta_x) = A(x) - S(x)$, so if X is K-semistable, then $A(x) \geq S(x)$ for all x. Further, the reverse implication also follows, due to convexity properties of L^* and E^*.

In summary, the Calabi–Yau theorem can be used to avoid techniques from the Minimal Model Program. Further, they apply also beyond the Fano situation. This gives rise to a stability criterion for the existence of *twisted* Kähler–Einstein metrics, as explored in [BBJ17].

Possible generalizations. There are several possible extensions of the work of this paper. On the one hand, one could try to prove all the results over an arbitrary non-Archimedean field. We lay the groundwork for doing so; in particular, our definition of the spaces $\mathrm{PSH}(L)$

[3] The Mabuchi functional is a variation of the Donaldson-Futaki invariant that has better functorial properties.

and $\mathcal{E}^1(L)$, as well as the Monge-Ampère operator make sense in general. However, we need to assume that k is discretely or trivially valued, of residue charateristic zero, for several of the fundamental results. It would be interesting to see if the tropical charts in [CD12, GK17] can replace our use of dual complexes; the latter are used to establish the compactness of $\mathrm{PSH}(L)$, and to solve the Monge-Ampère equation $\mathrm{MA}(\phi) = \mu$, by approximating μ by nicer measures.

We also use vanishing theorems for multiplier ideals to prove that the psh envelope of a continuous metric is continuous. This technique requires residue characteristic zero. Very recently, Gubler, Jell, Künnemann and Martin [GJKM17] instead used test ideals to prove continuity of envelopes in equicharacteristic p. This allows them to prove solve the Monge-Ampère equation for measures μ supported on a dual complex. For all of this they need to assume resolution of singularities, and that the variety is obtained by base change from a family over a perfect field.

As in the Archimedean case [BBGZ13, BBEGZ16], it would also be interesting to relax the assumptions on (X, L). For example, one could consider big line bundles L, singular varieties X, or pairs (X, B). This may require new ideas.

Organization. Following a general discussion of Berkovich analytifications in §1, where we in particular discuss the Gauss extension $k \subset k((\varpi))$, we discuss metrics on line bundles in §2. In particular, we define the classes $\mathrm{FS}(L)$ and $\mathrm{DFS}(L)$ that in our study play the role of smooth positive and positive metrics, respectively. In §3, we recall the definition of the Monge-Ampère operator on DFS metrics, the associated energy functionals, and state various estimate involving the Monge-Ampère operator on FS metrics. These estimates are proved in the appendix. In §4 we use dual complexes to study analytifications of smooth projective varieties over discretely or trivially valued fields of residue characteristic zero. The discretely valued case was studied in [BFJ16a], and we use the Gauss extension to treat the trivially valued case. Then, in §5, we introduce the class of $\mathrm{PSH}(L)$ of psh metrics on an ample line bundle L. When k is discretely valued, this class is shown to coincide with the class considered in [BFJ16a]. When k is discretely or trivially valued, of residue characteristic zero, we deduce from *loc. cit.* the compactness of $\mathrm{PSH}(L)/\mathbf{R}$. In §6 we introduce the class $\mathcal{E}^1(L)$ of metrics of finite energy and extend the Monge-Ampère operator and associated energy functionals to this class. Finally, in §7 we prove Theorem B.

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1. Berkovich analytifications

In this section we recall basic facts about the analytification of varieties in the sense of Berkovich, with particular attention to the case when the ground field is discretely or trivially valued.

1.1. **Setup.** A *non-Archimedean field* is a field k equipped with a complete multiplicative, non-Archimedean norm $|\cdot|$. We denote by $k^\circ := \{|a| \le 1\} \subset k$ the valuation ring of k, by $k^{\circ\circ} = \{|a| < 1\}$ its maximal ideal, and by $\tilde{k} := k^\circ/k^{\circ\circ}$ the residue field. The *value group* of k is $|k^\times| \subset \mathbf{R}_+^\times$, where $k^\times = k \setminus \{0\}$. We say k is *trivially valued* if $|k^\times| = \{1\}$ and *discretely valued* if $|k^\times| = r^{\mathbf{Z}}$ for some $r \in (0,1)$. For any subgroup $\Gamma \subset \mathbf{R}_+^\times$ we write $\sqrt{\Gamma} = \{r^{1/n} \in \mathbf{R}_+^\times \mid r \in \Gamma, n \in \mathbf{N}\}$.

The notation k'/k means that k' is a non-Archimedean field extension of k. Given k'/k, set $s(k'/k) := \operatorname{tr.deg}(\widetilde{k'}/\tilde{k})$, $t(k'/k) := \dim_{\mathbf{Q}}(\sqrt{|k'^\times|}/\sqrt{|k^\times|})$, and $d(k'/k) := s(k'/k) + t(k'/k)$.

All schemes will be separated. By an ideal on a scheme we mean a coherent ideal sheaf. By a variety over a field k, we mean an integral (separated) scheme of finite type over k.

1.2. **Analytification.** Let k be any non-Archimedean field. The analytification functor in [Berk90, 3] associates to any variety $\mathcal{X}$ (or, more generally, scheme locally of finite type) over k a (good, boundaryless) k-analytic space $\mathcal{X}^{\mathrm{an}}$ in the sense of Berkovich. Since our focus will be on the Berkovich spaces, we change notation and denote by X the analytification and by X^{sch} the underlying variety, viewed as a scheme. We will also write $\mathbf{A}^n = \mathbf{A}^n_k$, $\mathbf{P}^n = \mathbf{P}^n_k$ and $\mathbf{G}_m = \mathbf{G}_{m,k}$ for the analytifications of $\mathbb{A}^n = \mathbb{A}^n_k$, $\mathbb{P}^n = \mathbb{P}^n$, $\mathbb{G}_m = \mathbb{G}_{m,k}$, respectively.

As a set, the analytification X of X^{sch} consists of all pairs $x = (\xi, |\cdot|)$, where $\xi \in X^{\mathrm{sch}}$ is a point and $|\cdot| = |\cdot|_x$ is a multiplicative norm on the residue field $\kappa(\xi)$ extending the norm on k. We denote by $\mathcal{H}(x)$ the completion of $\kappa(\xi)$ with respect to this norm. The map $\ker\colon X \to X^{\mathrm{sch}}$ sending $(\xi, |\cdot|)$ to ξ is surjective and called the *kernel* map. If $X^{\mathrm{sch}} = \operatorname{Spec} A$ is affine, with A a finitely generated k-algebra, X consists of all multiplicative seminorms on A extending the norm on k.

The *Zariski topology* on X is the weakest topology in which $\ker\colon X \to X^{\mathrm{sch}}$ is continuous. Unless mentioned otherwise, we work with the *Berkovich* topology on X, the coarsest refinement of the Zariski topology for which the following holds: for any open affine subset $U = \operatorname{Spec} A \subset X^{\mathrm{sch}}$ and any $f \in A$, the function $\ker^{-1}(U) \ni x \to |f(x)| \in \mathbf{R}_+$ is continuous, where $f(x)$ denotes the image of f in $k(\xi) \subset \mathcal{H}(x)$, so that $|f(x)| = |f|_x$. Then X is Hausdorff, locally compact, and locally path connected. It is compact iff X^{sch} is proper. We say X is projective (resp. quasiprojective, smooth) if X^{sch} has the corresponding properties.

A point $x \in X$ is *rigid* if $\ker x$ is a closed point of X^{sch}. This is equivalent to $\mathcal{H}(x)$ being a finite extension of k. The set X^{rig} of rigid points is a dense subset of X unless k is trivially valued. The points in X whose kernel is the generic point of X^{sch} are the *valuations* (in multiplicative terminology) of the function field of X^{sch} extending the valuation on k. They form a dense subset $X^{\mathrm{val}} \subset X$.

If $x \in X$, we define $s(x) := s(\mathcal{H}(x)/k)$ and similarly $t(x)$, $d(x)$. The *Abhyankar inequality* says that $d(x) \le \dim \overline{\ker(x)} \le \dim X$. A point x is *quasimonomial* if $d(x) = \dim X$.[4] Write $X^{\mathrm{qm}} \subset X^{\mathrm{val}}$ for the set of quasimonomial points. In §4 we give a geometric description of

[4]Such points have many other names in the literature, including "monomial" or "Abhyankar" points.

quasimonomial point when k is trivially or discretely valued and of residue characteristic zero; see e.g. [Tem16, 7.2.4] for the general case.

Lemma 1.1. *If $f\colon X' \to X$ is induced by a surjective morphism of irreducible k-varieties, then $f(X'^{\mathrm{qm}}) \subset X^{\mathrm{qm}}$.*

Proof. Pick $x' \in X'^{\mathrm{qm}}$ and set $x := f(x')$. Since f is surjective, x is a valuation. Now $d(x') = n' := \dim X$, and we must show that $d(x) \geq n := \dim X$. By considering the non-Archimedean field extensions $k \subset \mathcal{H}(x) \subset \mathcal{H}(x')$, we see that $d(x') = d(x) + d(\mathcal{H}(x')/\mathcal{H}(x))$, so we must show that $d(\mathcal{H}(x')/\mathcal{H}(x)) \leq n' - n$. But $\mathcal{H}(x)$ and $\mathcal{H}(x')$ are the completions of $k(X)$ and $k(X')$ with respect to the norms defined by x and x', respectively, and $\mathrm{tr.deg}(k(X')/k(X)) = n' - n$, so $d(\mathcal{H}(x)/\mathcal{H}(x)) \leq n' - n$ by the Abhyankar inequality. □

We say that a point $x \in X$ is *Shilov* if $s(x) = \dim X$. When k is trivially valued, the only Shilov point is the trivial valuation on $k(X)$.

If k'/k is a non-Archimedean field extension, the ground field extension $X_{k'}$ is a k'-analytic space, and coincides with the analytification of the k'-variety $X^{\mathrm{sch}} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$. We have a continuous surjective map $\pi\colon X_{k'} \to X$ whose fiber over a point $x \in X$ can be identified with the Berkovich spectrum of the non-Archimedean Banach k-algebra $\mathcal{H}(x)\hat{\otimes}_k k'$.

1.3. **Models and Shilov points.** If X is projective, then a *model* of X is a flat and projective scheme $\mathcal{X}$ over $\mathrm{Spec}\, k^\circ$, together with an isomorphism of the generic fiber $\mathcal{X}_\eta$ onto X^{sch}. The special fiber $\mathcal{X}_0$ of $\mathcal{X}$ is a scheme over $\tilde{k}$.

To any model $\mathcal{X}$ of X is associated a *reduction map* $\mathrm{red}_{\mathcal{X}}\colon X \to \mathcal{X}_0$ defined as follows. Given $x \in X$, denote by R_x the corresponding valuation ring in the residue field $\kappa(\ker(x))$. By the valuative criterion of properness, the map $T_x := \mathrm{Spec}\, R_x \to \mathrm{Spec}\, k^\circ$ admits a unique lift $T_x \to \mathcal{X}$ mapping the generic point to $\ker(x)$. We then let $\mathrm{red}_{\mathcal{X}}(x)$ be the image of the closed point of T_x under this map.[5] Each $x \in X$ with $\mathrm{red}_{\mathcal{X}}(x)$ a generic point of $\mathcal{X}_0$ is a Shilov point, and the set $\Gamma(\mathcal{X}) \subset X^{\mathrm{Shi}}$ of *Shilov points for $\mathcal{X}$* so obtained is a finite set. Every Shilov point in X is in fact a Shilov point for some model $\mathcal{X}$, see [GM16, Proposition A.9].

Given any two models $\widetilde{\mathcal{X}}, \mathcal{X}$ of X, there is a canonical birational map $\widetilde{\mathcal{X}} \dashrightarrow \mathcal{X}$ induced by the isomorphisms of the generic fibers onto X^{sch}. We say that $\widetilde{\mathcal{X}}$ *dominates* $\mathcal{X}$ if this map is a morphism. Any two models can be dominated by a third.

1.4. **The discretely valued case.** Assume k is discretely valued, i.e. $|k^\times| = r^{\mathbf{Z}}$ for a unique $r \in \mathbf{R}_+^\times$, with $r < 1$. Then any model is noetherian, with special fiber $\mathcal{X}_0$ a Cartier divisor, of $\mathcal{X}$. If $\mathcal{X}$ (and hence X) is normal, $\mathcal{X}_0$ can be decomposed as a Weil divisor $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$, where E_i, $i \in I$ are the irreducible components of $\mathcal{X}_0$ and $b_i \geq 1$, and each E_i determines a unique Shilov point $x_i \in X^{\mathrm{Shi}}$ with $\mathrm{red}_{\mathcal{X}}(x_i)$ the generic point of E_i, given by $x_i = r^{\mathrm{ord}_{E_i}/b_i}$, where $\mathrm{ord}_{E_i}\colon k(\mathcal{X})^\times \to \mathbf{Z}$ is the order of vanishing along E_i.

1.5. **The trivially valued case.** When k is a trivially valued field, we have an embedding $X^{\mathrm{sch}} \hookrightarrow X$, defined by associating to $\xi \in X^{\mathrm{sch}}$ the point in X defined by the trivial norm on $\kappa(\xi)$. In this way we can view X^{sch} as a closed subset of X. It consists of all points $x \in X$ such that the norm on $\mathcal{H}(x)$ is trivial. Obviously the kernel map is the identity on X^{sch}. By

[5] If $x \in X^{\mathrm{val}}$, that is, $s(x)$ is the generic point of X^{sch}, then $\mathrm{red}_{\mathcal{X}}(x)$ is the *center* of x on $\mathcal{X}$ in valuative terminology [Vaq00]; this is also the terminology used in [BFJ16a].

using only closed points, we obtain the subset $X^{\rm rig} \subset X^{\rm sch} \subset X$ of rigid points. The closure of $X^{\rm rig}$ in X is equal to $X^{\rm sch}$. We will refer to the image in X of the generic point of $X^{\rm sch}$ as the generic point of X; it corresponds to the trivial valuation on $k(X)$.

There is a *scaling action* of multiplicative group $\mathbf{R}_+^\times$ on X defined by powers of norms. We denote by x^t the image of x by $t \in \mathbf{R}_+^\times$. If $x = (\xi, |\cdot|)$, then $x^t = (\xi, |\cdot|^t)$ for $t \in \mathbf{R}_+^\times$; in particular, $\ker(x^t) = \ker(x)$. The set of fixed points of the scaling action is exactly $X^{\rm sch} \subset X$. As $t \to 0$, we have $x^t \to \ker x \in X^{\rm sch}$.

Recall that a point $x \in X$ is divisorial, $x \in X^{\rm div}$, if $s(x) = \dim X - 1$ and $t(x) = 1$. All such points arise as follows. There exists a proper birational map $X' \to X$, with X' normal, a prime Weil divisor $E \subset X'$, and $r \in \mathbf{R}_+^\times$ with $r < 1$, such that $x = r^{\operatorname{ord}_E}$, with ord_E the order of vanishing along E.

Any ideal $\mathfrak{a}$ on $X^{\rm sch}$ induces a continuous function $|\mathfrak{a}| \colon X \to \mathbf{R}_+$. This function is defined by $|\mathfrak{a}|(x) := \max\{|f(x)| \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X^{\rm sch}, \operatorname{red}(x)}\}$ for $x \in X$, and is homogeneous: $|\mathfrak{a}|(x^t) = (|\mathfrak{a}|(x))^t$ for $x \in X$ and $t \in \mathbf{R}_+^\times$.

Now suppose X is projective. Since $k^\circ = k$, $X^{\rm sch}$ is the only model of X. The associated reduction map $\operatorname{red}\colon X \to X^{\rm sch}$ has the following properties. Let $x \in X$ and set $\xi := \ker x$, so that x defines valuation on $\kappa(\xi)$ that is trivial on k. If Y is the closure of ξ in X, then $\eta = \operatorname{red}(x)$ is the unique point in Y such that $|f(y)| \le 1$ for $y \in \mathcal{O}_{Y,\eta}$ and $|f(y)| < 1$ when further $f(\eta) = 0$.

Clearly $\operatorname{red} x$ is a specialization of $\ker x$, i.e. $\ker x$ belongs to the Zariski closure of $\operatorname{red} x$. We have $\operatorname{red} x = \ker x$ iff $x \in X^{\rm sch}$, and in this case $\operatorname{red} x = \ker x = x$. Note that $\operatorname{red} x^t = \operatorname{red} x$ for $t \in \mathbf{R}_+^\times$, and that $x^t \to \operatorname{red} x$ as $t \to \infty$.

The scaling action induces a dual action on continuous functions: if $\varphi \in C^0(X)$ and $t \in \mathbf{R}_+^\times$, then we define $t^*\varphi \in C^0(X)$ by $t^*\varphi(x) := \varphi(x^t)$. This, in turn, induces an action of $\mathbf{R}_+^\times$ on positive (resp. signed) Radon measures[6] on X by setting $\langle t_*\mu, \varphi\rangle := \langle \mu, t^*\varphi\rangle$ for μ a positive (resp. signed) Radon measure on X, $t \in \mathbf{R}_+^\times$, and $\varphi \in C^0(X)$. We have $t_*\mu = \int_X \delta_{x^t}\, d\mu(x)$.

1.6. **The Gauss extension.** If k is a non-Archimedean field and $r \in \mathbf{R}_+^\times$, the *circle algebra* k_r is a Banach k-algebra defined by

$$k_r := \{a = \sum_{-\infty}^{\infty} a_j \varpi^j \mid \lim_{j\to\pm\infty} |a_j| r^j = 0\},$$

equipped with the multiplicative norm

$$\|\sum_j a_j \varpi^j\| = \max_j |a_j| r^j. \tag{1.1}$$

The Berkovich spectrum $C_k(r) = \mathcal{M}(k_r)$ is the Berkovich circle of radius r over k. When $r \notin \sqrt{|k^\times|}$, k_r is a non-Archimedean field and $C_k(r)$ a singleton; such field extensions k_r/k play a crucial role in the general theory of Berkovich spaces.

Now let k be a trivially valued field and pick $r \in (0, 1)$. In this case,

$$k' := k_r \simeq k(\!(\varpi)\!)$$

[6]A positive (resp. signed) Radon measure on X is a positive (resp. bounded) linear functional on $C^0(X)$. Equivalently, it is a positive (resp. signed) Borel measure on X that is inner regular.

is the field of formal Laurent series with coefficients in k, equipped with the natural norm for which $|\varpi| = r$. Clearly $|k'^{\times}| = r^{\mathbf{Z}}$. The valuation ring of k' is the ring $k'^{\circ} = k[\![\varpi]\!]$ of formal power series and we identify the residue field of k' with k. The reduction map $k'^{\circ} \to k$ sends ϖ to 0. For any non-Archimedean extension K/k, we have $K\hat{\otimes}_k k' \simeq K_r$.

Let X be the analytification of a k-variety X^{sch}. Write $X' := X_{k'}$ for the ground field extension, and $\pi\colon X' \to X$ for the canonical map. For $x \in X$, we have $\pi^{-1}(x) \simeq C_{\mathcal{H}(x)}(r)$.

Lemma 1.2. *If $x' \in X'$ and $x := \pi(x') \in X$, then we have:*

$$s(x') = s(\mathcal{H}(x')/\mathcal{H}(x)) + s(x) \tag{1.2}$$

$$t(x') = t(\mathcal{H}(x')/\mathcal{H}(x)) + t(x) - 1 \tag{1.3}$$

$$d(x') = d(\mathcal{H}(x')/\mathcal{H}(x)) + d(x) - 1. \tag{1.4}$$

Proof. The invariant s is additive in towers of non-Archimedean field extensions; hence

$$s(\mathcal{H}(x')/k') + s(k'/k) = s(\mathcal{H}(x')/k) = s(\mathcal{H}(x')/\mathcal{H}(x)) + s(\mathcal{H}(x)/k).$$

By definition, $s(x') = s(\mathcal{H}(x')/k')$ and $s(x) = s(\mathcal{H}(x)/k)$, and (1.1) gives $s(k'/k) = 0$. This proves (1.2). The proofs of (1.3)–(1.4) are identical, using $t(k'/k) = 1$, and $d(k'/k) = 1$. □

Corollary 1.3. *We have $d(x) \geq d(x')$. In particular, if x' is quasimonomial, so is x.*

Proof. The preimage $\pi^{-1}(x) \simeq C_{\mathcal{H}(x)}(r)$ embeds in the Berkovich affine line over $\mathcal{H}(x)$, so $d(\mathcal{H}(x')/\mathcal{H}(x)) \leq 1$ by the Abhyankar inequality. We conclude using (1.4). □

As a special case of [Berk90, 5.2], we have a natural continuous section

$$\sigma = \sigma_r \colon X \to X'$$

of $\pi\colon X' \to X$. Namely, if $x \in X$, then $\pi^{-1}(x) \simeq C_{\mathcal{H}(x)}(r)$ is the circle of radius r over $\mathcal{H}(x)$, and $\sigma_r(x) \in \pi^{-1}(x)$ is the point corresponding to the norm on the circle algebra $\mathcal{H}(x)_r$. This type of norm is known as a Gauss norm, so we call σ_r the *Gauss extension*.

Concretely, if $x \in X$, then $x' = \sigma_r(x)$ is given as follows. Any $f \in \mathcal{H}(x') \simeq \mathcal{H}(x)_r$ can be written as $f = \sum_j f_j \varpi^j$ where $f_j \in \mathcal{H}(x)$. Then

$$|f(x')| = \max_j |f_j(x)| r^j. \tag{1.5}$$

Corollary 1.4. *For any $x \in X$, set $x' := \sigma_r(x) \in X'$. Then we have:*

(i) *if $r \in \sqrt{|\mathcal{H}(x)^{\times}|}$, then $s(x') = s(x) + 1$ and $t(x') = t(x) - 1$;*
(ii) *if $r \notin \sqrt{|\mathcal{H}(x)^{\times}|}$, then $s(x') = s(x)$ and $t(x') = t(x)$;*
(iii) *$d(x') = d(x)$.*

Proof. Clearly (iii) follows from (i) and (ii). To prove (i)–(ii) it suffices by Lemma 1.2 to compute $s := s(\mathcal{H}(x')/\mathcal{H}(x))$ and $t := t(\mathcal{H}(x')/\mathcal{H}(x))$.

First assume $r \notin \sqrt{|\mathcal{H}(x)^{\times}|}$. By (1.5) we have $\sqrt{|\mathcal{H}(x')^{\times}|}/\sqrt{|\mathcal{H}(x)^{\times}|} \simeq r^{\mathbf{Q}}$, so $t = 1$. To compute s, pick any $f \in \mathcal{H}(x')^{\circ}$ and write $f = \sum_j f_j \varpi^j$. Then $\tilde{f} = \tilde{f}_0 \in \widetilde{\mathcal{H}(x)}$, so $s = 0$.

Now suppose $r \in \sqrt{|\mathcal{H}(x)^{\times}|}$. Pick $n \geq 1$ minimal such that $r^n \in |\mathcal{H}(x)^{\times}|$, $g \in \mathcal{H}(x)^{\times}$ with $|g(x)| = r^n$, and set $f = \varpi^{-n} g$. Then $f \in \mathcal{H}(x')^{\circ}$ and it is easy to see that $\{\tilde{f}\}$ is a transcendence basis for $\widetilde{\mathcal{H}(x')}/\widetilde{\mathcal{H}(x)}$. Thus $s = 1$, which completes the proof. □

Corollary 1.5. *With the same notation as in Corollary 1.4, we have:*

(i) *x' is a valuation iff x is a valuation;*
(ii) *x' is quasimonomial iff x is quasimonomial;*
(iii) *x is the trivial valuation iff $x' = r^{\mathrm{ord}_{X\times\{0\}}}$;*
(iv) *x is divisorial iff x' is Shilov, $x' \neq r^{\mathrm{ord}_{X\times\{0\}}}$ and $r \in \sqrt{|\mathcal{H}(x)^\times|}$.*

Proof. To prove (i), pick any $f \in \mathcal{H}(x') \simeq \mathcal{H}(x)_r$ and write $f = \sum_j f_j \varpi^j$. From (1.5) it follows that $|f(x')| = 0$ iff $|f_j(x)| = 0$ for all j, which yields (i). The statements in (ii)–(iv) follow from Corollary 1.4 and our definition of quasimonomial and Shilov points. □

Let $\mathbb{G}_m$ be the multiplicative group over k. Then $\mathbb{G}_m(k) \simeq k^\times$ acts on k' by

$$\gamma \cdot \sum_j a_j \varpi^j = \sum_j \gamma^{-j} a_j \varpi^j,$$

and $k \subset k'$ is the set of $k^\times$-invariant points. We also have an induced continuous action on X'. If $x' \in X'$ and $\gamma \in k^\times$, the image $\gamma \cdot x' \in X'$ is characterized by $|f(\gamma \cdot x')| = |(\gamma \cdot f)(x')|$ for $f \in \mathcal{H}(x)_r$, where $x = \pi(x')$.

Proposition 1.6. *If k is an infinite field, then the image $\sigma_r(X) \subset X'$ is closed and consists of all $k^\times$-invariant points.*

Proof. Given $x' \in X'$, set $x := \pi(x')$, and let $\|\cdot\|$ be the norm on the circle algebra $\mathcal{H}(x)_r$. First suppose $x' = \sigma_r(x)$. For any $\gamma \in k^\times$, and $f = \sum_j \varpi^j f_j \in \mathcal{H}(x)_r$, we then have

$$|f(\gamma \cdot x')| = \|\gamma \cdot f\| = \max_j r^j |\gamma^{-j} f_j(x)| = \max_j r^j |f_j(x)| = \|f\| = |f(x')|,$$

so x' is $k^\times$-invariant.

Now suppose x' is $k^\times$-invariant, but that $x' \neq \sigma_r(x)$. We can then find $f \in \mathcal{H}(x)_r$ such that $|f(x')| < \|f\|$. Write $f = \sum_j f_j \varpi^j$, and let J be the finite set of indices j for which $|\varpi^j f_j(x)| = \|f\|$. We may assume $f_j = 0$ for $j \notin J$. For $\gamma \in k^\times$ we have $\gamma \cdot f = \sum_j \varpi^j \gamma^{-j} f_j$. By assumption, $|(\gamma \cdot f)(x')| = |f(x')|$, and hence $|\sum_j \varpi^j \gamma^{-j} f_j(x)| < \|f\|$ for all $\gamma \in k^\times$. Pick distinct elements γ_i, $i \in J$, of $k^\times$. The matrix $(\gamma_i^{-j})_{i,j\in J}$ is of Vandermonde type and therefore invertible. Let $(a_{ij})_{i,j\in J}$ be its inverse, so that $\sum_i a_{il}\gamma_i^{-j} = \delta_{jl}$. Thus $\sum_i a_{il}(\gamma_i \cdot f) = \varpi^l f_l$ for each $l \in J$, so

$$\|f\| = |\varpi^l f_l(x)| = |\sum_i a_{il}(\gamma_i \cdot f)(x')| \le \max_i |(\gamma_i \cdot f)(x')| = |f(x')| < \|f\|,$$

a contradiction. □

The base change X' can also be viewed as a closed subset of $X \times \mathbf{A}^1$ (i.e. the analytification of $X^{\mathrm{sch}} \times \mathbb{A}^1$ with respect to the trivial valuation on k) cut out by the equation $|\varpi| = r$. In this way, we can view the Gauss extension as a continuous map

$$\sigma = \sigma_r \colon X \to X \times \mathbf{A}^1.$$

The scaling actions by $\mathbf{R}_+^\times$ on X and $X \times \mathbf{A}^1$ interact with the Gauss extension as follows:

$$\sigma_{r^t}(x^t) = \sigma_r(x)^t$$

for $x \in X$, $r, t \in \mathbf{R}_+^\times$. Given $d \geq 1$, we have a selfmap $m_d \colon X \times \mathbf{A}^1 \to X \times \mathbf{A}^1$ induced by the base change $\varpi \to \varpi^d$. We then have

$$\sigma_{r^d}(x) = m_d(\sigma_r(x))$$

for any $x \in X$.

Following [Oda13] we define a *flag ideal* on $X^{\mathrm{sch}} \times \mathbb{A}^1$ to be a $\mathbb{G}_m$-invariant ideal cosupported on the central fiber $X^{\mathrm{sch}} \times \{0\}$.

1.7. **Test configurations.** Assume k is trivially valued and X projective. As already noted, X^{sch} is the only model of X. Instead of models, we use the notion of a test configuration, first introduced by Donaldson [Don02]. The definition below essentially follows [BHJ17].

Definition 1.7. *A test configuration $\mathcal{X}$ for X consists of the following data:*

(i) *a flat and projective morphism of k-schemes $\pi : \mathcal{X} \to \mathbb{A}_k^1$;*
(ii) *a $\mathbb{G}_m$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{A}_k^1$;*
(iii) *an isomorphism $\mathcal{X}_1 \xrightarrow{\sim} X$.*

The *trivial* test configuration of X is given by the product $X^{\mathrm{sch}} \times \mathbb{A}_k^1$, with the trivial $\mathbb{G}_m$-action on X. Given test configurations $\mathcal{X}$, $\mathcal{X}'$ for X there exists a unique $\mathbb{G}_m$-equivariant birational map $\mathcal{X}' \dashrightarrow \mathcal{X}$ extending the isomorphism $\widetilde{\mathcal{X}}_1 \simeq X \simeq \mathcal{X}_1$. We say that $\mathcal{X}'$ *dominates* $\mathcal{X}$ if this map is a morphism. Any two test configurations can be dominated by a third. Two test configurations that dominate each other will be identified. Any test configuration dominating the trivial test configuration is obtained by blowing up a flag ideal on $X^{\mathrm{sch}} \times \mathbb{A}_k^1$.

Now consider the non-Archimedean field extension $k' = k_r = k(\!(\varpi)\!)$ of k from §1.6. The inclusion $k[\varpi] \hookrightarrow k[\![\varpi]\!] = k'^\circ$ induces a morphism $\operatorname{Spec} k'^\circ \to \mathbb{A}_k^1$. Any test configuration $\mathcal{X}$ dominating the trivial test configuration therefore gives rise to a $\mathbb{G}_m$-invariant model $\mathcal{X}' := \mathcal{X} \times_{\mathbb{A}_k^1} \operatorname{Spec} k'^\circ$ of $X' := X_{k'}$ dominating $X_{k'^\circ}^{\mathrm{sch}} := X^{\mathrm{sch}} \times_{\operatorname{Spec} k} \operatorname{Spec} k'^\circ$, and whose special fiber $\mathcal{X}_0'$ can be identified with the central fiber $\mathcal{X}_0$ of $\mathcal{X}$. In fact, every $\mathbb{G}_m$-invariant model $\mathcal{X}'$ of X' dominating $X_{k'^\circ}^{\mathrm{sch}}$ arises in this way. Indeed, $\mathcal{X}'$ is the blowup of a $\mathbb{G}_m$-invariant ideal on $X_{k'^\circ}^{\mathrm{sch}}$ cosupported on the special fiber, and such an ideal comes from a flag ideal on $X^{\mathrm{sch}} \times \mathbb{A}_k^1$.

If $\mathcal{X}$ is a test configuration for X dominating the trivial test configuration, write $\mathrm{red}_{\mathcal{X}} \colon X \to \mathcal{X}_0$ for the composition $\mathrm{red}_{\mathcal{X}} \circ \sigma_r$, where $\sigma_r \colon X \to X'$ is the Gauss extension. When $\mathcal{X}$ is the trivial test configuration, $\mathcal{X}_0 = X^{\mathrm{sch}}$ and $\mathrm{red}_{\mathcal{X}}$ becomes the reduction map in §1.5.

Let $\mathcal{X}$ be a normal test configuration for X dominating the trivial test configuration, and write the central fiber as $\mathcal{X}_0 = \sum_{i \in I} b_i E_i$ where E_i are the irreducible components. Passing to the model $\mathcal{X}'$ of X', E_i defines a $k^\times$-invariant Shilov point $x_i' = r^{\mathrm{ord}_{E_i}/b_i} \in X'$. It follows from Corollary 1.5 that $x_i' = \sigma_r(x_i)$, where $x_i \in X$ is either the generic point or a Shilov point. Note that x_i depends on the choice of $r \in \mathbf{R}_+^\times$, but only up to scaling.

Given $d \geq 1$, define a new test configuration $\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}_d$ for X as the normalization of the fiber product of $\mathcal{X} \to \mathbb{A}_k^1$ and the map $\mathbb{A}_k^1 \to \mathbb{A}_k^1$ given by $\varpi \to \varpi^d$. The induced map $g = g_d \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is finite, of degree d. Let $\tilde{E}$ be an irreducible component of $\widetilde{\mathcal{X}}_0$. Then $E := g(\tilde{E})$ is an irreducible component of $\mathcal{X}_0$. Let $x_{\tilde{E}}$ and x_E be the associated points in X.

Lemma 1.8. *With the notation above, we have $x_{\tilde{E}} = x_E^d$.*

Proof. Set $e_{\tilde{E}} := \operatorname{ord}_{\tilde{E}}(g^*E)$. Then $\operatorname{ord}_{\tilde{E}}(g^*f) = e_{\tilde{E}} \operatorname{ord}_E(f)$ for every $f \in k(\mathcal{X}) = k(\widetilde{\mathcal{X}})$. Applying this to $f = \varpi$ yields $db_{\tilde{E}} = b_E$, where $b_E = \operatorname{ord}_E(\varpi)$ and $b_{\tilde{E}} = \operatorname{ord}_{\tilde{E}}(\varpi)$. Now suppose $f \in k(X)$. Then $g^*f = f$, and hence

$$|f(x_{\tilde{E}})| = |(g^*f)(x_{\tilde{E}})| = r^{\operatorname{ord}_{\tilde{E}}(g^*f)/b_{\tilde{E}}} = r^{e_{\tilde{E}} \operatorname{ord}_E(f)/b_{\tilde{E}}} = r^{d \operatorname{ord}_E(f)/b_E} = |f(x_E)|^d,$$

which completes the proof. □

2. Metrics on line bundles

This section contains a general discussion of metrics on line bundles, essentially following [CD12, 6.2]. We also introduce classes of metrics that, in our study, play the role of positive smooth and smooth metrics in complex geometry.

Let k be any non-Archimedean field. As we will in part work additively, fix a logarithm $\log\colon \mathbf{R}_+^\times \to \mathbf{R}$.[7] When k is nontrivially (resp. discretely) valued, we require $\log|k^\times| \supset \mathbf{Z}$ (resp. $\log|k^\times| = \mathbf{Z}$). When k is trivially valued, the logarithm is chosen compatible with the Gauss extension $\sigma = \sigma_r\colon k \to k_r = k(\!(\varpi)\!)$ in the sense that $\log r = -1$.

Let X be the analytification of a variety X^{sch} over k.

2.1. Line bundles and their skeleta.

Line bundles and their skeleta. By a *line bundle* L on X we mean the analytification of (the total space of) a line bundle L^{sch} on X^{sch}.[8] Thus we have a canonical map $p\colon L \to X$, the analytification of the corresponding map $L^{\mathrm{sch}} \to X^{\mathrm{sch}}$. Write $L^\times$ for the complement in L of the image of the zero section.

We can cover X by Zariski open subsets $U = (\operatorname{Spec} A)^{\mathrm{an}}$, where A is a finitely generated k-algebra, such that $p^{-1}(U) \simeq (\operatorname{Spec} A[T])^{\mathrm{an}}$ and $p^{-1}(U) \cap L^\times \simeq (\operatorname{Spec} A[T^{\pm 1}])^{\mathrm{an}}$; the map p is then given by restriction of seminorms from $A[T]$ to A. When k is trivially valued, this description shows that p commutes with the $\mathbf{R}_+^\times$-actions on L and X.

The fiber L_x over a point $x \in X$ is isomorphic to $(\operatorname{Spec} \mathcal{H}(x)[T])^{\mathrm{an}}$, the Berkovich affine line over $\mathcal{H}(x)$. Similarly, $L_x^\times = L^\times \cap L_x$ is isomorphic to $(\operatorname{Spec} \mathcal{H}(x)[T^{\pm 1}])^{\mathrm{an}}$, and comes with a multiplicative action of $\mathcal{H}(x)^\times$. The *skeleton* $\operatorname{Sk}(L_x) \subset L_x^\times$ is the set of fixed points under this action, and is isomorphic to $\mathbf{R}_+^\times$, with $r \in \mathbf{R}_+^\times$ corresponding to the multiplicative norm on $\mathcal{H}(x)[T^{\pm 1}]$ defined by $|\sum_i a_i T^i| = \max_i |a_i| r^i$. There is a continuous retraction $L_x^\times \to \operatorname{Sk}(L_x)$ that in this notation sends $v \in L_x^\times$ to $r := |T(v)|$.

We define the skeleton $\operatorname{Sk}(L) \subset L^\times$ of L as the union of the individual skeleta $\operatorname{Sk}(L_x)$. This is a closed subset of $L^\times$.[9] The retraction above defines a continuous global retraction $L^\times \to \operatorname{Sk}(L)$. With $U \subset X$ as above, we have $p^{-1}(U) \cap \operatorname{Sk}(L) \simeq U \times \mathbf{R}_+^\times$. If L and M are line bundles on X, we have a canonical map $L \times_X M \to L \otimes M$, where the fiber product on the left is in the category of k-analytic spaces. Using a variant of the $*$-multiplication in [Berk90, 5.2], this induces a continuous map $\operatorname{Sk}(L) \times_X \operatorname{Sk}(M) \to \operatorname{Sk}(L \otimes M)$. If $v \in \operatorname{Sk}(L_x)$ and $w \in \operatorname{Sk}(M_x)$, the image $v * w \in \operatorname{Sk}((L \otimes M)_x)$ is defined as follows: the norm on the Banach k-algebra $\mathcal{H}(v) \hat{\otimes}_{\mathcal{H}(x)} \mathcal{H}(w)$ is multiplicative, and hence defines a point in $L \times M$, whose image $v * w \in L \otimes M$ under the canonical map above is easily seen to belong to $\operatorname{Sk}(L \otimes M)$. Identifying the skeleta with $\mathbf{R}_+^\times$, the map $(v, w) \to v * w$ simply becomes the multiplication map $(r, s) \mapsto rs$. In this way, the skeleton $\operatorname{Sk}(L)$ of any line bundle L is a torsor for $\operatorname{Sk}(\mathcal{O}_X) = X \times \mathbf{R}_+^\times$.

Similarly, if L is a line bundle on X, we have a canonical isomorphism $L^\times \mapsto L^{-1\times}$ that restricts to a homeomorphism of $\operatorname{Sk}(L)$ onto $\operatorname{Sk}(L^{-1})$. For $x \in X$, we can identify $\operatorname{Sk}(L_x)$ and $\operatorname{Sk}(L_x^{-1})$ with $\mathbf{R}_+^\times$, and the map above is then given by $r \mapsto r^{-1}$.

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[7] Working multiplicatively throughout would obviate the choice of a logarithm, but the additive convention is very convenient when studying the Monge-Ampère operator and functionals on metrics.

[8] There is a notion of a line bundle on an arbitrary k-analytic space. When X^{sch} is projective, the GAGA results in [Berk90, 3] imply that all line bundles on X are analytifications of line bundles on X^{sch}.

[9] In the Archimedean case, that is, $k = \mathbf{C}$ with the usual norm, one can define the skeleton of a line bundle L as the quotient of $L^\times$ by the natural S^1-action.

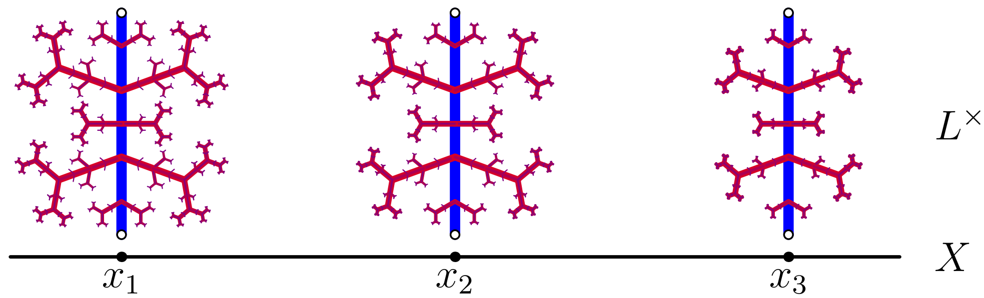

FIGURE 1. The skeleton of a line bundle L over X. The three trees are the fibers $L^\times_{x_i}$ over points x_i, $i = 1, 2, 3$. The vertical segment in each fiber is the skeleton $\mathrm{Sk}(L^\times_{x_i}) \sim \mathbf{R}^\times_+$.

If $f\colon Y \to X$ is a morphism of quasiprojective varieties, induced by a morphism $Y^{\mathrm{sch}} \to X^{\mathrm{sch}}$, and L is a line bundle on X, then f^*L is a line bundle on Y, and f induces a map $f^*L \to L$ that sends $\mathrm{Sk}(f^*L)$ to $\mathrm{Sk}(L)$.

Let k'/k be a non-Archimedean field extension and set $X' := X_{k'}$. For any line bundle L on X, $L' := L_{k'}$ is a line bundle on X', and the canonical map $L' \to L$ maps $\mathrm{Sk}(L')$ onto $\mathrm{Sk}(L)$. More precisely, for any $x' \in X'$, $\mathrm{Sk}(L'_{x'})$ maps homeomorphically onto $\mathrm{Sk}(L_x)$, where $x \in X$ is the image of x'. In the special case when k is trivially valued and $k' = k(\!(\varpi)\!)$, the Gauss extension $\sigma\colon L \to L'$ maps $\mathrm{Sk}(L)$ into $\mathrm{Sk}(L')$, More precisely, for every $x \in X$, σ maps $\mathrm{Sk}(L_x)$ maps homeomorphically onto $\mathrm{Sk}(L'_{x'})$, where $x' = \sigma(x)$.

2.2. **Metrics.** A *metric* on a line bundle L on X is a function $\|\cdot\|\colon L \to \mathbf{R}_{\ge 0}$ whose restriction to each fiber $L_x \simeq \mathbf{A}^1_{\mathcal{H}(x)}$ is of the form $\|v\| = c|T(v)|$, where $c = c(x) \in (0, \infty)$. In particular, the restriction of $\|\cdot\|$ to L_x is continuous, and we have $\|av\| = |a|\|v\|$ for $a \in \mathcal{H}(x)^\times$, $v \in L_x$, and $\|v\| > 0$ for $v \in L^\times_x$. The restriction of the metric to L_x is uniquely determined by its value at any point in $L^\times_x$. Choosing a metric on L amounts to specifying the unit circle $\|v\| = 1$ inside $L_x \simeq \mathbf{A}^1_{\mathcal{H}(x)}$ for each x, just like in the complex case.

A metric on $\mathcal{O}_X$ will be identified with a function on X with values in $\mathbf{R}^\times_+$, the identification arising by evaluating the metric on the constant section 1.

A metric $\|\cdot\|$ on L is uniquely determined by its restriction to the skeleton $\mathrm{Sk}(L)$. In fact, if $q\colon L^\times \to \mathrm{Sk}(L)$ denotes the retraction, then $\|v\| = \|q(v)\|$ for every $v \in L^\times$. The metric is affine on the skeleton: if we identify $\mathrm{Sk}(L_x) \simeq \mathbf{R}^\times_+$ for $x \in X$, then $\|r\| = ar$, where $a \in \mathbf{R}^\times_+$.

Given metrics on line bundles L, M, there is a unique metric on $L \otimes M$ such that $\|v \otimes w\| = \|v\| \cdot \|w\|$ for $v \in L_x(\mathcal{H}(x))$, $w \in M_x(\mathcal{H}(x))$. We also have $\|v * w\| = \|v\| \cdot \|w\|$ for $(v, w) \in \mathrm{Sk}(L) \times_X \mathrm{Sk}(M)$. Similarly, a metric on L induces a metric on L^{-1} satisfying analogous properties.

It will be convenient to use *additive* notation for both line bundles and metrics. Thus we write $L + M := L \otimes M$, $-L := L^{-1}$, and describe a metric $\|\cdot\|$ on L via

$$\phi := -\log\|\cdot\|\colon L^\times \to \mathbf{R}.$$

In fact, by a metric on L we will, from now on, mean such a function ϕ.[10] When needed, the corresponding multiplicative metric will be denoted by $\|\cdot\|_\phi = e^{-\phi}$.

[10]The term "weight" is sometimes used in the literature for such a function ϕ.

With this convention, if ϕ is a metric on L, any other metric is of the form $\phi+\varphi$, where φ is a function on X, that is, a metric on $\mathcal{O}_X$. If ϕ_i is a metric on L_i, $i=1,2$, then $\phi_1+\phi_2$ is a metric on L_1+L_2. If ϕ is a metric on L, then $-\phi$ is a metric on $-L$. If ϕ is a metric on mL, where $m\geq 1$, then $m^{-1}\phi$ is a metric on L. If ϕ_1,ϕ_2 are metrics on L, so is $\max\{\phi_1,\phi_2\}$.

If ϕ_1 and ϕ_2 are metrics on L that induce the same metric on mL for some $m\geq 1$, then $\phi_1=\phi_2$. We can therefore define a metric on a $\mathbf{Q}$-line bundle L as the choice of a metric ϕ_m on mL for m sufficiently divisible, with the compatibility condition $l\phi_m=\phi_{ml}$.

If ϕ is a metric on a line bundle L on X, and $f: Y\to X$ is a morphism, induced by a morphism $Y^{\mathrm{sch}}\to X^{\mathrm{sch}}$, then f lifts to a map $f^*L\to L$. For any metric ϕ on L, $f^*\phi=\phi\circ f$ is then a metric on f^*L.

Let k'/k be a non-Archimedean field extension. Given a line bundle L on X, set $X'=X_{k'}$ and $L':=L_{k'}$, and write π for the canonical maps $L'\to L$ and $X'\to X$. If ϕ is a metric on L, then $\phi\circ\pi$ is a metric on L'. In the special case when k is trivially valued and $k'=k(\!(\varpi)\!)$, we have that $\phi'\circ\sigma$ is a metric on L for every metric ϕ' on L', where $\sigma\colon L\to L'$ denotes the Gauss extension.

If k is trivially valued and L is a line bundle on X, then the canonical map $L^\times\to X$ is equivariant for the scaling action of $\mathbf{R}_+^\times$ on $L^\times$ and X, and induces an action on metrics on L: if ϕ is a metric on L and $t\in\mathbf{R}_+^\times$, then $\phi_t\colon L^\times\to\mathbf{R}$ defined by

$$\phi_t(v):=t\phi(v^{1/t}) \tag{2.1}$$

is a metric on L.

A metric on a line bundle L is *continuous* if it is continuous as a function on $L^\times$. This is equivalent to the restriction to $\mathrm{Sk}(L)$ being continuous. If $s\in H^0(X,L)$ is a nowhere vanishing global section, then ϕ is continuous iff the function $\phi\circ s$ is continuous on X. A simple partition of unity argument shows that every line bundle admits a continuous metric, see also 2.4. Continuity of metrics is preserved by all the operations above.

If X is proper, we say that a metric ϕ on L is *bounded* if for some/any continuous metric ψ on L, the function $\phi-\psi$ is bounded.

A *singular metric* on a line bundle L is a function $\phi\colon L^\times\to\mathbf{R}\cup\{-\infty\}$ such that, for every $x\in X$, the restriction $\phi|_{L_x^\times}$ is either $\equiv-\infty$ or a metric on L_x. All the remarks above about metrics (except continuity) carry over to singular metrics. Note that any metric is a singular metric.

As an important example, any global section $s\in H^0(X,L)$ defines a singular metric

$$\phi:=\log|s|$$

on L as follows: for any $x\in X$ we have $\phi(s(x))=0$ if $s(x)\neq 0$, whereas $\phi\equiv-\infty$ on $L_x^\times$ if $s(x)=0$. In other words, $\phi(v)=\log|v/s(x)|$ for $v\in L_x^\times$. Note that $\log|s|$ restricts to a continuous metric over the Zariski open set $\{s\neq 0\}\subset X$. When k is trivially valued, any singular metric $\phi=\log|s|$ is *homogeneous* in the sense that $\phi_t=\phi$ for $t\in\mathbf{R}_+^\times$.

Consider a non-Archimedean field extension k'/k, and set $X'=X_{k'}$, $L'=L_{k'}$. If $s\in H^0(X,L)$ and $s'\in H^0(X',L')$ is defined from s by ground field extension, then $\log|s'|=\log|s|\circ\pi$, where $\pi\colon L'\to L$ is the canonical map. When k is trivially valued and $k'=k(\!(\varpi)\!)$, this implies $\log|s|=\log|s'|\circ\sigma$, where $\sigma\colon L\to L'$ is the Gauss extension.

2.3. **FS metrics.** Then next class of metrics will, for the purposes of this paper, play the role of positive smooth metrics in complex geometry.

Definition 2.1. *A* Fubini-Study metric *(*FS metric *for short) on a line bundle L on X is a metric ϕ of the form*

$$\phi := \frac{1}{m} \max_{\alpha} (\log |s_\alpha| + \lambda_\alpha), \tag{2.2}$$

where $m \geq 1$, (s_α) is a finite set of global sections of mL without common zero, and $\lambda_\alpha \in \mathbf{Z}$.

We write $\mathrm{FS}(L)$ for the space of FS metrics on L. The terminology stems from the observation that if $X = \mathbf{P}^N$, $m = 1$, $\lambda_\alpha = 0$ and the s_α are homogenous coordinates, then ϕ is the (non-Archimedean counterpart of the) Fubini-Study metric on $\mathcal{O}(1)$. The constants λ_α are only necessary when k is trivially valued, since otherwise the assumption $\log |k^\times| \supset \mathbf{Z}$ allows us to absorb λ_α into s_α.

Remark 2.2. *If L is ample and $m \gg 0$, so that mL is very ample, we may in* (2.2) *assume that the sections $(s_\alpha)_{\alpha \in A}$ of mL define an embedding of X into projective space. Indeed, pick additional sections $(s_\alpha)_{\alpha \in A'}$ of mL such that $(s_\alpha)_{\alpha \in A \cup A'}$ gives an embedding into $\mathbf{P}^N$, and set $\lambda_\alpha \ll 0$ for $\alpha \in A'$. Then $\max_{\alpha \in A}(\log |s_\alpha| + \lambda_\alpha) = \max_{\alpha \in A \cup A'}(\log |s_\alpha| + \lambda_\alpha)$.*

Remark 2.3. *The terminology differs slightly from* [BE17]*, where real coefficients λ_α are allowed in the definition of FS metrics. The present convention is, however, sufficient for the purposes of this article*

Note that any FS metric is continuous, and that L admits an FS metric iff L is semiample. The following lemma follows easily from the definition.

Lemma 2.4. *With notation as above, we have:*

(i) *if $\phi \in \mathrm{FS}(L)$, then $\phi + c \in \mathrm{FS}(L)$ for any $c \in \mathbf{Q}$;*
(ii) *if $\phi_1, \phi_2 \in \mathrm{FS}(L)$, then $\max\{\phi_1, \phi_2\} \in \mathrm{FS}(L)$;*
(iii) *if $\phi_i \in \mathrm{FS}(L_i)$ for $i = 1, 2$, then $\phi_1 + \phi_2 \in \mathrm{FS}(L_1 + L_2)$;*
(iv) *if ϕ is a metric on L, and $m\phi \in \mathrm{FS}(mL)$ for some $m \geq 1$, then $\phi \in \mathrm{FS}(L)$;*
(v) *if $\phi_1, \phi_2 \in \mathrm{FS}(L)$, $\theta_1, \theta_2 \in \mathbf{Q}_+$, and $\theta_1 + \theta_2 = 1$, then $\theta_1\phi_1 + \theta_2\phi_2 \in \mathrm{FS}(L)$;*
(vi) *if $f : X' \to X$ is (induced by) a morphism of k-varieties, and $\phi \in \mathrm{FS}(L)$ for some line bundle L on X, then $f^*\phi \in \mathrm{FS}(f^*L)$.*

2.4. **DFS metrics.** Then next class of metrics will, for us, play the role of smooth metrics in complex geometry.

Definition 2.5. *A* DFS metric *on a line bundle L on X is a metric of the form $\phi_1 - \phi_2$, with ϕ_i an FS metrics on L_i, $i = 1, 2$, where $L = L_1 - L_2$. A* DFS-function *on X is a DFS metric on $\mathcal{O}_X$.*

Write $\mathrm{DFS}(L)$ for the set of DFS metrics on a line bundle L, and $\mathrm{DFS}(X)$ for the set of DFS functions on X. Every DFS metric is continuous. If X is quasiprojective, every line bundle on X admits a DFS metric.

Lemma 2.6. *If L is ample, every DFS function on X is a difference of FS metrics on L.*

Proof. Consider $\varphi \in \mathrm{DFS}(X)$. By definition, there exists a semiample line bundle M and $\psi_1, \psi_2 \in \mathrm{FS}(M)$ such that $\varphi = \psi_1 - \psi_2$. Pick $b \geq 1$ large enough so that $bL - M$ is base point free, and pick any $\phi \in \mathrm{FS}(bL - M)$. Then $\phi_i := b^{-1}(\psi_i + \phi) \in \mathrm{FS}(L)$ and $\varphi = \phi_1 - \phi_2$. ☐

Theorem 2.7. *The set* $\mathrm{DFS}(X)$ *is a* $\mathbf{Q}$*-vector subspace of* $C^0(X)$ *which is stable under max and contains all constants. If X is quasiprojective,* $\mathrm{DFS}(X)$ *separates points. If X is projective,* $\mathrm{DFS}(X)$ *is dense in* $C^0(X)$*.*

Corollary 2.8. *If X is projective, then for every line bundle L,* $\mathrm{DFS}(L)$ *is dense in the set of continuous metrics on L.*

Proof of Theorem 2.7. It follows easily from Lemma 2.4 that $\mathrm{DFS}(X)$ is a $\mathbf{Q}$-vector space which is stable under max and contains all constant functions.

Now assume X is quasiprojective. It only remains to prove $\mathrm{DFS}(X)$ separates points. Indeed, if X is further projective, then X is compact, so the Boolean version of the Stone-Weierstrass theorem implies that $\mathrm{DFS}(X)$ is dense in $C^0(X)$.

Being quasiprojective, X embeds into some projective space, so since DFS-functions are clearly stable under pull-back, we are reduced to the case $X = \mathbf{P}^n_k$. Now let $x, y \in X$ be two distinct points. By considering a hyperplane not containing x or y, we may further assume $x, y \in \mathbf{A}^n_k \subset \mathbf{P}^n_k$. Since $x \neq y$, there exists a polynomial $f \in k[t_1, \dots, t_n]$, say of degree $d \geq 0$, such that $|f(x)| \neq |f(y)|$. Suppose $|f(x)| < |f(y)|$ for definiteness and write $f = s/t_0^d$, where $s \in H^0(\mathbf{P}^n_k, \mathcal{O}(d))$. Given $m \in \mathbf{Z}$ and $\lambda_\alpha \in \mathbf{Z}$, $0 \leq j \leq n$, set

$$\psi = d \max_{0 \leq j \leq n} (\log |t_\alpha| - \lambda_\alpha)$$

and

$$u := \max\{\log |s|, \psi - m\} - \psi = \max\{\log |s| - \psi, -m\}.$$

Then ψ is an FS metric on $\mathcal{O}(1)$ and u a DFS-function on $\mathbf{P}^n_k$. Assume $\lambda_0 = 0$ and $\lambda_\alpha \gg 1$ for $j > 0$. Then $\psi(x) = d \log |t_0(x)|$ and $\psi(y) = d \log |t_0(y)|$. If $m > -\log |f(y)|$, this implies

$$\begin{aligned} u(x) = \max\{\log |s(x)| - d \log |t_0(x)|, -m\} &= \max\{\log |f(x)|, -m\} \\ < \max\{\log |f(y)|, -m\} &= \max\{\log |s(y)| - d \log |t_0(y)|, -m\} = u(y), \end{aligned}$$

completing the proof. □

2.5. **Smooth and regularizable metrics.** We now comment on the relation of FS metrics and DFS metrics to the point of view of Chambert-Loir and Ducros in [CD12].

In our study, DFS (resp. FS) metrics, play the role of smooth (resp. smooth positive) metrics in complex geometry. This is in line with earlier work by Zhang [Zha95], Gubler [Gub98], Chambert-Loir [CL06] and others, but differs from the point of view by Chambert-Loir and Ducros in [CD12].

Indeed, a metric ϕ on a line bundle L is *smooth* (resp. *smooth psh*) in the sense of [CD12] iff for every $x \in X$, there exists an open subset $U \subset X$ of x, nonvanishing sections $s_1, \dots, s_N \in \Gamma(U, L)$, and a smooth (resp. smooth convex) function $\chi \colon \mathbf{R}^N \to \mathbf{R}$ such that $\phi = \chi(\log |s_1|, \dots, \log |s_N|)$ on $p^{-1}(U)$. A continuous metric ϕ is (locally) *psh-regularizable* it is locally a uniform limit of smooth psh metrics. Finally, a continuous metric is *regularizable* if it is a difference of psh-regularizable metrics.

Lemma 2.9. *Any FS metric is psh-regularizable, and any DFS metric is regularizable.*

The second statement is indeed a consequence of the first, which is proved exactly as in [CD12, 6.3.2].

2.6. **Ground field extension and scaling.** Next we consider how FS metrics and DFS metrics behave under ground field extension, and under scaling in the trivially valued case.

Proposition 2.10. *Consider a non-Archimedean field extension k'/k. Let L be a line bundle on X, set $L' = L_{k'}$, and let $\pi \colon L' \to L$ be the canonical map.*

(i) *If* $\phi \in \mathrm{FS}(L)$*, then* $\phi \circ \pi \in \mathrm{FS}(L')$*.*
(ii) *If* $\phi \in \mathrm{DFS}(L)$*, then* $\phi \circ \pi \in \mathrm{DFS}(L')$*.*

Proof. It suffices to prove (i). Write $\phi = \frac{1}{m}\max_\alpha(\log|s_\alpha| + \lambda_\alpha)$, where $\lambda_\alpha \in \mathbf{Z}$ and s_α are global sections of mL without common zero. The sections s_α induce global sections s'_α of mL' without common zero, and $\phi \circ \pi = \frac{1}{m}\max_\alpha(\log|s'_\alpha| + \lambda_\alpha)$, so $\phi \circ \pi$ is an FS metric. □

Proposition 2.11. *Suppose k is trivially valued and set $k' = k((\varpi))$. Let L be a line bundle on X, $L' = L_{k'}$, $\pi\colon L' \to L$ the canonical map, and $\sigma\colon L \to L'$ the Gauss extension.*

(i) *If* $\phi' \in \mathrm{FS}(L')$*, then* $\phi' \circ \sigma \in \mathrm{FS}(L)$*; further,* $\phi' \circ \sigma \circ \pi \ge \phi'$*.*
(ii) *If* $\phi' \in \mathrm{DFS}(L')$*, then* $\phi' \circ \sigma \in \mathrm{DFS}(L)$*.*

Proof. It suffices to prove (i). Since $\log|k'^\times| = \mathbf{Z}$, we can write any FS metric on L' as $\phi' = m^{-1}\max_{\alpha \in A}\log|s'_\alpha|$, where the s'_α are finitely many sections of mL' without common zero. For $\alpha \in A$, we can write $s'_\alpha = \sum_j \varpi^j s'_{\alpha,j}$, where each $s'_{\alpha,j}$ arises from a section $s_{\alpha,j} \in H^0(X, mL)$ via ground field extension. For each α we then have

$$\log|s'_\alpha| \circ \sigma = \max_j\{\log|s'_{\alpha,j}| \circ \sigma - j\} = \max_j\{\log|s_{\alpha,j}| - j\}$$

on L, and hence

$$\phi' \circ \sigma = m^{-1}\max_{\alpha,j}\{\log|s_{\alpha,j}| - j\}. \tag{2.3}$$

Since the sections $s_{\alpha,j}$, $\alpha \in A$, $j \in \mathbf{Z}$ have no common zero on X, there exists a finite subset $J \subset \mathbf{Z}$ such that the same is true for $\alpha \in A$, $j \in J$, and the right-hand side of (2.3) is unchanged if we only take the max over $\alpha \in A$ and $j \in J$. Thus $\phi' \circ \sigma \in \mathrm{FS}(L)$. Further,

$$\phi' \circ \sigma \circ \pi = m^{-1}\max_{\alpha,j}(\log|s'_{\alpha,j}| - j) \ge m^{-1}\max_\alpha \log|\sum_j \varpi^j s'_{\alpha,j}| = \phi',$$

which completes the proof. □

Proposition 2.12. *Assume k is trivially valued. Let L be a line bundle on X and ϕ a metric on L. For $t \in \mathbf{R}_+^\times$, let ϕ_t be the scaled metric on L as in* (2.1)*.*

(i) *If* $\phi \in \mathrm{FS}(L)$*, then* $\phi_t \in \mathrm{FS}(L)$ *for* $t \in \mathbf{Q}_+^\times$*.*
(ii) *If* $\phi \in \mathrm{DFS}(L)$*, then* $\phi_t \in \mathrm{DFS}(L)$ *for* $t \in \mathbf{Q}_+^\times$*.*

Thus $\mathbf{Q}_+^\times$ *acts on* $\mathrm{FS}(L)$ *and* $\mathrm{DFS}(L)$*.*

Proof. It suffices to prove (i). We can write

$$\phi = \tfrac{1}{m}\max_\alpha(\log|s_\alpha| + \lambda_\alpha), \tag{2.4}$$

where $m \ge 1$, the s_α are sections on mL without common zero, and $\lambda_\alpha \in \mathbf{Z}$. Since $L \to X$ is equivariant for the $\mathbf{R}_+^\times$-action, we have $s_\alpha(x^t) = s_\alpha(x)^t$ for $x \in X$, $t \in \mathbf{R}_+^\times$; hence

$$\phi_t = \tfrac{1}{m}\max_\alpha(\log|s_\alpha| + t\lambda_\alpha), \tag{2.5}$$

and the result follows. □

2.7. **Model metrics.** For the rest of §2, assume X is *projective*. We will give geometric interpretations of FS metrics and DFS metrics.

A *model* of a line bundle L on X consists of a model $\mathcal{X}$ of X together with a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ and an isomorphism $\mathcal{L}|_{\mathcal{X}_\eta} \xrightarrow{\sim} L^{\mathrm{sch}}$. A model $\mathcal{L}$ is *ample* (resp. *semiample*, *nef*) if it is relatively ample (resp. semiample, relatively nef) for the morphism $\mathcal{X} \to \operatorname{Spec} k^\circ$.

Any model $\mathcal{L}$ defines a metric $\phi_{\mathcal{L}}$ on L as follows. First suppose $\mathcal{L}$ is a line bundle. Given $x \in X$, set $\xi = \mathrm{red}_{\mathcal{X}}(x) \in \mathcal{X}_0$ and pick a local section $s \in \Gamma(\mathcal{U}, \mathcal{L})$ defined in a Zariski open neighborhood $\mathcal{U}$ of ξ, such that $s(\xi) \neq 0$. If $U = (\mathcal{U} \cap X^{\mathrm{sch}})^{\mathrm{an}}$, then $x \in U$, s defines a section of L on U, and we declare $\phi_{\mathcal{L}}(s(x)) = 0$. When $\mathcal{L}$ is a $\mathbf{Q}$-line bundle, we set $\phi_{\mathcal{L}} := m^{-1}\phi_{m\mathcal{L}}$ for m sufficiently divisible.

A metric of the form $\phi_{\mathcal{L}}$ is called a *model metric*. If ϕ_i is a model metric on L_i, $i = 1, 2$, then $\phi_1 - \phi_2$ is a model metric on $L_1 - L_2$. If ϕ is a metric on L and $a \in \mathbf{Z} \setminus \{0\}$, then ϕ is a model metric on L iff $a\phi$ is a model metric on aL.

Proposition 2.13. *Assume k is nontrivially valued, and let L be a line bundle on X.*

(i) *A DFS metric on L is the same as a model metric.*
(ii) *If L is semiample, then an FS metric on L is the same thing as a model metric defined by a semiample model $\mathcal{L}$ of L*
(iii) *If L is ample, then an FS metric on L is the same thing as a model metric defined by an ample model $\mathcal{L}$ of L.*

Proof. Note that (i) follows from (iii), since any DFS metric is a difference of FS metrics on ample line bundles, and any model metric is a difference between model metrics defined by ample models.

To prove (ii), assume L is semiample and consider a metric ϕ on L. First suppose $\phi = \phi_{\mathcal{L}}$ is a model metric, with $\mathcal{L}$ a semiample $\mathbf{Q}$-line bundle. To prove that $\phi_{\mathcal{L}}$ is an FS metric, we may, after replacing L and $\mathcal{L}$ by a suitable multiple, assume $\mathcal{L}$ is a base point free line bundle. Let $(s'_j)_{0 \le j \le N}$ be global sections of $\mathcal{L}$ without common zero on $\mathcal{X}$. For each j, let $s_j \in H^0(X, L) = H^0(X^{\mathrm{sch}}, L^{\mathrm{sch}})$ be the restriction of s'_j to X^{sch}. Then $(s_j)_{0 \le j \le N}$ have no common zero on X, so $\psi := \max_j \log|s_j|$ is an FS metric on L. We claim that $\psi = \phi$, which will complete the proof. Pick any point $x \in X$, and set $\xi := \mathrm{red}(x) \in \mathcal{X}_0$. We may assume $s'_0(\xi) \neq 0$. By the definition of ϕ, this implies $\phi(s_0(x)) = 0$. On the other hand, for any j, we have $s'_j/s'_0 \in \mathcal{O}_{\mathcal{X},\xi}$, so s_j/s_0 is a regular function at x, and $|s_j(x)/s_0(x)| \le 1$, by the definition of the reduction map. It follows that $\psi = \log|s_0|$ on L_x, and hence $\psi(s_0(x)) = 0 = \phi(s_0(x))$, which completes the proof.

Conversely, suppose ϕ is an FS metric, say $\phi := m^{-1} \max_{0 \le j \le N}(\log|s_\alpha| + \lambda_j)$, where $\lambda_j \in \mathbf{Z}$ and the s_j are global sections of mL without common zero. Replacing L and ϕ by mL and $m\phi$, respectively, we may assume $m = 1$ and that L is very ample. Since $\log|k^\times| \supset \mathbf{Z}$, we may further assume $\lambda_j = 0$ for all j. Thus $\phi := \max_{0 \le j \le N} \log|s_j|$. The sections s_j define a morphism $f\colon X^{\mathrm{sch}} \to \mathbb{P}^N_k$, and $L = f^*\mathcal{O}_{\mathbb{P}^N_k}(1)$. Pick any model $\mathcal{X}'$ of X and let $\mathcal{X}$ be the scheme theoretic closure of the graph of f in $\mathcal{X} \times_{\operatorname{Spec} k^\circ} \mathbb{P}^N_{k^\circ}$. Then f extends to a morphism $\mathcal{X} \to \mathbb{P}^N_{k^\circ}$, and if we set $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^N_{k^\circ}}(1)$, then $(\mathcal{X}, \mathcal{L})$ is a semiample model of (X, L). We claim that $\phi = \phi_{\mathcal{L}}$. But by construction, the sections s_j extend to global sections s_j of $\mathcal{L}$ without common zeros. The same argument as above now shows that $\phi = \phi_{\mathcal{L}}$, completing the proof of (ii).

Finally assume L is ample. We only need to show any FS metric ϕ on L is a model metric associated to an ample model of L. Reasoning as above, and also invoking Remark 2.2, we may assume that L is very ample and that $\phi := \max_{0\le j\le N}\log|s_\alpha|$, where the s_j define an embedding $X^{\mathrm{sch}} \hookrightarrow \mathbb{P}^N_k$ into projective space. Let $\mathcal{X}$ be the closure of the image of X^{sch} in $\mathbb{P}^N_{k^\circ}$, and $\mathcal{L} := \mathcal{O}_{\mathbb{P}^N_{k^\circ}}(1)|_{\mathcal{X}}$. Then $(\mathcal{X},\mathcal{L})$ is an ample model of (X,L) and $\phi = \phi_{\mathcal{L}}$. □

If $(\mathcal{X},\mathcal{L})$ and $(\widetilde{\mathcal{X}},\widetilde{\mathcal{L}})$ are models of (X,L) such that $\widetilde{\mathcal{X}}$ dominates $\mathcal{X}$, and $\widetilde{\mathcal{L}}$ is the pullback of $\mathcal{L}$ under the morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$, then $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ give rise to the same model metric on L, $\phi_{\mathcal{L}} = \phi_{\widetilde{\mathcal{L}}}$. We say that a metric $\phi \in \mathrm{DFS}(L)$ is *determined* on a model $\mathcal{X}$ of X if $\phi = \phi_{\mathcal{L}}$ for a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$. Two $\mathbf{Q}$-line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$ define the same metric on L iff $\mathcal{L}_1 - \mathcal{L}_2$ is numerically trivial [GM16, 4.5].

2.8. **The trivial metric.** If k is trivially valued, the only model of (X,L) is $(X^{\mathrm{sch}}, L^{\mathrm{sch}})$. This gives rise to the *trivial metric* $\phi_{\mathrm{triv}} = \phi_{\mathrm{triv},L}$ on L. The assignment $L \to \phi_{\mathrm{triv},L}$ is linear in L, in the sense that $\phi_{\mathrm{triv},L_1} + \phi_{\mathrm{triv},L_2} = \phi_{\mathrm{triv},L_1+L_2}$ and $\phi_{\mathrm{triv},mL} = m\phi_{\mathrm{triv},L}$ for $m \in \mathbf{Z}$. The following result therefore shows that the trivial metric on a semiample (resp. arbitrary) line bundle is an FS metric (resp. a DFS metric).

Lemma 2.14. *If L is base point free and (s_α) are finitely many global sections of L without common zero, then the trivial metric on L is given by $\phi_{\mathrm{triv}} = \max_\alpha \log|s_\alpha|$.*

Proof. Set $\phi := \max_\alpha \log|s_\alpha|$. Given $x \in X$, set $\xi := \mathrm{red}(x) \in X$, and consider a local section s of L defined in a Zariski open neighborhood of ξ such that $s(\xi) \neq 0$. We must prove that $\phi(s(x)) = \max_\alpha \log|s_\alpha(x)/s(x)| = 0$. But since $\xi = \mathrm{red}(x)$ and $s(\xi)\neq 0$, we have $|s_\alpha(x)/s(x)| \le 1$ for all α, with equality iff $s_\alpha(\xi) \neq 0$. □

The trivial metric allows us to identify metrics on L with functions on X: if ϕ is a metric on L, then $\phi - \phi_{\mathrm{triv}}$ is a function on X; the trivial metric itself corresponds to the zero function. The trivial metric is homogeneous, i.e. invariant under the scaling action by $\mathbf{R}_+^\times$.

2.9. **Metrics from test configurations.** In this section, k is trivially valued. We shall show that DFS metrics are exactly metrics arising from test configurations. These are the only metrics considered in [BHJ17]. The next definition follows *loc. cit.*

Definition 2.15. *A test configuration $\mathcal{L}$ for L consists of a test configuration $\mathcal{X}$ of X in the sense of Definition 1.7, together with the following data:*

(iv) *a $\mathbb{G}_m$-linearized $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;*
(v) *an isomorphism $\mathcal{L}|_{\mathcal{X}_1} \simeq L$.*

If more precision is needed, we say that $(\mathcal{X},\mathcal{L})$ is a test configuration for (X,L). A test configuration $(\mathcal{X},\mathcal{L})$ is said to be *ample* (resp. semiample, nef) if $\mathcal{L}$ is relatively ample (resp. semiample, nef) for the canonical morphism $\mathcal{L} \to \mathbb{A}^1$.

The *trivial* test configuration of (X,L) is $(X^{\mathrm{sch}} \times \mathbb{A}^1, L^{\mathrm{sch}} \times \mathbb{A}^1)$, where the action of $\mathbb{G}_m$ occurs on the $\mathbb{A}^1$-factors. A test configuration $(\widetilde{\mathcal{X}},\widetilde{\mathcal{L}})$ *dominates* another test configuration $(\mathcal{X},\mathcal{L})$ for (X,L), if $\widetilde{\mathcal{X}}$ dominates $\mathcal{X}$ and $\widetilde{\mathcal{L}}$ is the pullback of $\mathcal{L}$ under $\widetilde{\mathcal{X}} \to \mathcal{X}$.

To explain how a test configuration induces a metric, set $k = k(\!(\varpi)\!)$ and let $(X',L') = (X_{k'}, L_{k'})$ be the ground field extension. Suppose $(\mathcal{X},\mathcal{L})$ is a test configuration for (X,L) dominating the trivial test configuration. Then the base change $(\mathcal{X}',\mathcal{L}')$ of $(\mathcal{X},\mathcal{L})$ via the map $\operatorname{Spec} k[\![\varpi]\!] \to \operatorname{Spec} k[\varpi]$ is a model of (X', L'), and hence defines a model metric $\phi_{\mathcal{L}'}$ on L'. Using the Gauss extension $\sigma\colon L \to L'$, this gives rise to a metric $\phi_{\mathcal{L}} = \phi_{\mathcal{L}'} \circ \sigma$ on L.

When $\mathcal{L}$ is a line bundle, any section $s \in H^0(X, L)$ has a $\mathbb{G}_m$-invariant extension $\bar{s} \in H^0(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L})$. Conversely, any section $s \in H^0(\mathcal{X}, \mathcal{L})$ decomposes as $s = \sum_j \varpi^j \bar{s}_j$, where $s_j \in H^0(X, L)$.

Remark 2.16. *When $\mathcal{L}$ (and hence L) is a base point free line bundle, the induced metric $\phi_{\mathcal{L}}$ on L can be described as follows. Consider any point $x \in X$, set $\xi := \operatorname{red}_{\mathcal{X}}(x) \in \mathcal{X}_0$, and pick a global section $s \in H^0(\mathcal{X}, \mathcal{L})$ such that $s(\xi) \neq 0$. Write $s = \sum_j \varpi^j \bar{s}_j$, where $s_j \in H^0(X, L)$. For any j such that $\varpi^j \bar{s}_j$ is regular and does not vanish at ξ, we then have $\phi_{\mathcal{L}}(s_j(x)) = -j$; this determines $\phi_{\mathcal{L}}$ on $L_x^{\times}$.*

If $(\mathcal{X}, \mathcal{L})$ and $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ are test configurations for (X, L), with $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ dominating $(\mathcal{X}, \mathcal{L})$, then the model $(\widetilde{\mathcal{X}}', \widetilde{\mathcal{L}}')$ dominates $(\mathcal{X}', \mathcal{L}')$, so $\phi_{\widetilde{\mathcal{L}}'} = \phi_{\mathcal{L}'}$ and finally $\phi_{\widetilde{\mathcal{L}}} = \phi_{\mathcal{L}}$. If $\mathcal{L}$ is an arbitrary test configuration for L, we can therefore define the metric $\phi_{\mathcal{L}}$ on L as the metric defined by any test configuration dominating both $(\mathcal{X}, \mathcal{L})$ and the trivial test configuration. We say that an DFS metric ϕ is *determined* on a test configuration $\mathcal{X}$ for X, if $\phi = \phi_{\mathcal{L}}$ for some test configuration $(\mathcal{X}, \mathcal{L})$.

When L is ample, we say, following [BHJ17], that a metric on L is *positive* if it is defined by a semiample test configuration for L. By [BHJ17, Lemma 6.3], every positive metric is associated to a unique normal ample test configuration for L, but the latter may not dominate the trivial test configuration in general.

Proposition 2.17. *Let L be a line bundle on X.*

(i) *A DFS metric on L is the same thing as a metric defined by a test configuration.*
(ii) *If L is semiample, then an FS metric on L is the same thing as a metric defined by a semiample test configuration for L.*
(iii) *If L is ample, then an FS metric on L is the same thing as a positive metric on L.*

Proof. Most of the statements follows from Proposition 2.13. Set $k' = k(\!(\varpi)\!)$, $X' = X_{k'}$, $L' = L_{k'}$, and let $\pi\colon L' \to L$ denote the canonical map. Consider a metric ϕ on L and set $\phi' := \phi \circ \pi$,

If $\phi = \phi_{\mathcal{L}}$ for a test configuration $\mathcal{L}$, then $\phi' = \phi_{\mathcal{L}'}$, where $\mathcal{L}'$ is the associated model of L'. By Proposition 2.13, ϕ' is a DFS metric, hence so is $\phi = \phi' \circ \sigma$ by Proposition 2.11. The same argument shows that if $\mathcal{L}$ is semiample, then ϕ is an FS metric.

It only remains to prove that any FS metric ϕ is defined by a semiample test configuration. Passing to multiples of L and ϕ, respectively, we may assume $\phi = \max_{0 \le j \le N}(\log|s_j| - \lambda_j)$, where $N \ge 0$, $s_j \in H^0(X, L) = H^0(X^{\mathrm{sch}}, L^{\mathrm{sch}})$, $\lambda_j \in \mathbf{Z}$ for each j, and the sections s_j have no common zero. Define sections s'_j of $L^{\mathrm{sch}} \times \mathbb{A}^1$ over $X^{\mathrm{sch}} \times \mathbb{G}_m$ by $s'_j := \varpi^{\lambda_j} s_j$. Then $(s'_j)_{0 \le j \le N}$ define a rational map $\tau\colon X^{\mathrm{sch}} \times \mathbb{A}^1 \dashrightarrow \mathbb{P}^N \times \mathbb{A}^1$ that is regular outside the central fiber. Let $\mathcal{X}$ be the graph of this rational map and $\mathcal{L}$ the pullback of $\mathcal{O}(1) \times \mathbb{A}^1$. Then $(\mathcal{X}, \mathcal{L})$ is a semiample test configuration for (X, L), and $\phi = \phi_{\mathcal{L}}$. This completes the proof. □

We later need an alternative description of DFS functions on X, analogous to [BFJ16a, Proposition 2.2]. Recall that a *flag ideal* is a $\mathbb{G}_m$-invariant ideal on $X^{\mathrm{sch}} \times \mathbb{A}^1$ cosupported on the central fiber. Any flag ideal $\mathfrak{a}$ defines continuous function $\varphi_{\mathfrak{a}} := \log|\mathfrak{a}| \circ \sigma$ on X, where $\sigma\colon X \to X \times \mathbf{A}^1$ is the Gauss extension. Note that $\varphi_{\mathfrak{a}} \equiv -1$ for $\mathfrak{a} = (\varpi)$.

Proposition 2.18. *The* $\mathbf{Q}$*-vector space generated by all functions* $\varphi_{\mathfrak{a}}$ *coincides with* $\mathrm{DFS}(X)$*, and is hence dense in* $C^0(X)$*.*

Proof. It suffices to prove the following two statements:

(i) if $\phi \in \mathrm{FS}(L)$ and $\phi \le \phi_{\mathrm{triv}}$, then $\phi - \phi_{\mathrm{triv}} = m^{-1}\varphi_{\mathfrak{a}}$ for some $m \ge 1$ and flag ideal $\mathfrak{a}$;

(ii) if $\mathfrak{a}$ is a flag ideal, then there exist L and $\phi \in \mathrm{FS}(L)$ such that $\phi = \phi_{\mathrm{triv}} + \varphi_{\mathfrak{a}}$.

Here ϕ_{triv} denotes the trivial metric on L.

To prove (i), write $\phi = m^{-1}\max_{0\le j\le N}\{-j + \max_{s\in A_j}\log|s|\}$, where $m, N \ge 1$ and A_j is a finite set of global sections of mL, and the sections in A_N have no common zero. Let $\mathfrak{a}_j$ be the base ideal defined by the sections in A_j, Then $\mathfrak{a}_N = \mathcal{O}_{X^{\mathrm{sch}}}$, $\mathfrak{a} := \sum_0^N \varpi^j \mathfrak{a}_j$ is a flag ideal, and $\phi = \phi_{\mathrm{triv}} + m^{-1}\varphi_{\mathfrak{a}}$.

Now suppose $\mathfrak{a} = \sum_{j=0}^N (\varpi^j)\mathfrak{a}_j$, is a flag ideal, where the $\mathfrak{a}_j$ are ideals on X^{sch} and $\mathfrak{a}_N = \mathcal{O}_{X^{\mathrm{sch}}}$. Pick L ample such that $L \otimes \mathfrak{a}_j$ is globally generated, say by sections $(s_{j,l})_{l\in L_j}$ for all j. Then $\varphi_{\mathfrak{a}} = \max_j(\max_l \log|s_{j,l}| - j) - \phi_{\mathrm{triv}}$, proving (ii). □

2.10. **Scaling and base change.** Assume k is trivially valued. The scaling action on metrics can then be interpreted geometrically as a base change, as in [BHJ17, §6.3]. Indeed, consider a normal test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) and an integer $d \ge 1$. As in §1.7, let $\widetilde{\mathcal{X}}$ be the normalization of the base change under $\varpi \mapsto \varpi^d$, and $g\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the induced map. Set $\widetilde{\mathcal{L}} = g^*\mathcal{L}$. Then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is a normal test configuration for (X, L).

Proposition 2.19. *If* $\phi = \phi_{\mathcal{L}}$ *and* $\tilde{\phi} = \phi_{\widetilde{\mathcal{L}}}$ *are the model metrics associated to* $\mathcal{L}$ *and* $\widetilde{\mathcal{L}}$*, respectively, then* $\tilde{\phi} = \phi_d$*,* i.e. ϕ *scaled by a factor* d*.*

Proof. The statement is true when $(\mathcal{X}, \mathcal{L})$ is the trivial test configuration, and $\phi = \phi_{\mathrm{triv}}$. By linearity it therefore suffices to treat the case when $L = \mathcal{O}_X$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(D)$ for a divisor D supported on the central fiber. We then view $\phi = \phi_D$ as a function on X and must show that $\phi_{g^*D}(x^d) = d\phi_D(x)$ for all $x \in X$. Since ϕ_D is continuous, it suffices to do this for $x \in X^{\mathrm{Shi}}$. Passing to a higher model, we may assume that $x = x_E$ for an irreducible component $E \subset \mathcal{X}_0$. Let $\tilde{E}$ be an irreducible component of $\widetilde{\mathcal{X}}_0$ such that $g(\tilde{E}) = E$. By Lemma 1.8 we have $x_{\tilde{E}} = x_E^d$. Further, $b_E e_{\tilde{E}} = d b_{\tilde{E}}$, where $b_E = \mathrm{ord}_E(\mathcal{X}_0)$, $b_{\tilde{E}} = \mathrm{ord}_{\tilde{E}}(\widetilde{\mathcal{X}}_0)$ and $e_{\tilde{E}} = \mathrm{ord}_{\tilde{E}}(g^*E)$. This leads to

$$\phi_{g^*D}(x_E^d) = \phi_{g^*D}(x_{\tilde{E}}) = b_{\tilde{E}}^{-1}\mathrm{ord}_{\tilde{E}}(g^*D) = b_{\tilde{E}}^{-1} e_{\tilde{E}}\,\mathrm{ord}_E(D) = d b_E^{-1}\mathrm{ord}_E(D) = d\phi_D(x_E),$$

completing the proof. □

2.11. **Shilov points and the Izumi inequality.** We will later need two results on the uniform behavior of functions on X of the form $\phi - \phi_{\mathrm{ref}}$, where $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$ is a reference metric (assumed to be the trivial metric when k is trivially valued) and ϕ ranges over metrics in $\mathrm{FS}(L)$.

The first result concerns the suprema of such functions.

Lemma 2.20. *For any* $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$*, there exists a finite subset* $Z \subset X^{\mathrm{Shi}}$ *of Shilov points such that* $\sup_X(\phi - \phi_{\mathrm{ref}}) = \max_Z(\phi - \phi_{\mathrm{ref}})$ *for every* $\phi \in \mathrm{FS}(L)$*.*

Proof. By Proposition 2.13, ϕ_{ref} is a model metric, associated to a model $\mathcal{X}$ of X. In this case, we can take $Z = \Gamma(\mathcal{X}) \subset X^{\mathrm{Shi}}$, the finite set of Shilov points associated to $\mathcal{X}$. Indeed, this follows from [BE17, Lemma 4.18] when $\phi = \log|s|$ for a nonzero section $s \in H^0(X, L)$, or more generally, $\phi = \frac{1}{m}\log|s|$ for $s \in H^0(X, mL)$. Taking maxima yields the same result for FS metrics ϕ. □

The second result is deeper, and can be viewed as a global version of the *Izumi inequality*, see [BFJ14] for a discussion. A proof when k is discretely valued case of residue characteristic zero, can be found in [BFJ16a, §6.1]. The result in the trivially valued case is equivalent to [BKMS16, Proposition 2.12]. Our proof is completely different from the ones in *loc. cit.*

Theorem 2.21. *For any metric $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$ and any two quasimonomial points $x_1, x_2 \in X^{\mathrm{qm}}$, there exists a constant $C = C(L, x_1, x_2, \phi_{\mathrm{ref}})$ such that*

$$|(\phi - \phi_{\mathrm{ref}})(x_1) - (\phi - \phi_{\mathrm{ref}})(x_2)| \le C \tag{2.6}$$

for any metric $\phi \in \mathrm{FS}(L)$.

Proof. We may assume $\dim X \ge 1$. We first prove that if the statement is true over a non-Archimedean extension k' of k, then it is true over k. Indeed, set $X' = X_{k'}$, $L' = L_{k'}$, and write π for the canonical maps $X' \to X$ and $L' \to L$. Since $x_i \in X^{\mathrm{qm}}$, we have $d(x_i) = n$. By [Berk90, 9.1.4], there exist points $x'_i \in \pi^{-1}(x_i)$, $i = 1, 2$, such that $d(x'_i) \ge d(x_i) = n$. But then $d(x'_i) = n$ by the Abhyankar inequality, so $x'_i \in X'^{\mathrm{qm}}$. Now $\phi'_{\mathrm{ref}} := \phi_{\mathrm{ref}} \circ \pi \in \mathrm{DFS}(L')$ and $\phi' := \phi \circ \pi \in \mathrm{FS}(L)$ for any $\phi \in \mathrm{FS}(L)$. Further, $(\phi' - \phi'_{\mathrm{ref}})(x'_i) = (\phi - \phi_{\mathrm{ref}})(x_i)$, for $i = 1, 2$, so we can pick $C = C(L', x'_1, x'_2, \phi'_{\mathrm{ref}})$.

We may therefore assume that k is algebraically closed and nontrivially valued. We may of course also assume $x_1 \ne x_2$. Let us first treat the case when X is the analytification of a smooth projective curve. The result can then be proved using potential theory, as systematically developed by Thuillier [Thu05]; see also [BR10, BPR11, Jon12].

We can equip $\mathbf{H} := X \setminus X^{\mathrm{rig}} \supset X^{\mathrm{qm}}$ with a natural structure of a metric space. For any strictly semistable model $\mathcal{X}$ of X, the dual graph (or skeleton) $\Delta_{\mathcal{X}}$ is a connected metrized graph that embeds isometrically into $\mathbf{H}$. Further, any point $x \in X^{\mathrm{qm}}$ belongs to some dual graph $\Delta_{\mathcal{X}}$. Pick a strictly semistable model $\mathcal{X}$ of X such that $x_1, x_2 \in \Delta_{\mathcal{X}}$ and ϕ_{ref} is associated to a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$. Let $V \subset X^{\mathrm{Shi}}$ be the set of vertices of $\Delta := \Delta_{\mathcal{X}}$. Each $x \in V$ corresponds to an irreducible component E of the special fiber of $\mathcal{X}$, and we set $r(x) := \deg(\mathcal{L}_{\mathrm{ref}}|_E) \in \mathbf{Q}$. (The measure $\sum_{x \in V} r(x)\delta_x$ on X is the Monge-Ampère measure $dd^c\phi_{\mathrm{ref}}$, see §3.1.)

Given $\phi \in \mathrm{FS}(L)$, the restriction of the function $\varphi := \phi - \phi_{\mathrm{ref}}$ to Δ is then continuous and has the following properties:

(i) φ is convex and piecewise affine (with rational slopes) on each edge of Δ;
(ii) for each vertex $x \in V$, we have $r(x) + \sum_{\vec{v} \in E_x} D_{\vec{v}}\varphi \ge 0$, where E_x is the set of edges of Δ emanating from x and $D_{\vec{v}}\varphi$ is the slope of φ at x along each such edge.

It is now easy to see from (i)–(ii) that $|D_{\vec{v}}\varphi|$ is uniformly bounded independently of φ. From the convexity statement in (ii) it now follows that $\varphi|_\Delta$ is uniformly Lipschitz. Since $x_1, x_2 \in \Delta$, the estimate (2.6) follows.

The case of curves being settled, the general case of Theorem 2.21 is now a consequence of Lemma 2.22 below. Indeed, the line bundle $M := f^*L$ on Y is ample. If we write $f\colon M \to L$ for the canonical map, then $\psi_{\mathrm{ref}} := \phi_{\mathrm{ref}} \circ f \in \mathrm{DFS}(M)$. Further, if $\phi \in \mathrm{FS}(L)$, then $\psi := \phi \circ f \in \mathrm{FS}(M)$, and $(\phi - \phi_{\mathrm{ref}})(x_i) = (\psi - \psi_{\mathrm{ref}})(y_i)$. We can therefore pick $C = C(M, y_1, y_2, \psi_{\mathrm{ref}})$. □

Lemma 2.22. *For any two quasimonomial points $x_1, x_2 \in X^{\mathrm{qm}}$, there exists an algebraically closed extension k'/k, a morphism $f : Y \to X_{k'}$ from a smooth projective k'-analytic curve and quasimonomial points $y_1, y_2 \in Y^{\mathrm{qm}}$ such that $f(y_i) = x_i$.*

The proof relies on the next result.

Lemma 2.23. *Let k be a non-Archimedean field, X a k-analytic space of pure dimension $n \geq 1$, and $x \in X^{\mathrm{qm}}$ a quasimonomial point. We can then find a non-Archimedean field extension k'/k such that the preimage of x under the induced map $\pi : X_{k'} \to X$ has nonempty topological interior.*

Proof of Lemma 2.23. We are grateful to A. Ducros and J. Poineau for help with the following argument.

For any ℓ/k, the ℓ-analytic space X_ℓ has dimension n, the map $\pi_\ell \colon X_\ell \to X$ is continuous and surjective, and by [Berk90, 9.1.4] there exists a quasimomonial point $y \in \pi_\ell^{-1}(x)$. We may therefore replace k by any non-Archimedean extension.

In particular, we may assume k algebraically closed and nontrivially valued. By [Poi13, Corollaire 3.14], for any non-Archimedean extension k'/k, we have a canonical section $\sigma \colon X \to X'$ of π. Indeed, the norm on the Banach k-algebra $\mathcal{M}(\mathcal{H}(x)\hat{\otimes}_k k')$ is multiplicative and defines a point $\sigma(x) =: x'$ in $\pi^{-1}\{x\} = \mathcal{M}(\mathcal{H}(x)\hat{\otimes}_k k') \subset X'$. It follows from [Berk90, 9.1.4, 9.3.2] that x' is also quasimonomial. If we choose $k' := \mathcal{H}(x)$, then $|\mathcal{H}(x')^\times| = |k'^\times|$, so $t(x') = 0$, and hence $s(x') = n$, i.e. x' is Shilov.

We may therefore assume that k is algebraically closed, nontrivially valued, and that x is Shilov. We will prove that any algebraically closed extension k' of $\mathcal{H}(x)$ yields the desired conclusion.

By [Poi13, 4.16], there exists a strictly k-affinoid Laurent domain $V = \mathcal{M}(A) \subset X$ whose Shilov boundary is the singleton $\{x\}$. Then $V' = V_{k'} = \mathcal{M}(A')$ is a Laurent domain in $X' = X_{k'}$, with Shilov boundary $\{x'\}$, where $x' = \sigma(x)$, and $A' = A\hat{\otimes}_k k'$. Consider the reduction maps $\mathrm{red} \colon V \to \tilde{V}$ and $\mathrm{red}' \colon V' \to \tilde{V}' = \operatorname{Spec} \tilde{A}'$, where $\tilde{A}' = \tilde{A} \otimes_{\tilde{k}} \tilde{k}'$. If $\pi \colon V' \to V$ is the canonical map, and $\tilde{\pi} \colon \tilde{V}' \to \tilde{V}$ the reduction, then $\tilde{\pi} \circ \mathrm{red}' = \mathrm{red} \circ \pi$.

The canonical character $A \to \mathcal{H}(x) \to k'$ and the identity on k' induce a character $\chi \colon A' \to k'$. Let $y' \in V(k')$ be the rigid point defined as the image of the corresponding map $\mathcal{M}(k') \to \mathcal{M}(A_{V'}) = V'$. Its reduction $\tilde{y}' = \mathrm{red}'(y')$ is a closed point, given by the image of the morphism $\operatorname{Spec} \tilde{k}' \to \operatorname{Spec} \tilde{A}' = \tilde{V}'$. On the one hand, $\tilde{\pi}(\tilde{y}')$ is the generic point of $\tilde{V}$, so $\pi(y') = x$. On the other hand, the preimage under red' of any closed point in $\tilde{V}'$ is open in V', and hence so is its preimage in X'. Thus y' lies in the topological interior of $\pi^{-1}(x)$ in X'. □

Proof of Lemma 2.22. Lemma 2.23 yields an algebraically closed k'/k such that $\pi^{-1}(x_i) \subset X'$ has nonempty interior for $i = 1, 2$. Since k' is nontrivially valued, X'^{rig} is dense in X', so we can pick $x'_i \in X'^{\mathrm{rig}} \cap \pi^{-1}(x_i)$ for $i = 1, 2$, and it is then easy to construct an irreducible curve on the variety $\mathcal{X}_{k'}$ passing through x'_1, x'_2 (see for instance [Mum70, Lemma, §2.6, p.53]). Let $\mathcal{Y}$ be the normalization of this curve and Y be the corresponding k'-analytic space. We then have points $z_i \in Y(k')$, $i = 1, 2$, and a map $f' \colon Y \to X'$ such that $f'(z_i) = x'_i$, $i = 1, 2$. Write $f := \pi \circ f' \colon Y \to X$. Since $f^{-1}(x_i) = (f')^{-1}\pi^{-1}(x_i)$ is open in Y and nonempty, it contains some point $y_i \in Y^{\mathrm{qm}}$, $i = 1, 2$. □

3. The Monge-Ampère operator on DFS metrics

In this section we recall the definition of the mixed Monge-Ampère operator on DFS metrics, as well as the energy functionals E, I and J on FS metrics. We also prove estimates that are, for the most part, well known in the Archimedean setting. Unless otherwise stated, k is an arbitrary non-Archimedean field, and X is a geometrically integral projective variety over k.

3.1. **The mixed Monge-Ampère operator.** Let ϕ_i be a DFS metric on a line bundle L_i for $1 \le i \le n$. To this data is associated a *mixed Monge-Ampère measure*

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n.$$

This was defined by Chambert-Loir [CL06] when k is discretely valued and contains a countable dense subfield, and extended by Gubler [Gub08] to the general nontrivially valued case.

Later, Chambert-Loir and Ducros [CD12] developed a general theory of forms and currents on Berkovich spaces using ideas from tropical geometry, and used their theory to define mixed Monge-Ampère measures of regularizable metrics, and hence of DFS metrics.[11] See §2.5 for the terminology used in what follows, and [CD12] for details.

First, when the ϕ_i are smooth metrics, $dd^c\phi_i$ is defined as a $(1,1)$-form, and the wedge product $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is a smooth (n,n)-form on X, which is then identified with a measure on X. It is a positive measure when the ϕ_i are smooth psh.

Second, when the ϕ_i are psh-regularizable, i.e. locally given as uniform limits of smooth psh metrics $(\phi_i^j)_j$, the locally defined measures $dd^c\phi_1^j \wedge \cdots \wedge dd^c\phi_n^j$ converge as $j \to \infty$ to a measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ independent on the choice of local regularization ϕ_i^j.

The assigment $(\phi_1, \dots, \phi_n) \mapsto dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is symmetric and multilinear on the semigroup of n-tuples of psh-regularizable metrics. We can therefore define $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ when the ϕ_i are regularizable (i.e. differences of psh-regularizable metrics).

By Lemma 2.9, any FS metric (resp. DFS metric) is psh-regularizable (resp. regularizable), so the mixed Monge-Ampère measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is a positive Radon measure (resp. Radon measure) on X when the ϕ_i are FS metrics (resp. DFS metrics).

We need the following result, which in implicit in [CD12] (see [BE17, Lemma 6.2]).

Lemma 3.1. *For $1 \le i \le n$, let ϕ_i be a regularizable metric on a line bundle L_i. Let k'/k be a non-Archimedean field extension, write π for the canonical maps $X_{k'} \to X$ and $L_{i,k'} \to L_i$, and set $\phi_i' := \phi_i \circ \pi$ for $1 \le i \le n$. Then*

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \pi_*(dd^c\phi_1' \wedge \cdots \wedge dd^c\phi_n'). \tag{3.1}$$

As a consequence, we have

$$\int_X \psi \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \int_{X'} \psi' \, dd^c(\phi_1' \wedge \cdots \wedge dd^c\phi_n') \tag{3.2}$$

for any continuous function ψ on X, where $\psi' = \psi \circ \pi$.

Proposition 3.2. *If ϕ_i is a DFS (resp. FS) metrics on a line bundle L_i, $1 \le i \le n$, then the mixed Monge-Ampère measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is a finite atomic signed (resp. positive) measure on X^{qm} of total mass $(L_1 \cdot \ldots \cdot L_n)$.*

[11] Alternatively, the approach in [GK17] treats DFS metrics more directly, without regularization.

Proof. When k is nontrivially valued, this was proved in [CD12, 6.9.2], see §3.2 below. The mixed Monge-Ampère measure is supported on Shilov points in this case.

When k is trivially valued, we get the same result using Lemma 2.22 with the ground field extension $k' = k((\varpi))$. In this case, the Monge-Ampère measure is now supported on Shilov points and the generic point of X, see §3.4. □

3.2. **Geometric description.** Suppose k is nontrivially valued. Then each DFS metric ϕ_i, $1 \le i \le n$, is a model metric associated to a model $\mathcal{L}_i$ of L. We may assume that the $\mathcal{L}_i$ are all $\mathbf{Q}$-line bundles on a common model $\mathcal{X}$ of X. Let E_i, $i \in I$ be the irreducible components of the special fiber $\mathcal{X}_0$, and let $x_j \in X^{\mathrm{Shi}}$ be the Shilov point associated to E_j. In this case, we have

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \sum_j \lambda_j \delta_{x_j}, \tag{3.3}$$

where $\lambda_j \in \mathbf{R}$, see [CD12, 6.9.2]. If the non-Archimedean field k is *stable*, then

$$\lambda_j = [|\mathcal{H}(x)^\times| : |k^\times|](\mathcal{L}_1|_{E_j} \cdot \ldots \cdot \mathcal{L}_n|_{E_j}), \tag{3.4}$$

see [CD12, 6.9.3]. Important cases of stable fields include algebraically closed fields, and trivially or discretely valued fields.[12]

Let us say a few more words about Gubler's construction in [Gub08], adapted (and specialized) to our situation. Assume k is algebraically closed. For $0 \le i \le n$, let $\hat{\phi}_i = (L_i, \phi, s_i)$ be the datum of a line bundle L_i of X, a model (i.e. DFS) metric ϕ_i on L_i, and a rational section s_i of L_i. Assuming that the divisors of the s_i have empty intersection, Gubler [Gub98] defines a *local height* $\lambda_{\hat{\phi}_0,\ldots,\hat{\phi}_n}$ of X. This is symmetric and multilinear in $(\hat{\phi}_0, \ldots, \hat{\phi}_n)$. Now suppose that $L_0 = \mathcal{O}_X$, so that ϕ_0 is a DFS-function, and that $s_0 \equiv 1$. In this case, the local height $\lambda_{\hat{\phi}_0,\ldots,\hat{\phi}_n}$ is independent of $s_1, \ldots, s_n$. Further, $\phi_0 \mapsto \lambda_{\hat{\phi}_0,\ldots,\hat{\phi}_n}$ is linear and given by integration against the Monge-Ampère measure. In other words,

$$\lambda_{\hat{\phi}_0,\ldots,\hat{\phi}_n} = \int \phi_0 \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n. \tag{3.5}$$

3.3. **The discretely valued case.** When k is discretely valued, the special fiber of $\mathcal{X}$ is a Cartier divisor $\mathcal{X}_0 = \sum_j b_j E_j$, where $b_j \in \mathbf{Z}_{>0}$. In this case, (3.3)–(3.4) give

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \sum_j b_j(\mathcal{L}_1|_{E_j} \cdot \ldots \cdot \mathcal{L}_n|_{E_j})\delta_{x_j}, \tag{3.6}$$

consistent with the construction in [CL06]. The formula in (3.5) can be interpreted as follows. Consider any $\phi_0 \in \mathrm{DFS}(X)$. We may assume ϕ_0 is determined on $\mathcal{X}$ by a Cartier divisor $D = \sum_j r_j E_j$ on $\mathcal{X}$ supported on the special fiber. We then have

$$\int \phi_0 \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \lambda_{\hat{\phi}_0,\ldots,\hat{\phi}_n} = (D \cdot \mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_n) = \sum_j r_j(\mathcal{L}_1|_{E_j} \cdot \ldots \cdot \mathcal{L}_n|_{E_j}).$$

[12]See [BGR, 3.6] for details on stable fields. Recall that we require non-Archimedean fields to be complete.

3.4. **The trivially valued case.** When k is trivially valued, we can interpret the mixed Monge-Ampère measure of DFS metrics in terms of test configurations, in line with [BHJ17]. Indeed, consider metrics $\phi_i \in \mathrm{DFS}(L_i)$, $1 \le i \le n$. These are defined by normal test configurations $(\mathcal{X}, \mathcal{L}_i)$ for (X, L_i), with $\mathcal{X}$ independent of i and dominating the trivial test configuration.

Now apply Lemma 3.1 to $k' = k(\!(\varpi)\!)$. It follows from the discussion in §3.3 that if we write the central fiber as $\mathcal{X}_0 = \sum_j b_j E_j$, then

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \sum_j b_j(\mathcal{L}_1|_{E_j} \cdot \ldots \cdot \mathcal{L}_n|_{E_j})\delta_{x_j}, \tag{3.7}$$

where $x_j \in X$ is the point associated to E_j. Further, if $\psi \in \mathrm{DFS}(X)$ is defined by a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor $D = \sum_j r_j E_j$ on $\mathcal{X}$, supported on the central fiber, then

$$\int \psi \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = (D \cdot \mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_n).$$

Here the intersection number on the right is a sum of intersection numbers on the E_j. There is also another useful expression. As in [BHJ17, §2.2], we can define canonical *compactifications* $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ of $(\mathcal{X}, \mathcal{L}_i)$ (with $\bar{\mathcal{X}}$ still independent of i). We then have

$$\int \psi \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = (D \cdot \bar{\mathcal{L}}_1 \cdot \ldots \cdot \bar{\mathcal{L}}_n),$$

this being an ordinary intersection number on the projective k-scheme $\bar{\mathcal{X}}$.

Proposition 3.3. *Assume k is trivially valued. For $1 \le i \le n$, let ϕ_i be a DFS metric on a line bundle L_i on X. If $\phi_{i,t}$ denotes the scaling of ϕ_i by $t \in \mathbf{Q}_+^\times$ as in* (2.1)*, then*

$$dd^c\phi_{1,t} \wedge \cdots \wedge dd^c\phi_{n,t} = t_*(dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n). \tag{3.8}$$

As a consequence, for any $\psi \in \mathrm{DFS}(X)$, we have

$$\int \psi_t \, dd^c\phi_{1,t} \wedge \cdots \wedge dd^c\phi_{n,t} = t \int \psi \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n. \tag{3.9}$$

Proof. Note that (3.9) follows from (3.8), since $t^*\psi_t(x) = \psi_t(x^t) = t\psi(x)$ for any $\psi \in \mathrm{DFS}(X)$. Conversely (3.9) implies (3.8), since $\mathrm{DFS}(X)$ is dense in $C^0(X)$.

Thus it suffices to prove (3.9). Since we are dealing with group actions of $\mathbf{Q}_+^\times$, it suffices to consider the case when $t = d$ is a positive integer. To do this, we follow [BHJ17, §6.6]. Pick a normal test configuration $\mathcal{X}$ for X, dominating the trivial test configuration, Let $\mathcal{X}'$ be the pullback of $\mathcal{X}$ by the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $\varpi \mapsto \varpi^d$, and let $g_d \colon \mathcal{X}' \to \mathcal{X}$ be the resulting map. Set $\mathcal{L}'_i = g_d^*\mathcal{L}_i$, $1 \le i \le n$ and $D' := g_d^* D$. By Proposition 2.19, $\phi_{i,d}$ and ψ_d are the model metrics associated to $\mathcal{L}'_i$ and D', respectively.

The morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ above extends to a morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$, and the compactification $\bar{\mathcal{X}}'$ of $\mathcal{X}'$ is the base change of $\bar{\mathcal{X}} \to \mathbb{P}^1_k$ with respect to this morphism, so g_d extends to a morphism $g_d \colon \bar{\mathcal{X}}' \to \bar{\mathcal{X}}$, and the compactification $\bar{\mathcal{L}}'_i$ of $\mathcal{L}'_i$ is equal to $g_d^*\bar{\mathcal{L}}_i$. Thus

$$\begin{aligned}\int \psi_d \, dd^c\phi_{1,d} \wedge \cdots \wedge dd^c\phi_{n,d} &= (\mathcal{D}' \cdot \bar{\mathcal{L}}'_1 \cdot \ldots \cdot \bar{\mathcal{L}}'_n) \\ &= (g_d^*\mathcal{D} \cdot g_d^*\bar{\mathcal{L}}_1 \cdot \ldots \cdot g_d^*\bar{\mathcal{L}}_n) = d(\mathcal{D} \cdot \bar{\mathcal{L}}_1 \cdot \ldots \cdot \bar{\mathcal{L}}_n) = d \int \psi \, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n,\end{aligned}$$

where the third equality follows from the projection formula. □

3.5. **Fundamental properties.** We now state two fundamental properties of the Monge-Ampère operator on DFS metrics. First, we have the following *integration by parts* formula.

Proposition 3.4. *For any DFS-functions f, g on X and DFS metrics $\phi_1, \dots, \phi_{n-1}$ on line bundles $L_1, \dots, L_{n-1}$ on X, we have*

$$\int f dd^c g \wedge dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{n-1} = \int g dd^c f \wedge dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{n-1}. \tag{3.10}$$

Proof. After ground field extension, we may assume that k is nontrivially valued. In this case, we can express both sides of (3.10) as local heights, and the equality follows from (3.5) and the symmetry of the local height. Alternatively, one can argue using [CD12, 3.12.2]. □

Second, we have the following *Cauchy-Schwarz inequality.* It is a special case of the local Hodge index theorem (Theorem 2.1) in [YZ16]. The case when k is discretely or trivially valued is also treated in [BFJ15, Proposition 2.21] and [BHJ17, Lemma 6.14], respectively.

Proposition 3.5. *Let $\phi_1, \dots, \phi_{n-1}$ be FS metrics on semiample line bundles $L_1, \dots, L_{n-1}$.*

(i) *The symmetric bilinear form*

$$(f, g) \mapsto -\int_X f dd^c g \wedge dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{n-1}$$

on $\mathrm{DFS}(X)$ *is positive semidefinite.*

(ii) *If $f \in \mathrm{DFS}(X)$ is a nonconstant function, and there exists $\varepsilon \in \mathbf{Q}_+^\times$ such that $\phi_i \pm \varepsilon f \in$ $\mathrm{FS}(L_i)$ for $1 \le i \le n$, then*

$$\int_X f dd^c f \wedge dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{n-1} < 0.$$

Remark 3.6. *The local Hodge index theorem in* [YZ16] *is much more general than Proposition 3.5. For example, it holds when f is associated to a vertical $\mathbf{R}$-Cartier $\mathbf{R}$-divisor on a model of X.*

Propositions 3.4 and 3.5 form the engine that drives the Monge-Ampère vehicle.

3.6. **Mixed Monge-Ampère measures and Shilov points.** The following conjecture states that there are "many" mixed Monge-Ampère measures.

Conjecture 3.7. *Assume that k is nontrivially valued, X is geometrically integral, and L is ample. Let $f\colon X^{\mathrm{Shi}} \to \mathbf{R}$ be a function such that*

$$\int_{X^{\mathrm{Shi}}} f\, dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_n = 0$$

for any FS metrics $\psi_i \in \mathrm{FS}(L)$, $1 \le i \le n$. Then $f \equiv 0$.

Proposition 3.8. *If k is discretely valued, of residue characteristic zero, and X is smooth, then Conjecture 3.7 is true.*

Proof. Given any point $x \in X^{\mathrm{Shi}}$, pick a regular model $\mathcal{X}$ of X such that x corresponds to an irreducible component of $\mathcal{X}_0$, and such that L admits an ample model $\mathcal{L}$ on $\mathcal{X}$. This is possible by the assumptions on k and X. Let $\psi \in \mathrm{FS}(L)$ be the metric induced by $\mathcal{L}$. Let $E_1, \dots, E_N$ be the irreducible components of $\mathcal{X}_0$, with associated points $x_1, \dots, x_N \in X^{\mathrm{Shi}}$. We may assume $x_1 = x$. Write $\mathcal{X}_0 = \sum_1^N b_i E_i$, where $b_i \ge 1$.

Set $\mathcal{L}_i = \mathcal{L}$ and $\psi_i = \psi$ for $2 \le i \le N$. Given $t \in \mathbf{Q}^N$, set $\mathcal{M}_t = \mathcal{L} + t_1 E_1 + \cdots + t_N E_N$, $\chi_t = \phi_{\mathcal{M}_t}$, and $\mu_t = dd^c\chi_t \wedge dd^c\psi_2 \wedge \cdots \wedge dd^c\psi_N$. We claim that we can pick t such that $\mu_t = (L^n)\delta_{x_1}$. Then $(L^n)f(x) = \int f\mu_t = 0$, completing the proof.

Define a map $g = (g_1, \dots, g_N)\colon \mathbf{Q}^N \to \mathbf{Q}^N$ by $g_i(t) = \mu_t\{x_i\}$. We claim that the image of g is the set W of $z \in \mathbf{Q}^N$ such that $\sum_i z_i = (L^n)$. Clearly the image of g is contained in W. To prove the converse, note that g is an affine function, with linear part given by $h\colon \mathbf{Q}^N \to \mathbf{Q}^N$, where $h_i(t) = b_i(\mathcal{M}_t|_{E_i} \cdot \mathcal{L}|_{E_i}^{n-1})$. It suffices to prove that h has 1-dimensional kernel. But $h(t) = 0$ implies that the divisor $D_t = \sum t_i E_i$ satisfies $(D_t^2 \cdot \mathcal{L}^{n-1}) = 0$. By the Hodge Index Theorem, this implies that $t_i = cb_i$, $1 \le i \le N$, for some $c \in \mathbf{Q}$, completing the proof. □

3.7. **Normalized Monge-Ampère operator.** For the rest of this section, we fix a semi-ample line bundle L that we furthermore suppose is big, i.e.

$$V := (L^n) > 0.$$

It is convenient to normalize the Monge-Ampère operator. Given FS metrics $\phi_1, \dots, \phi_n$ on L, we set

$$\mathrm{MA}(\phi_1, \dots, \phi_n) := V^{-1} dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n.$$

This is then a probability measure on X. We also write

$$\mathrm{MA}(\phi) := \mathrm{MA}(\phi, \dots, \phi) = V^{-1}(dd^c\phi)^n.$$

When k is trivially valued, Proposition 3.3 implies $\mathrm{MA}(\phi_t) = t_* \mathrm{MA}(\phi)$ for $\phi \in \mathrm{DFS}(L)$ and $t \in \mathbf{Q}_+^\times$. Further, (3.6) shows that $\mathrm{MA}(\phi_{\mathrm{triv}})$ is a Dirac mass at the generic point of X.

3.8. **The Monge-Ampère energy functional.** Define the *Monge-Ampère energy*[13] of a metric $\phi \in \mathrm{DFS}(L)$ relative to another metric $\psi \in \mathrm{DFS}(L)$ as

$$E(\phi, \psi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}. \tag{3.11}$$

The energy functional has many nice properties. First, it is evidently *antisymmetric*:

$$E(\psi, \phi) = -E(\phi, \psi)$$

for $\phi, \psi \in \mathrm{DFS}(L)$. Second, it satisfies the following *cocycle property*

$$E(\phi, \psi) = E(\phi, \chi) + E(\chi, \psi), \tag{3.12}$$

for $\phi, \psi, \chi \in \mathrm{DFS}(L)$, as follows from integration by parts (Proposition 3.4). Since evidently $E(\phi + c, \phi) = c$ for $\phi \in \mathrm{DFS}(L)$ and $c \in \mathbf{Q}$, this implies the following *translation property*

$$E(\phi + c, \psi) = E(\phi, \psi) + c \quad \text{for any } c \in \mathbf{Q}.$$

Third, it is antiderivative of the Monge-Ampère operator in the sense that

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi + tf, \psi) = \int f\, \mathrm{MA}(\phi), \tag{3.13}$$

for any metrics $\phi, \psi \in \mathrm{DFS}(L)$ and any function $f \in \mathrm{DFS}(X)$. Note that $\phi + tf$ is a priori only a DFS metric for $t \in \mathbf{Q}$. However, $t \to E(\phi + tf, \psi)$ is a polynomial in t of degree at most $n+1$, and the left hand side of (3.13) means the derivative of this polynomial at $t = 0$.

[13] This functional is known under several different names (and notation), e.g. the Aubin-Mabuchi energy.

Differentiating once more, we see that

$$\frac{d^2}{dt^2}\Big|_{t=0} E(\phi + tf) = V^{-1} \int f dd^c f \wedge (dd^c\phi)^{n-1}. \tag{3.14}$$

When restricted to FS metrics, the Monge-Ampère operator has even more properties. First, we have $E(\phi_1, \phi_2) \le 0$ when $\phi_1, \phi_2 \in \mathrm{FS}(L)$ and $\phi_1 \le \phi_2$. By the cocycle property, this implies that E is increasing in the first argument: $E(\phi_1, \psi) \le E(\phi_2, \psi)$, for $\phi_1, \phi_2 \in \mathrm{FS}(L)$ and $\psi \in \mathrm{DFS}(L)$ when $\phi_1 \le \phi_2$.

Second, if $\phi, \psi \in \mathrm{FS}(L)$, the terms in (3.11) are nonincreasing in j, since

$$\int (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j} - \int (\phi - \psi)(dd^c\phi)^{j+1} \wedge (dd^c\psi)^{n-1-j}$$
$$= - \int (\phi - \psi) dd^c(\phi - \psi) \wedge (dd^c\phi)^{j-1} \wedge (dd^c\psi)^{n-1-j} \ge 0.$$

by the Cauchy-Schwartz inequality (Proposition 3.5). This implies that

$$\int (\phi - \psi)\,\mathrm{MA}(\phi) \le E(\phi, \psi) \le \int (\phi - \psi)\,\mathrm{MA}(\psi) \tag{3.15}$$

for $\phi, \psi \in \mathrm{FS}(L)$. Third, the right-hand side of (3.14) is nonpositive when $\phi \in \mathrm{FS}(L)$. This implies that for any $\psi \in \mathrm{FS}(L)$, $E(\cdot, \psi)$ is $\mathbf{Q}$-*concave* on $\mathrm{FS}(L)$, in the sense that

$$E(\theta_1\phi_1 + \theta_2\phi_2, \psi) \ge \theta_1 E(\phi_1, \psi) + \theta_2 E(\phi_2, \psi)$$

for $\phi_1, \phi_2 \in \mathrm{FS}(L)$, $\psi \in \mathrm{DFS}(L)$, and $\theta_1, \theta_2 \in \mathbf{Q}_+$ such that $\theta_1 + \theta_2 = 1$.

3.9. **The I and J-functionals.** Next we introduce two other functionals based on the Monge-Ampère operator. Given two FS metrics ϕ and ψ on L, set

$$I(\phi, \psi) = \int (\phi - \psi)(\mathrm{MA}(\psi) - \mathrm{MA}(\phi)).$$

Clearly, I is symmetric, $I(\phi, \psi) = I(\psi, \phi)$, and translation invariant, $I(\phi + c, \psi + d) = I(\phi, \psi)$ for any constants c, d. By (3.15), we have $I(\phi, \psi) \ge 0$, as can also be seen from

$$I(\phi, \psi) = - \sum_{j=0}^{n-1} V^{-1} \int (\phi - \psi) dd^c(\phi - \psi) \wedge (dd^c\phi)^j \wedge (dd^c\psi)^{n-1-j}. \tag{3.16}$$

The closely related J-functional is defined by

$$J_\psi(\phi) := -E(\phi, \psi) + \int (\phi - \psi)\,\mathrm{MA}(\psi).$$

By (3.11), this is also nonnegative, and it can be written as

$$J_\psi(\phi) = - \sum_{j=0}^{n-1} \frac{j+1}{n+1} V^{-1} \int (\phi - \psi) dd^c(\phi - \psi) \wedge (dd^c\phi)^{n-1-j} \wedge (dd^c\psi)^j. \tag{3.17}$$

The $\mathbf{Q}$-concavity of $\phi \mapsto E(\phi, \psi)$ implies that $\phi \mapsto J_\psi(\phi)$ is $\mathbf{Q}$-convex on $\mathrm{FS}(L)$.

Lemma 3.9. *For any FS metrics $\phi, \psi \in \mathrm{FS}(L)$ we have*
$$I(\phi,\psi) = J_\psi(\phi) + J_\phi(\psi) \tag{3.18}$$
and
$$n^{-1}J_\phi(\psi) \le J_\psi(\phi) \le I(\phi,\psi) \le (n+1)J_\psi(\phi). \tag{3.19}$$

Proof. The equality (3.18) is immediate from the definition, and implies $J_\psi(\phi) \le I(\phi,\psi)$ since $J_\phi(\psi) \ge 0$. The last inequality in (3.19) follows from (3.16) and (3.17). It remains to prove the first inequality. But
$$nJ_\phi(\psi) - J_\psi(\phi) = (n+1)(E(\phi) - E(\psi)) + n\int(\psi-\phi)\,\mathrm{MA}(\phi) + \int(\psi-\phi)\,\mathrm{MA}(\psi),$$
which is nonnegative since the terms in (3.12) are decreasing. □

We will prove in Corollary 6.28 that $I(\phi,\psi) = 0$ iff $\phi - \psi$ is a constant.

3.10. **Reference metric.** The functionals E, I and J above depend on two arguments. It will be convenient to fix a reference metric $\phi_{\mathrm{ref}} \in \mathrm{FS}(L)$ and use this as the second argument.[14] In the trivially valued case, we will always pick $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$, the trivial metric on L. In the discretely valued case, there is no canonical choice of a reference metric.

Thus we set
$$E(\phi) := E(\phi,\phi_{\mathrm{ref}}) = \frac{1}{n+1}\sum_{j=0}^{n} V^{-1}\int(\phi-\phi_{\mathrm{ref}})(dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-j}.$$
$$J(\phi) := J_{\phi_{\mathrm{ref}}}(\phi) = \int(\phi-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}}) - E(\phi). \tag{3.20}$$
and
$$I(\phi) := I(\phi,\phi_{\mathrm{ref}}) = \int(\phi-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}}) - \int(\phi-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi).$$
From (3.18) and (3.19) it then follows that
$$n^{-1}J \le I - J \le nJ. \tag{3.21}$$

3.11. **Ground field extension and scaling.** Consider a non-Archimedean field extension k'/k. Write $X' = X_{k'}$ and $L' := L_{k'}$, and let $\pi\colon X' \to X$ and $\pi\colon L' \to L$ be the canonical maps. From Lemma 3.1 we get

Corollary 3.10. *If $\phi,\psi \in \mathrm{FS}(L)$ and $\phi' := \phi\circ\pi$, $\psi' := \psi\circ\pi$, then*
$$E(\phi',\psi') = E(\phi,\psi), \quad I(\phi',\psi') = I(\phi,\psi) \quad \text{and} \quad J_{\psi'}(\phi') = J_\psi(\phi).$$
With $\psi = \phi_{\mathrm{ref}}$ and $\psi' = \phi_{\mathrm{ref}}\circ\pi$, we get $E(\phi') = E(\phi)$, $I(\phi') = I(\phi)$, and $J(\phi') = J(\phi)$.

Similarly, Proposition 3.3 implies the following result, which generalizes Lemma 7.3 and Proposition 7.8 of [BHJ17], and shows that the energy functionals are *homogeneous* with respect to the scaling action by $\mathbf{Q}_+^\times$.

Corollary 3.11. *If k is trivially valued, $\phi,\psi \in \mathrm{FS}(L)$ and $t \in \mathbf{Q}_+^\times$, then*
$$E(\phi_t,\psi_t) = tE(\phi,\psi), \quad I(\phi_t,\psi_t) = tI(\phi,\psi), \quad \text{and} \quad J_{\psi_t}(\phi_t) = tJ_\psi(\phi).$$
Setting $\psi = \phi_{\mathrm{triv}}$, this yields $E(\phi_t) = tE(\phi)$, $I(\phi_t) = tI(\phi)$, and $J(\phi_t) = tJ(\phi)$.

[14] Alternatively, the functionals could be considered as metrics on suitable line bundles over a point. This would remove the need for a reference metric; see e.g. [BHJ16] or [BE17].

3.12. **Estimates.** The results in this section will be used for studying the Monge-Ampère operator on the class of metrics of finite energy. They typically have a closely related analogue in the complex setting; for the convenience of the reader we indicate some of the precise results from [BBGZ13, BBEGZ16] below. The proofs, which are surprisingly similar in the complex and non-Archimedean case, appear in Appendix A.

We will let C_n denote various constants (that can easily be made explicit) depending only on n. Similarly, $\mathrm{D}_{\mathrm{ref}}$ is a constant that depends on n, X, L, and on the choice of reference metric $\phi_{\mathrm{ref}} \in \mathrm{FS}(L)$. When k is trivially valued and $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$, we can choose $\mathrm{D}_{\mathrm{ref}} = 0$.

Lemma 3.12. *If $\phi \in \mathrm{FS}(L)$, then $0 \le \sup(\phi - \phi_{\mathrm{ref}}) - \int(\phi - \phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}}) \le \mathrm{D}_{\mathrm{ref}}$.*

Corollary 3.13. *If $\phi \in \mathrm{FS}(L)$ and $\sup(\phi - \phi_{\mathrm{ref}}) = 0$, then $-E(\phi) - \mathrm{D}_{\mathrm{ref}} \le J(\phi) \le -E(\phi)$.*

Lemma 3.14. [BBGZ13, Proposition 2.3]. *If $\phi, \psi \in \mathrm{FS}(L)$ and $t \in [0,1] \cap \mathbf{Q}$, then*

$$I(t\phi + (1-t)\psi, \psi) \le nt^2 I(\phi, \psi).$$

Lemma 3.15. [BBEGZ16, Lemma 1.9]. *For any FS metrics ϕ_1, ϕ_2, ψ on L we have*

$$-V^{-1}\int(\phi_1 - \phi_2)dd^c(\phi_1 - \phi_2) \wedge (dd^c\psi)^{n-1} \le C_n I(\phi_1, \phi_2)^{\frac{1}{2^{n-1}}} \max_{i=1,2} I(\phi_i, \psi)^{1-\frac{1}{2^{n-1}}}.$$

Lemma 3.16. [BBEGZ16, Theorem 1.8]. *For any FS metrics ϕ_1, ϕ_2, ϕ_3 on L, we have*

$$I(\phi_1, \phi_3) \le C_n \max\{I(\phi_1, \phi_2), I(\phi_2, \phi_3)\}.$$

Corollary 3.17. *For any FS metrics ϕ, ψ on L we have*

$$I(\phi, \psi) \le C_n \max\{J(\phi), J(\psi)\}.$$

Corollary 3.18. *For any FS metrics $\phi_1, \phi_2, \psi_1, \dots, \psi_{n-1}$ on L we have*

$$\begin{aligned} &-V^{-1}\int(\phi_1 - \phi_2)dd^c(\phi_1 - \phi_2) \wedge dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_{n-1} \\ &\qquad \le C_n I(\phi_1, \phi_2)^{\frac{1}{2^{n-1}}} \max\{\max_{i=1,2} J(\phi_i), \max_{1\le i\le n-1} J(\psi_i)\}^{1-\frac{1}{2^{n-1}}}. \end{aligned}$$

Corollary 3.19. *For any FS metrics ψ_1, ψ_2 and ϕ_i, ϕ_i', $1 \le i \le n$ on L we have*

$$\begin{aligned} &V^{-1}\left|\int(\psi_1 - \psi_2)dd^c\phi_1' \wedge \cdots \wedge dd^c\phi_n' - \int(\psi_1 - \psi_2)dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n\right| \\ &\qquad \le C_n I(\psi_1, \psi_2)^{\frac{1}{2^n}} \max_{1\le p\le n} I(\phi_p, \phi_p')^{\frac{1}{2^n}} M^{1-\frac{1}{2^{n-1}}}, \end{aligned}$$

where $M = \max\{J(\psi_1), J(\psi_2), \max_p J(\phi_p), \max_p J(\phi_p')\}$.

Corollary 3.20. *For any FS metrics $\phi_1, \phi_2, \psi_1, \psi_2$ on L we have*

$$\int(\psi_1 - \psi_2)(\mathrm{MA}(\phi_1) - \mathrm{MA}(\phi_2)) \le C_n I(\psi_1, \psi_2)^{\frac{1}{2^n}} I(\phi_1, \phi_2)^{\frac{1}{2^n}} M^{1-\frac{1}{2^{n-1}}},$$

where $M = \max\{J(\phi_1), J(\phi_2), J(\psi_1), J(\psi_2)\}$.

Corollary 3.21. *For any FS metrics ϕ, ψ on L, we have*

$$\max\{|I(\phi) - I(\psi)|, |J(\phi) - J(\psi)|\} \le C_n I(\phi, \psi)^{\frac{1}{2^n}} \max\{J(\phi), J(\psi)\}^{1-\frac{1}{2^n}}.$$

Lemma 3.22. [BBGZ13, Lemma 2.7]. *If $\phi_0, \phi_1, \dots, \phi_n$ are FS metrics on L, then*

$$\int (\phi_0 - \phi_{\mathrm{ref}}) \operatorname{MA}(\phi_1, \dots, \phi_n) \geq \int (\phi_0 - \phi_{\mathrm{ref}}) \operatorname{MA}(\phi_{\mathrm{ref}}) - C_n(\mathrm{D}_{\mathrm{ref}} + \max_i J(\phi_i)).$$

Lemma 3.23. [BBGZ13, Lemma 3.13]. *For any FS metrics ϕ_1, ϕ_2, ψ on L, we have*

$$\left| \int (\psi - \phi_{\mathrm{ref}}) \operatorname{MA}(\phi_1) - \int (\psi - \phi_{\mathrm{ref}}) \operatorname{MA}(\phi_2) \right| \\ \leq C_n I(\phi_1, \phi_2)^{\frac{1}{2}} (\mathrm{D}_{\mathrm{ref}} + \max\{J(\psi), J(\phi_1), J(\phi_2)\})^{\frac{1}{2}}.$$

4. Dual complexes

In this section, *k is discretely or trivially valued, of residue characteristic zero, and X is smooth and projective.* Following [BFJ16a], we will show how X can be described using dual complexes. This technique will be crucial to understanding the deeper properties of singular semipositive metrics on ample line bundles.

4.1. **Snc models and test configurations.** If k is discretely valued, a model $\mathcal{X}$ of X is an *snc model* if $\mathcal{X}$ is regular and the special fiber $\mathcal{X}_0$ has strict normal crossing support. Let E_i, $i \in I$ be the irreducible components of $\mathcal{X}_0$. A *stratum* of $\mathcal{X}_0$ is a connected component of a nonempty intersection $\bigcap_{j\in J} E_j$, where $J \subset I$. By Hironaka's theorem, the set $\mathrm{SNC}(X)$ of snc models is directed and cofinal in the set of all models.

Similarly, if k is trivially valued, a test configuration $\mathcal{X}$ for X is an *snc test configuration* if $\mathcal{X}$ dominates the trivial test configuration and the central fiber $\mathcal{X}_0$ has strict normal crossing support. As above we define the notion of a stratum of $\mathcal{X}_0$. Again, Hironaka's theorem implies that the set $\mathrm{SNC}(X)$ of snc test configurations is directed and cofinal in the set of all test configurations.

If k is trivially valued and $\mathcal{X}$ is an snc test configuration for X, then the base change $\mathcal{X}' = \mathcal{X} \times_{\mathbb{A}^1_k} \operatorname{Spec} k'^{\circ}$ is an snc model of $X' = X_{k'}$. Further, the strata of $\mathcal{X}_0$ and $\mathcal{X}'_0$ are in 1-1 correspondence.

4.2. **Dual complexes of snc models.** First assume k is discretely valued, i.e. $|k^\times| = r^{\mathbf{Z}}$, where $0 < r < 1$. The discussion below follows [BFJ16a, §2.4]. Let $\mathcal{X}$ be an snc model, with special fiber $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$. The *dual complex* $\Delta_{\mathcal{X}}$ of $\mathcal{X}$ is the (generalized) simplicial complex with one vertex for each E_i and a p-dimensional face for each stratum of $\mathcal{X}_0$ of codimension p. We equip $\Delta_{\mathcal{X}}$ with an *integral affine structure*, and define an *embedding* $\mathrm{emb}_{\mathcal{X}} \colon \Delta_{\mathcal{X}} \hookrightarrow X$ as follows.

Consider the simplex σ_Y of $\Delta_{\mathcal{X}}$ corresponding to a stratum Y of $\mathcal{X}_0$. Let $E_0, \dots, E_p$ be the irreducible components of $\mathcal{X}_0$ containing Y. Identify σ_Y with the simplex $\{w \in \mathbf{R}_+^{p+1} \mid \sum_0^p b_i w_i = 1\}$ and equip it with the integral affine structure induced by $\mathbf{Z}^{p+1} \subset \mathbf{R}^{p+1}$.

Now let $\eta \in \mathcal{X}_0$ be the generic point of Y and pick a system $(z_j)_{0\le j\le p}$ of regular parameters for $\mathcal{O}_{\mathcal{X},\eta}$ with z_j defining E_j. By Cohen's structure theorem, $\widehat{\mathcal{O}}_{\mathcal{X},\eta} \simeq \kappa(\eta)[\![z_0, \dots, z_p]\!]$. Let val_w be the restriction to $\mathcal{O}_{\mathcal{X},\eta}$ of the monomial valuation on this power series ring, taking value w_j on z_j, i.e. $\mathrm{val}_w\left(\sum_{\alpha\in\mathbf{N}^{p+1}} c_\alpha z^\alpha\right) = \min\{\sum_0^p w_j \alpha_j \mid c_\alpha \neq 0\}$, and set $\mathrm{emb}_{\mathcal{X}}(w) := r^{\mathrm{val}_w}$. Then $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}}) \subset X^{\mathrm{qm}}$. If $w \in \Delta_{\mathcal{X}}$, then $\mathrm{emb}_{\mathcal{X}}(w) \in X^{\mathrm{Shi}}$ iff w is a rational point; in particular, Shilov points are dense in $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$.

There is also an *evaluation map* $\mathrm{ev}_{\mathcal{X}} : X \to \Delta_{\mathcal{X}}$, defined as follows. Given $x \in X$, set $\xi := \mathrm{red}_{\mathcal{X}}(x) \in \mathcal{X}_0$, so that $|f(x)| \le 1$ for $f \in \mathcal{O}_{\mathcal{X},\xi}$ and $|f(x)| < 1$ for $f \in \mathfrak{m}_{\mathcal{X},\xi}$. Let $E_0, \dots, E_p$ be the irreducible components of $\mathcal{X}_0$ containing ξ, and let Y be the minimal stratum of $\mathcal{X}_0$ containing ξ. Then $\mathrm{ev}_{\mathcal{X}}(x) \in \sigma_Y$ is the point with weight $w = (w_0, \dots, w_p)$ given by $w_j = \log|z_j(x)|/\log r$, where z_j is a local equation for E_j, $0 \le j \le p$.

The composition $p_{\mathcal{X}} := \mathrm{emb}_{\mathcal{X}} \circ \mathrm{ev}_{\mathcal{X}} \colon X \to X$ is continuous and equal to the identity on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$; in other words, it is a *retraction* of X onto $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$.

4.3. **Dual complexes of snc test configurations.** Now assume k is trivially valued. For any snc test configuration $\mathcal{X}$ of X, we define the dual complex $\Delta_{\mathcal{X}}$ exactly as in the discretely valued case. For example, the dual complex of the trivial test configuration has a single vertex, corresponding to $X \times \{0\}$.

Pick any $r \in (0,1)$ and set $k' = k_r = k(\!(\varpi)\!)$. The base change $\mathcal{X}'$ of $\mathcal{X}$ is then an snc model of $X' = X_{k'}$ and we can identify the dual complexes of $\mathcal{X}$ and $\mathcal{X}'$.

The image of $\Delta_{\mathcal{X}}$ under $\mathrm{emb}_{\mathcal{X}'}\colon \Delta_{\mathcal{X}} \hookrightarrow X'$ consists of $k^\times$-invariant points. Hence there exists an embedding $\mathrm{emb}_{\mathcal{X}}\colon \Delta_{\mathcal{X}} \hookrightarrow X$ such that $\mathrm{emb}_{\mathcal{X}'} = \sigma \circ \mathrm{emb}_{\mathcal{X}}$, where $\sigma = \sigma_r\colon X \to X'$ is the Gauss extension. Here $\mathrm{emb}_{\mathcal{X}}(w) \in X^{\mathrm{div}}$ iff $w \in \Delta_{\mathcal{X}}$ is a rational point, so divisorial points are dense in $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$.

We also define the evaluation map $\mathrm{ev}_{\mathcal{X}}\colon X \to \Delta_{\mathcal{X}}$ as the composition $\mathrm{ev}_{\mathcal{X}} = \mathrm{ev}_{\mathcal{X}'} \circ \sigma$. As before, the composition $p_{\mathcal{X}} := \mathrm{emb}_{\mathcal{X}} \circ \mathrm{ev}_{\mathcal{X}}\colon X \to X$ is a retraction onto $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. The maps $\mathrm{emb}_{\mathcal{X}}$ and $\mathrm{ev}_{\mathcal{X}}$ depend on r, but only up to scaling. Note that $\sigma \circ p_{\mathcal{X}} = p_{\mathcal{X}'} \circ \sigma$.

4.4. **Limits of dual complexes.** In both the discretely valued and trivially valued case we shall, for convenience, view a dual complex $\Delta_{\mathcal{X}}$, where $\mathcal{X} \in \mathrm{SNC}(X)$, as a *subset* of the Berkovich space X. Thus $\mathrm{emb}_{\mathcal{X}}\colon \Delta_{\mathcal{X}} \hookrightarrow X$ is simply the inclusion, $p_{\mathcal{X}} = \mathrm{ev}_{\mathcal{X}}$ is a retraction of X onto $\Delta_{\mathcal{X}}$, and Shilov points are dense in $\Delta_{\mathcal{X}}$. For example, the dual complex of the trivial test configuration of X is a singleton, consisting of the generic point of X.

Theorem 4.1. *The following properties hold in both the discretely and trivially valued case.*

(i) *We have $X^{\mathrm{qm}} = \bigcup_{\mathcal{X}} \Delta_{\mathcal{X}}$, where $\mathcal{X}$ runs over elements of* $\mathrm{SNC}(X)$.
(ii) *If $\mathcal{X}, \mathcal{Y} \in \mathrm{SNC}(X)$ and $\mathcal{X}$ dominates $\mathcal{Y}$, then:*
  (a) *$p_{\mathcal{Y}} \circ p_{\mathcal{X}} = p_{\mathcal{Y}}$ on X;*
  (b) *$\Delta_{\mathcal{Y}} \subset \Delta_{\mathcal{X}}$, and hence $p_{\mathcal{X}} = \mathrm{id}$ on $\Delta_{\mathcal{Y}}$.*
(iii) *For every $x \in X$, we have $\lim_{\mathcal{X}} p_{\mathcal{X}}(x) = x$.*

Proof. In the discretely valued case, all the statements are proved in [BFJ16a, §3.2]. Now suppose k is trivially valued.

The statements in (ii) follow from the discretely valued case. Indeed, if $\mathcal{X} \geq \mathcal{Y}$, then the model $\mathcal{X}'$ dominates $\mathcal{Y}'$. This first gives $p_{\mathcal{Y}'} \circ p_{\mathcal{X}'} = p_{\mathcal{Y}'}$ on X', and hence

$$\sigma \circ p_{\mathcal{Y}} \circ p_{\mathcal{X}} = p_{\mathcal{Y}'} \circ \sigma \circ p_{\mathcal{X}} = p_{\mathcal{Y}'} \circ p_{\mathcal{X}'} \circ \sigma = p_{\mathcal{Y}'} \circ \sigma = \sigma \circ p_{\mathcal{Y}}.$$

This implies $p_{\mathcal{Y}} \circ p_{\mathcal{X}} = p_{\mathcal{Y}}$ since σ is injective. Similarly, $\Delta_{\mathcal{Y}'} \subset \Delta_{\mathcal{X}'}$ as subsets of X', and hence $\Delta_{\mathcal{Y}} \subset \Delta_{\mathcal{X}}$, again by the injectivity of σ. This proves (ii).

To prove (i) and (iii) we need to revisit the arguments in [BFJ16a]. First consider (i). If $x \in \Delta_{\mathcal{X}}$, then $\sigma(x) \in \Delta_{\mathcal{X}'} \subset X'^{\mathrm{qm}}$, and hence $x \in X^{\mathrm{qm}}$ by Corollary 1.5. Now suppose $x \in X^{\mathrm{qm}}$. We have $x' := \sigma(x) \in X'^{\mathrm{qm}}$, and hence $x' \in \Delta_{\mathcal{X}'}$ for some $\mathcal{X}' \in \mathrm{SNC}(X)$, but there is a priori no reason for the model $\mathcal{X}'$ to be $\mathbb{G}_m$-invariant and hence arise from a test configuration. However, we can adapt the proof of [JM12, Proposition 3.7] (itself based on the proof of [ELS03, Proposition 2.8]) to $x' = \sigma(x)$, viewed as a $k^\times$-invariant valuation in $X \times \mathbf{A}^1$, to construct a test configuration $\mathcal{X} \in \mathrm{SNC}(X)$ such that $x' \in \Delta_{\mathcal{X}'}$, where $\mathcal{X}'$ is the associated snc model, and then $x \in \Delta_{\mathcal{X}}$.

Finally consider (iii). By Proposition 2.18 it suffices to prove that $\lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} = \varphi$ pointwise on X for every function φ of the form $\varphi = \log|\mathfrak{a}| \circ \sigma$, where $\mathfrak{a}$ is a flag ideal on $X \times \mathbf{A}^1$. But $\varphi = \varphi \circ p_{\mathcal{X}}$ as soon $\mathcal{X} \in \mathrm{SNC}(X)$ dominates the blowup of $X^{\mathrm{sch}} \times \mathbb{A}^1_k$ along $\mathfrak{a}$. □

4.5. **Valuations and dual cone complexes.** When k is trivially valued, we can, as a special case of [JM12], describe both X^{val} and X^{qm} in terms of cone complexes.

Let $\pi\colon Y \to X^{\mathrm{sch}}$ be a proper birational morphism with Y smooth, and D a reduced simple normal crossings divisor on Y such that π is an isomorphism outside the support of D. To such a *log smooth pair* (Y, D) *over* X^{sch} we associate a *dual cone complex* $\Delta(Y, D)$, and embed the latter into X^{val} as the set of monomial points with respect to coordinates associated to irreducible components of D. The apex of $\Delta(Y, D)$ is the trivial valuation on X. We have $x \in X^{\mathrm{qm}}$ iff $x \in \Delta(Y, D)$ for some log smooth pair (Y, D) over X^{sch}, see [JM12, §3.2], and X^{val} is naturally an inverse limit of cone complexes over all log smooth pairs.

Thus we have two different descriptions of X^{qm}: as a union of simplicial complexes or as a union of simplicial cone complexes. Figure 2 illustrates the situation in dimension 1.

Question 4.2. *Suppose $\Delta = \Delta_{\mathcal{X}} \subset X$ is a dual complex in the sense of §4.3. Does there exist a log smooth pair (Y, D) over X^{sch} such that $\Delta \subset \Delta(Y, D)$? Conversely, suppose K is a compact subset of some punctured cone complex $\Delta(Y, D)^{\times}$. Does there exist an snc test configuration $\mathcal{X}$ such that $K \subset \Delta_{\mathcal{X}}$.*

We expect the answers to be 'yes', but cannot prove this. In this paper, we shall work with dual complexes of test configurations rather than dual cone complexes.

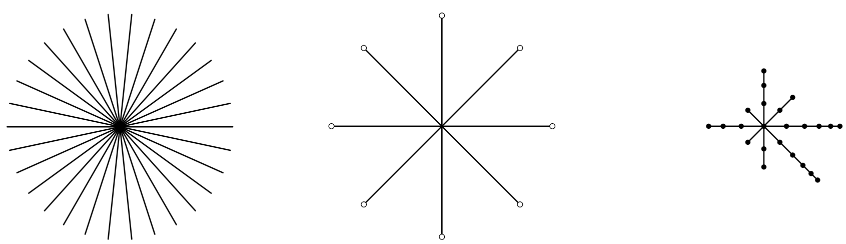

FIGURE 2. The left picture shows the Berkovich analytification X of a smooth curve X^{sch} over a trivially valued field. The middle picture shows the dual cone cone complex $\Delta(X, D)$ of a reduced divisor D of degree 8 on X^{sch}. The right picture shows the dual complex $\Delta_{\mathcal{X}}$ associated to an snc test configuration $\mathcal{X}$ for X

Finally, various subsets of Berkovich analytifications can be analyzed using a different kind of dual complexes. For example, consider the affine space $\mathbf{A}^n_k$ over a trivially valued field. In [BFJ08b], it was shown that the open subset of points centered at a given closed point $\xi \in \mathbb{A}^n_k$ is a cone over a compact space V_ξ, the valuation space at ξ. This space is homeomorphic to the inverse limit of dual complexes of log smooth pairs (Y, D) over $\mathbb{A}^n_k$, with D lying over ξ. In dimension $n = 2$, the valuation space has a natural tree structure and is called the *valuative tree* [FJ04]. These spaces, as well as their cousins "at infinity" have proved useful in dynamics [FJ07, FJ11] and singularity theory [FJ05a, FJ05b], but we shall not consider them further here.

5. Plurisubharmonic metrics

Let k be a non-Archimedean field, X the analytification of a geometrically integral projective k-variety, and L a semiample line bundle on X. In this section, we define and study a class $\mathrm{PSH}(L)$ of singular metrics on L that we call *plurisubharmonic* (psh) or *semipositive*. They are defined as decreasing limits of FS metrics.

When k is discretely or trivially valued and of residue characteristic zero, X is smooth, and L is ample, we use dual complexes to prove that $\mathrm{PSH}(L)/\mathbf{R}$ is compact, and that the psh envelope of a continuous metric is continuous. In the discretely valued case, we show that $\mathrm{PSH}(L)$ coincides with the class of singular semipositive metrics used in [BFJ16a, BFJ15].

5.1. Definitions and basic properties.

Let k, X and L be as above.

Definition 5.1. *A singular metric ϕ on L is* plurisubharmonic *(psh) if it is the pointwise limit of a decreasing net of FS metrics on L, and $\phi \not\equiv -\infty$ on $L^\times$. We write* $\mathrm{PSH}(L)$ *for the set of psh singular metrics on L.*

We sometimes say *semipositive* instead of plurisubharmonic to conform with usage elsewhere in the literature.

If $\phi \in \mathrm{PSH}(L)$, then a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ such that $\phi = \lim_j \phi_j$ is called a *regularization* of ϕ. Such nets always exist by our definition of $\mathrm{PSH}(L)$.

For brevity we will often refer to the elements of $\mathrm{PSH}(L)$ as metrics rather than singular metrics, unless the distinction is important. In any case, there are no singularities above quasimonomial points.

Proposition 5.2. *If $\phi \in \mathrm{PSH}(L)$, then $\phi > -\infty$ on $p^{-1}(X^{\mathrm{qm}})$, where $p\colon L \to X$ is the canonical map.*

Proof. Pick any metric $\psi \in \mathrm{DFS}(L)$. It suffices to prove that $\phi - \psi > -\infty$ on X^{qm}. But this follows the Izumi inequality (Theorem 2.21) applied to a regularization of ϕ. □

The topology on $\mathrm{PSH}(L)$ is defined in terms of pointwise convergence on X^{qm}: $\phi_j \to \phi$ iff the functions $\phi_j - \phi$ on X^{qm} converge pointwise to 0. We sometimes call this the *weak topology*, and refer to pointwise convergence on X^{qm} as *weak convergence*. It plays the role of L^1-convergence in the Archimedean case.[15] We will see in §5.2 that $\mathrm{PSH}(L)$ is Hausdorff.

When k is discretely or trivially valued, and of residue characteristic zero, $\mathrm{PSH}(L)/\mathbf{R}$ is compact, see Corollary 5.35. In the discretely valued case, $\mathrm{PSH}(L)$ coincides with the class of semipositive singular metrics defined in [BFJ16a], see Corollary 5.36.

Remark 5.3. *Our definition of* $\mathrm{PSH}(L)$ *is global. One could define a (potentially) more general notion of singular semipositive limits using decreasing limits of smooth psh metrics in the sense of* [CD12], *see §2.5. However, the class* $\mathrm{PSH}(L)$ *is sufficiently large for the purposes of this paper.*

Definition 5.4. *A function $\varphi\colon X \to [-\infty, +\infty)$ is quasi-plurisubharmonic, or qpsh, if there exists a semiample line bundle L such that $\varphi = \phi - \psi$, where $\phi \in \mathrm{PSH}(L)$ and $\psi \in \mathrm{DFS}(L)$. We write* $\mathrm{QPSH}(X)$ *for the set of quasi-psh functions on X.*

In the trivially valued case we have the following more precise notion.

[15] In contrast, the convergence in Definition 5.1 is supposed to hold pointwise on all of X (or $L^\times$).

Definition 5.5. *When k is trivially valued, a function $\varphi\colon X\to[-\infty,+\infty)$ is L-psh if $\phi_{\mathrm{triv}}+\varphi\in\mathrm{PSH}(L)$, where ϕ_{triv} is the trivial metric on L.*

Note that any semipositive metric is upper semicontinuous (usc) since it is the decreasing limit of continuous metrics. Thus any qpsh function is usc.

Proposition 5.6. *The following properties hold:*

- (i) *if $\phi\in\mathrm{PSH}(L)$, then $\phi+c\in\mathrm{PSH}(L)$ for any $c\in\mathbf{R}$;*
- (ii) *if $\phi_i\in\mathrm{PSH}(L_i)$, $i=1,2$, then $\phi_1+\phi_2\in\mathrm{PSH}(L_1+L_2)$;*
- (iii) *if $\phi_1,\phi_2\in\mathrm{PSH}(L)$, then $\max\{\phi_1,\phi_2\}\in\mathrm{PSH}(L)$;*
- (iv) *if ϕ is a singular metric on L, $m\geq 1$, and $m\phi\in\mathrm{PSH}(mL)$, then $\phi\in\mathrm{PSH}(L)$;*
- (v) *if $\phi_1,\phi_2\in\mathrm{PSH}(L)$, $\theta_1,\theta_2\in\mathbf{R}_+$, and $\theta_1+\theta_2=1$, then $\theta_1\phi_1+\theta_2\phi_2\in\mathrm{PSH}(L)$;*
- (vi) *If $(\phi_j)_j$ is a decreasing net in $\mathrm{PSH}(L)$ and $\phi:=\lim_j\phi_j\not\equiv-\infty$, then $\phi\in\mathrm{PSH}(L)$;*
- (vii) *If $(\phi_j)_j$ is a net of psh metrics in $\mathrm{PSH}(L)$ and ϕ_j converges uniformly to a metric ϕ, then $\phi\in\mathrm{PSH}(L)$.*

Proof. The statements in (i)–(iv) follow from the corresponding statements in Lemma 2.4. Using (i), we see that (vi) implies (vii). Further, (vi) and (vii) imply (v). Indeed, Lemma 2.4 shows that (v) holds when $\phi_i\in\mathrm{FS}(L_i)$ and $\theta_i\in\mathbf{Q}_+$. Using (vii) and that $\mathbf{Q}_+$ is dense in $\mathbf{R}_+$, we conclude (v) when $\phi_i\in\mathrm{FS}(L_i)$ and $\theta_i\in\mathbf{R}_+$; the general case then follows from (vi).

It remains to prove (vi). For each j, let $(\phi_{j,l})_{l\in L_j}$ be a decreasing net in $\mathrm{FS}(L)$ converging to ϕ_j, and for $m\in\mathbf{N}^\times$, set $\phi_{j,l,m}:=\phi_{j,l}+m^{-1}\in\mathrm{FS}(L)$. Let I be the set of triples (j,l,m), where $j\in J$, $l\in L_j$ and $m\in\mathbf{N}^\times$. Define a partial order on I by $(j_1,l_1,m_1)\geq(j_2,l_2,m_2)$ iff $\phi_{j_1,l_1,m_1}\leq\phi_{j_2,l_2,m_2}$. Let us show that I is directed, i.e. any two elements (j_1,l_1,m_1) and (j_2,l_2,m_2) can be dominated by a third. To see this, first pick $j\geq j_1,j_2$ so that $\phi_j\leq\phi_{j_1}$ and $\phi_j\leq\phi_{j_2}$. Also pick $m=2\max\{m_1,m_2\}$. We claim that $\phi_{j,l,m}\leq\phi_{j_1,l_1,m_1}$ and $\phi_{j,l,m}\leq\phi_{j_2,l_2,m_2}$ for $l\in L_j$ large enough. To see this, note that the set

$$U_l:=\{x\in X\mid(\phi_{j,l}-\phi_{j_i,l_i})(x)<(2m_i)^{-1}\text{ for }i=1,2\}$$

is open for all $l\in L_j$. Further, $(U_l)_l$ is an increasing family and $\bigcup_l U_l=X$, so by compactness of X there exists $l\in L_j$ such that $U_l=X$, and then $\phi_{j,l,m}\leq\max_{i=1,2}\phi_{j_i,l_i,m_i}$. By construction, $(\phi_i)_{i\in I}$ is now a decreasing net in $\mathrm{FS}(L)$ converging to ϕ, so since $\phi\not\equiv-\infty$, we have $\phi\in\mathrm{PSH}(L)$. ☐

Proposition 5.7. *Let $f\colon X'\to X$ be induced by a morphism of projective k-varieties, and set $L':=f^*L$. If $\phi\in\mathrm{PSH}(L)$ and $\phi':=\phi\circ f$, then either $\phi'\equiv-\infty$ on $L'^\times$ or $\phi'\in\mathrm{PSH}(L')$. When f is surjective, $\phi'\in\mathrm{PSH}(L')$.*

Proof. Let $(\phi_j)_j$ be a decreasing net in $\mathrm{FS}(L)$ converging to ϕ. By Lemma 2.4, $(f^*\phi_j)$ is a decreasing net in $\mathrm{FS}(L')$ converging to ϕ'. Thus either $\phi'\in\mathrm{PSH}(L')$ or $\phi'\equiv-\infty$ on $L'^\times$ by Proposition 5.6. Now suppose f is surjective and pick $x'\in X'^{\mathrm{qm}}$. By Lemma 1.1, $x:=f(x')\in X^{\mathrm{qm}}$, so since $\phi\neq-\infty$ on $L_x^\times$, see Proposition 5.14, it follows that $\phi'\neq-\infty$ on $L'^\times_{x'}$. ☐

Proposition 5.8. *Any (possibly singular) metric of the form*

$$\phi:=\max_{\alpha\in A}(m^{-1}\log|s_\alpha|+\lambda_\alpha),\tag{5.1}$$

where $m\geq 1$, $(s_\alpha)_{\alpha\in A}$ is a finite set of nonzero global sections of mL, and $\lambda_\alpha\in\mathbf{R}$, is semipositive.

Here the sections s_α are allowed to have common zeros.

Proof. After replacing m by a multiple, and the sections s_α by the corresponding tensor powers, we may assume L is base point free. Pick global sections s_α, $\alpha \in A'$ of L such that $(s_\alpha)_{\alpha\in A\cup A'}$ have no common zero, and set

$$\phi_j := \max\{\max_{\alpha\in A}(m^{-1}\log|s_\alpha| + 2^{-j}\lceil 2^j\lambda_\alpha\rceil), \max_{\alpha\in A'}(m^{-1}\log|s_\alpha| - j)\}$$

for $j \in \mathbf{Z}_{>0}$. Then $(\phi_j)_j$ is a decreasing sequence of FS metrics on L converging to ϕ. □

Corollary 5.9. *If k is trivially valued and $\mathfrak{a}$ is any ideal on X, then the function $\log|\mathfrak{a}|$ is qpsh. More precisely, it is L-psh for any L such that the sheaf $L \otimes \mathfrak{a}$ is globally generated.*

Proof. Suppose $L \otimes \mathfrak{a}$ is globally generated, say by sections s_α, $\alpha \in A$. If ϕ_{triv} denotes the trivial metric on L, then $\phi_{\mathrm{triv}} + \log|\mathfrak{a}| = \max_\alpha \log|s_\alpha| \in \mathrm{PSH}(L)$ by Proposition 5.8. □

We can now show that the space $\mathrm{PSH}(L)$ can be described as in the introduction.

Corollary 5.10. $\mathrm{PSH}(L)$ *is the smallest class of singular metrics on L that:*

(i) *contains all singular metrics of the form $m^{-1}\log|s|$, where $m \geq 1$ and s is a nonzero section of mL;*
(ii) *is closed under maxima, addition of constants, and decreasing limits.*

By "closed under decreasing limits" we mean that (vi) of Proposition 5.6 holds.

Proof. That $\mathrm{PSH}(L)$ satisfies (i) follows from Proposition 5.8. Similarly, Proposition 5.6 implies that $\mathrm{PSH}(L)$ satisfies (ii). Conversely, any class of singular metrics satisfying (i) and (ii) must contain all FS metrics, and, more generally, all decreasing limits of FS metrics. Hence it must contain $\mathrm{PSH}(L)$. □

Corollary 5.11. *If $\psi \in \mathrm{DFS}(L)$, the function $\phi \mapsto \sup_X(\phi - \psi)$ is continuous on* $\mathrm{PSH}(L)$.

Proof. By Lemma 2.20 there exists a finite subset $Z = Z(\psi) \subset X^{\mathrm{Shi}}$ such that $\sup_X(\phi-\psi) = \max_Z(\phi - \psi)$ for every $\phi \in \mathrm{FS}(L)$. Considering regularizations, the same equality holds for all $\phi \in \mathrm{PSH}(L)$. This completes the proof, since the topology on $\mathrm{PSH}(L)$ is defined by pointwise convergence on $X^{\mathrm{qm}} \supset X^{\mathrm{Shi}}$. □

Proposition 5.12. *Consider a non-Archimedean field extension k'/k and set $L' = L_{k'}$.*

(i) *If $\phi \in \mathrm{PSH}(L)$, then $\phi\circ\pi \in \mathrm{PSH}(L')$, where $\pi\colon L' \to L$ is the canonical map.*
(ii) *if k is trivially valued, $k' := k(\!(\varpi)\!)$, and $\sigma\colon L \to L'$ is the Gauss extension, then:*
   (a) *$\phi'\circ\sigma \in \mathrm{PSH}(L)$ and $\phi' \leq \phi'\circ\sigma\circ\pi$ for any $\phi' \in \mathrm{PSH}(L')$.*
   (b) *the maps $\phi \to \phi\circ\pi$ and $\phi' \to \phi'\circ\sigma$ above are continuous.*

Proof. Proposition 2.10 shows that (i) holds when $\phi \in \mathrm{FS}(L)$. This implies the general case: if $(\phi_j)_j$ is a decreasing net in $\mathrm{FS}(L)$, then $(\phi_j\circ\pi)_j$ is a decreasing net in $\mathrm{FS}(L')$, and $\lim_j(\phi_j\circ\pi) = (\lim_j\phi_j)\circ\pi$. The argument for (ii) (a) is similar, using Proposition 2.11 (i), and the continuity statements in (b) follows from the fact that $\pi(X'^{\mathrm{qm}}) \subset X^{\mathrm{qm}}$ and $\sigma(X^{\mathrm{qm}}) \subset X'^{\mathrm{qm}}$, see Corollary 1.3 and Corollary 1.5 (ii). □

Finally we prove that in the trivially valued case, $\mathrm{PSH}(L)$ is invariant under the scaling action on metrics on line bundles defined in (2.1).

Proposition 5.13. *If k is trivially valued and $\phi \in \mathrm{PSH}(L)$, then $\phi_t \in \mathrm{PSH}(L)$ for all $t \in \mathbf{R}_+^\times$. Further, the function $t \mapsto \phi_t(v)$ is convex for all $v \in L^\times$. It is decreasing if $\phi \le \phi_{\mathrm{triv}}$.*

Proof. By definition, there exists a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ converging to ϕ. For any $t \in \mathbf{R}_+^\times$, the net $(\phi_{j,t})_j$ of scaled metrics decreases to ϕ_t. By (2.5) and Proposition 5.8 we see that $\phi_{j,t} \in \mathrm{PSH}(L)$, so Proposition 5.6 (vi) implies that $\phi_t \in \mathrm{PSH}(L)$.

The fact that $t \to \phi_t(v)$ is convex is clear from (2.5) if $\phi \in \mathrm{FS}(L)$, and the general case follows by approximation. Now suppose that $\phi \le \phi_{\mathrm{triv}}$. If $\phi \in \mathrm{FS}(L)$, this means that ϕ is of the form (2.2) with $\lambda_j \le 0$ for all j. From (2.5) it is then clear that $t \mapsto \phi_t(v)$ is decreasing. For a general $\phi \in \mathrm{PSH}(L)$ with $\phi \le \phi_{\mathrm{triv}}$, pick a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ converging to ϕ. Pick $\varepsilon_j \in \mathbf{Q}_+^\times$ such that $\varepsilon_j \to 0$ and $\phi_j - \phi_{\mathrm{triv}} \le \varepsilon_j$. Then $\phi_j - \varepsilon_j \in \mathrm{FS}(L)$, so by what precedes, we get that $t \mapsto \phi_{j,t} - t\varepsilon_j$ is pointwise decreasing. Since $\phi_{j,t}$ decreases pointwise to ϕ_t, and $\varepsilon_j \to 0$, this completes the proof. □

5.2. **Hausdorff property.** It is not obvious that the topology on $\mathrm{PSH}(L)$ is Hausdorff, since it is defined in terms of pointwise convergence on X^{qm} only. Nevertheless, we have

Proposition 5.14. *For any semiample line bundle L, the weak topologies on $\mathrm{PSH}(L)$ and $\mathrm{PSH}(L)/\mathbf{R}$ are Hausdorff.*

Corollary 5.15. *If ϕ_j is a decreasing net in $\mathrm{PSH}(L)$ converging weakly to $\phi \in \mathrm{PSH}(L)$, then the convergence holds pointwise everywhere on $L^\times$.*

Proof. Let $\tilde\phi$ be the pointwise limit of ϕ_j. We know from Proposition 5.25 (vii) that $\tilde\phi \in \mathrm{PSH}(L)$. Since $\tilde\phi - \phi = 0$ on X^{qm}, it follows that $\tilde\varphi = \tilde\phi$. □

The proof of Proposition 5.14 relies on the following result.

Lemma 5.16. *Given any function $\varphi \in \mathrm{QPSH}(X)$ and any point $x \in X$, there exists a net $(x_\alpha)_{\alpha\in A}$ in X^{qm} such that $\lim_\alpha x_\alpha = x$ and $\lim_\alpha \varphi(x_\alpha) = \varphi(x)$.*

Proof. First assume k is nontrivially valued. There exists an ample line bundle L such that $\varphi = \phi - \psi$, where $\psi \in \mathrm{DFS}(L)$ and $\phi \in \mathrm{PSH}(L)$. Then ψ is a model metric, determined by a $\mathbf{Q}$-line bundle $\mathcal{L}$ on a model $\mathcal{X}$ of X. After replacing ψ and ϕ by multiples, we may assume $\mathcal{L}$ is an actual line bundle. We may cover $\mathcal{X}$ by open affine subsets $\mathcal{U}$ on which $\mathcal{L}$ admits trivializing sections. Then X is covered by the corresponding Zariski open sets $U = (\mathcal{U} \cap X^{\mathrm{sch}})^{\mathrm{an}}$. Pick $\mathcal{U}$ such that $x \in U$ and let τ be a trivializing section of $\mathcal{L}$ on $\mathcal{U}$. Given $s \in H^0(X, mL)$, we can write $s = f\tau^m$ on U, where $f \in \mathcal{O}_X(U)$, and then $m^{-1}\log|s| - \psi = m^{-1}\log|f|$ on U.

We can write ϕ as the limit of a decreasing net $(\phi_j)_{j\in J}$ of FS metric on L, for some directed set J. For each j, write $\phi_j = m_j^{-1}\max_{l\in L_j}\log|s_{j,l}|$, where $m_j \ge 1$ and $s_{j,l}$, $l \in L_j$, is a finite set of global sections of m_jL without common zero. As above we can write $m_j^{-1}\log|s_{j,l}| - \psi = m_j^{-1}\log|f_{j,l}|$, where $f_{j,l} \in \mathcal{O}_X(U)$. Hence $\varphi_j := \phi_j - \psi = m_j^{-1}\max_{l\in L_{j,l}}\log|f_{j,l}|$ on U.

Now let $(V_\alpha)_{\alpha\in A}$ be a decreasing net of strictly affinoid neighborhoods of x in U, with intersection equal to $\{x\}$. Let $\Gamma_\alpha \subset X^{\mathrm{Shi}}$ be the Shilov boundary of V_α. This is a finite set, and for each (j,l) there exists a point in Γ_α at which the maximum of $|f_{j,l}|$ over V_α occurs. Thus, for each j, the maximum of φ_j over V_α occurs at some point $x_{j,\alpha} \in \Gamma_\alpha$; in particular, $\varphi_j(x_{j,\alpha}) \ge \varphi_j(x)$. Passing to a subnet, we may assume that $x_\alpha := x_{j,\alpha}$ is independent of

j. This implies $\varphi(x_\alpha) \ge \varphi(x)$ for all α, which completes the proof when k is nontrivially valued.

Now suppose k is trivially valued. Set $k' = k((\varpi))$ and $X' := X_{k'}$. Write $\pi\colon X' \to X$ for the canonical map and $\sigma\colon X \to X'$ for the Gauss extension. Then $\varphi' := \varphi \circ \pi \in \mathrm{QPSH}(X)$, so by what precedes there exists a net $(x'_\alpha)_\alpha$ in X'^{qm} such that $x'_\alpha \to x' := \sigma(x)$ and $\lim_\alpha \varphi'(x'_\alpha) = \varphi'(x')$. By Corollary 1.3, we have $x_\alpha := \pi(x'_\alpha) \in X^{\mathrm{qm}}$. Further, $\varphi'(x'_\alpha) = \varphi(x_\alpha)$ and $\varphi'(x') = \varphi(x)$, and since π is continuous, we have $\lim_\alpha x_\alpha = x$. This completes the proof. □

Proof of Proposition 5.14. Consider metrics $\phi_1, \phi_2 \in \mathrm{PSH}(L)$ with $\phi_1 \ne \phi_2$. Pick any $\psi \in \mathrm{DFS}(L)$, and set $\varphi_i := \phi_i - \psi \in \mathrm{QPSH}(X)$, $i = 1, 2$. Then there exists $x \in X$ such that $\varphi_1(x) \ne \varphi_2(x)$, say $\varphi_1(x) > \varphi_2(x)$. By Lemma 5.16 there exists a net $(x_\alpha)_\alpha$ in X^{qm} such that $x_\alpha \to x$ and $\varphi_1(x_\alpha) \to \varphi(x)$. Now φ_2 is usc, so $\limsup_\alpha \varphi_2(x_\alpha) \le \varphi_2(x)$. Thus there exists α such that $\varphi_1(x_\alpha) > \varphi_2(x_\alpha)$. This implies that ϕ_1 and ϕ_2 belong to disjoint open subsets of $\mathrm{PSH}(L)$.

Similarly, suppose that $\phi_1 \ne \phi_2$ in $\mathrm{PSH}(L)/\mathbf{R}$. We may assume there exists $x \in X$ and $y \in X^{\mathrm{qm}}$ such that $\varphi_1(y) = \varphi_2(y)$ but $\varphi_1(x) \ne \varphi_2(x)$. As above, there must then exist $x' \in X^{\mathrm{qm}}$ such that $\varphi_1(x') \ne \varphi_2(x')$, which means ϕ_1 and ϕ_2 belong to disjoint open neighborhhods of $\mathrm{PSH}(L)/\mathbf{R}$. □

5.3. **Continuous plurisubharmonic metrics.** The subset of $\mathrm{PSH}(L)$ consisting of *continuous* metrics play an important role in the theory. Following Gubler and Martin [GM16], we now characterize this class. In particular, we show that the continuous metrics in $\mathrm{PSH}(L)$ are exactly the semipositive metrics studied by Zhang [Zha95] and Gubler [Gub98].

We start by characterizing psh DFS metrics on ample line bundles.

Proposition 5.17. *Let $\phi \in \mathrm{PSH}(L)$ be a psh metric, where L is ample.*

- (i) *If k is nontrivially valued, then $\phi \in \mathrm{DFS}(L)$ iff ϕ is a model metric associated to a nef model of L.*
- (ii) *If k is trivially valued, then $\phi \in \mathrm{DFS}(L)$ iff ϕ is a metric defined by a nef test configuration for L.*

As a direct consequence of (i) we have:

Corollary 5.18. *When k is discretely valued and of residue characteristic zero, X is smooth, and L is ample, the class $\mathrm{DFS}(L) \cap \mathrm{PSH}(L)$ coincides with the class of semipositive model metrics defined in* [BFJ16a].

Proof of Proposition 5.17. First consider the nontrivially valued case. In the terminology of [GM16], a model metric on L is then semipositive if it is defined by a nef model of L.

Suppose $\phi \in \mathrm{DFS}(L)$. In particular, ϕ is a model metric by Proposition 2.13 (i). Now ϕ is the limit of a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$. By Proposition 2.13 (ii), each ϕ_j is a model metric associated to an semiample (and hence nef) model of L. Thus [GM16, Theorem 1.3] implies that ϕ is a model metric associated to a nef model of L.

Conversely, suppose ϕ is a model metric defined by a nef model $(\mathcal{X}, \mathcal{L})$ for $(\mathcal{X}, \mathcal{L})$. Arguing as in the proof of [GM16, Proposition 4.14] we may (after passing to a different model) assume that for every $m \ge 1$ there exists a model metric ϕ_m on L associated to an ample model $(\mathcal{X}, \mathcal{L}_m)$ on L, such that $0 \le \phi_m - \phi \le 2^{-m}$. Then $\phi_m \in \mathrm{FS}(L)$ by Proposition 2.13 (iii). Now $(\phi_m + 2^{-m})_m$ is a decreasing sequence in $\mathrm{FS}(L)$ converging to ϕ, so $\phi \in \mathrm{PSH}(L)$.

Now consider the case when k is trivially valued. First assume ϕ is defined by a nef test configuration. Then $\phi' := \phi \circ \pi \in \mathrm{PSH}(L')$ is defined by the nef model $\mathcal{L}'$ of L'. By what precedes, $\phi' \in \mathrm{DFS}(L')$, and this implies $\phi \in \mathrm{DFS}(L)$ by Proposition 2.10.

Conversely, suppose $\phi \in \mathrm{DFS}(L)$. By Proposition 2.12 (i), $\phi = \phi_{\mathcal{L}}$, where $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L). The base change $(\mathcal{X}', \mathcal{L}')$ is a model of (X', L'), and $\phi' := \phi \circ \pi = \phi_{\mathcal{L}'}$. Since $\phi' \in \mathrm{PSH}(L)$ by Proposition 5.12, the $\mathbf{Q}$-line bundle $\mathcal{L}'$ must be relatively nef. Since the central fiber $\mathcal{X}_0$ of $\mathcal{X}$ can be identified with the special fiber $\mathcal{X}'_0$ of $\mathcal{X}'$, it follows that $\mathcal{L}$ is also relatively nef, completing the proof. □

The following theorem is the main result of [GM16].

Theorem 5.19. *Let $(\phi_j)_j$ be a net in $\mathrm{PSH}(L) \cap \mathrm{DFS}(L)$ converging in $\mathrm{PSH}(L)$ to a metric $\phi \in \mathrm{PSH}(L)$. Then $\phi \in \mathrm{PSH}(L) \cap \mathrm{DFS}(L)$.*

Proof. If k is nontrivially valued, this is a special case of [GM16, Theorem 1.3]. Now suppose k is trivially valued and consider the non-Archimedean field extension $k' = k_r = k((\varpi))$ of k. Write $L' = L_{k'}$, let $\pi \colon L' \to L$ be the canonical map and $\sigma \colon L \to L'$ the Gauss extension. Write $\phi' := \phi \circ \pi$ and $\phi'_j := \phi_j \circ \pi$. By Proposition 5.12, we have $\phi', \phi'_j \in \mathrm{PSH}(L')$ and $\phi'_j \to \phi'$. Since $\phi'_j \in \mathrm{DFS}(L')$ by Proposition 2.10, we have $\phi' \in \mathrm{DFS}(L)$ by what precedes. This implies $\phi = \phi' \circ \sigma \in \mathrm{DFS}(L)$ by Proposition 2.11. □

We now turn to more general continuous psh metrics.

Proposition 5.20. *If ϕ is a continuous metric on L, the following conditions are equivalent:*

(i) *$\phi \in \mathrm{PSH}(L)$;*
(ii) *ϕ is a uniform limit of metrics in $\mathrm{FS}(L)$;*
(iii) *ϕ is a uniform limit of metrics in $\mathrm{PSH}(L) \cap \mathrm{DFS}(L)$.*
(iv) *ϕ is a uniform limit of continuous metrics in $\mathrm{PSH}(L)$;*

In view of Proposition 5.17, it follows that ϕ is a continuous semipositive metric in the sense of Zhang [Zha95] iff it is a continuous psh metric. See also [GM16] for a discussion.

Proof. That (i) implies (ii) follows from Dini's theorem, and the reverse implication is elementary: if $\phi_j \in \mathrm{FS}(L)$ and ϕ_j converges uniformly to ϕ, then we can find $\varepsilon_j \in \mathbf{Q}_+$ such that $\varepsilon_j \to 0$ and $\phi_j + \varepsilon_j$ decreases to ϕ. Knowing that (i) and (ii) are equivalent, we immediately see that (iv) implies (i). Since the implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are obvious, this completes the proof. □

5.4. **Envelopes.** The class $\mathrm{PSH}(L)$ is closed under decreasing nets. It is useful to also consider increasing nets. This leads to the various types of *envelopes*.

Definition 5.21. *We define the* psh envelope *of a continuous metric ψ on L by*

$$P(\psi) = \sup\{\phi \in \mathrm{PSH}(L) \mid \phi \le \psi\}.$$

Despite the terminology, it is not obvious that $P(\psi)$ is psh, see the discussion below. At any rate, the psh envelope can be computed using only FS metrics.

Lemma 5.22. *For any continuous metric ψ on L, we have $P(\psi) = \sup\{\phi \in \mathrm{FS}(L) \mid \phi \le \psi\}$.*

Proof. Set $P'(\psi) = \sup\{\phi \in \mathrm{FS}(L) \mid \phi \leq \psi\}$. Clearly $P'(\psi) \leq P(\psi)$. For the reverse inequality, fix $\varepsilon \in \mathbf{Q}_+^\times$. Given $x \in X$, pick $\phi \in \mathrm{PSH}(L)$ such that $\phi \leq \psi$ and $(\phi - P(\psi))(x) \geq -\varepsilon$. Then pick a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ converging to ϕ. Since $(\phi_j - \psi)_j$ is a decreasing net of usc functions on a compact space, whose limit is nonpositive, there exists j such that $\phi_j - \psi \leq \varepsilon$. Then $\phi_j - \varepsilon \in \mathrm{FS}(L)$, $\phi_j - \varepsilon \leq \psi$, and $(\phi_j - \varepsilon - P(\psi))(x) \geq -2\varepsilon$. It follows that $P'(\psi) \geq P(\psi) - 2\varepsilon$, which completes the proof. □

Following [BE17], we now discuss whether the psh envelope is indeed psh. This question admits several equivalent reformulations. Given a bounded metric ψ on L, write ψ^* for the usc regularization of ψ, i.e. the smallest usc metric bounded below by ψ.

Proposition 5.23. *Given k, X and L, the following conditions are equivalent:*

(i) *for any continuous metric ψ on L, the psh envelope $P(\psi)$ is continuous and psh;*
(ii) *for any continuous metric ψ on L, the psh envelope $P(\psi)$ is psh;*
(iii) *for any DFS metric ψ on L, the psh envelope $P(\psi)$ is psh;*
(iv) *for any DFS metric ψ on L, the psh envelope $P(\psi)$ is continuous and psh;*
(v) *for any family $(\phi_\alpha)_{\alpha\in A}$ of metrics in* $\mathrm{PSH}(L)$ *that is uniformly bounded from above, the usc upper envelope $(\sup_\alpha \phi_\alpha)^*$ is psh.*
(vi) *for any increasing net $(\phi_j)_j$ of metrics in* $\mathrm{PSH}(L)$ *that is uniformly bounded from above, the usc limit $(\lim_j \phi_j)^*$ is psh.*

We shall show in Theorem 5.27 that (i)–(vi) hold when k is discretely or trivially valued, of residue characteristic zero, X is smooth, and L is ample It would be very interesting to know if they hold more generally.

To prove Proposition 5.23, we need the following elementary but useful lemma.

Lemma 5.24. *The psh envelope has the following properties:*

(i) *if ψ_1 and ψ_2 are continuous metrics on L and $\psi_1 \leq \psi_2$, then $P(\psi_1) \leq P(\psi_2)$;*
(ii) *if ψ is a continuous metric on L and $c \in \mathbf{R}$, then $P(\psi + c) = P(\psi) + c$;*
(iii) $\sup_X |P(\psi_1) - P(\psi_2)| \leq \sup_X |\psi_1 - \psi_2|$ *for any continuous metrics ψ_1, ψ_2 on L.*

Proof. That (i) holds is obvious, and (ii) follows from the fact that $\mathrm{PSH}(L)$ is closed under addition of constants. Finally (iii) is a formally consequence of this remark and of (i)–(ii). □

Proof of Proposition 5.23. From Lemma 5.22 it follows that $P(\psi)$ is lsc, since FS metrics are continuous. Hence (i) and (ii) are equivalent, as are (iii) and (iv). Further, (v) obviously implies (vi), and the reverse implication holds by considering the net $(\psi_j)_{j\in J}$, where J is the set of finite subsets of A and $\psi_j := \max_{\alpha\in j} \varphi_\alpha$. Clearly, (i) implies (iii) and the reverse implication follows from the Lipschitz property of Lemma 5.24 together with the fact that DFS metrics are dense in the set of continuous metrics on L, see Corollary 2.8 and also Proposition 5.20. We shall complete the proof by showing that (ii) and (v) are equivalent.

First assume (ii) holds and consider a family as in (v). Since $\phi := (\sup_\alpha \phi_\alpha)^*$ is usc, it can be written as a decreasing net $(\tau_j)_j$ of continuous (not necessarily psh) metrics. By (ii) the metric $\psi_j := P(\tau_j)$ is psh for each j. Since $\phi_\alpha \leq \tau_j$ and ϕ_α is psh, it follows that $\phi_\alpha \leq \psi_j$ for each α and j. Thus $\sup_\alpha \phi_\alpha \leq \psi_j$, and since ψ_j is usc we get $\phi \leq \psi_j$ for all j. The net $(\psi_j)_j$ is clearly decreasing. Since $\phi \leq \psi_j \leq \tau_j$ and $\lim_j \tau_j = \phi$, we must have $\lim_j \psi_j = \phi$, so by Proposition 5.6 (vii) we conclude $\phi \in \mathrm{PSH}(L)$.

Conversely, suppose (v) holds and let ψ be a continuous metric on L. We apply (v) to the family of all metrics $\phi \in \mathrm{PSH}(L)$ such that $\phi \leq \psi$. Thus $P(\psi)^*$ is psh. But since $P(\psi) \leq \psi$

and ψ is usc, we have $P(\psi)^* \le \psi$. Thus $P(\psi)^*$ is a competitor in the definition of $P(\psi)$ so $P(\psi)^* \le P(\psi)$, and hence $P(\psi)^* = P(\psi)$, since the reverse inequality is obvious. □

Proposition 5.25. *Consider a non-Archimedean field extension k'/k and set $L' := L_{k'}$.*

(i) *If ψ is a continuous metric on L, then $P(\psi \circ \pi) \ge P(\psi) \circ \pi$, where $\pi\colon L' \to L$ is the canonical map.*
(ii) *if k is trivially valued, $k' = k((\varpi))$ and $\sigma\colon L \to L'$ is the Gauss extension, then:*
  (a) *$P(\psi' \circ \sigma) \ge P(\psi') \circ \sigma$ for every continuous metric ψ' on L'.*
  (b) *$P(\psi \circ \pi) = P(\psi) \circ \pi$ for every continuous metric ψ on L.*

Proof. If $\phi \in \mathrm{PSH}(L)$ and $\phi \le \psi$, then $\phi \circ \pi \in \mathrm{PSH}(L')$ by Proposition 5.12 (i), and $\phi \circ \pi \le \psi \circ \pi$. Thus $P(\psi \circ \pi) \ge P(\psi) \circ \pi$, proving (i).

Now assume $k' = k((\varpi))$. The proof of (a) is the same as that of (i), but using (ii) of Proposition 5.12. It remains to prove (b), so let ψ be a continuous metric on L. Then

$$P(\psi \circ \pi) \ge P(\psi) \circ \pi = P(\psi \circ \pi \circ \sigma) \circ \pi \ge P(\psi \circ \pi) \circ \sigma \circ \pi \ge P(\psi \circ \pi).$$

Here the first two inequalities follow from (i) and (a), whereas the last inequality follows from from Proposition 5.12 (b) and the definition of $P(\psi \circ \pi)$. □

Proposition 5.26. *If k is trivially valued, then the scaling operation on metrics commutes with the psh envelope: if ψ is a continuous metric on L, then $P(\phi_t) = P(\phi)_t$ for all $t \in \mathbf{R}_+^\times$.*

Proof. This is a formal consequence of the definition and of Proposition 5.13. □

5.5. **Continuity of envelopes.** We now study finer properties of psh metrics. *For the rest of this section, assume that k is discretely or trivially valued, of residue characteristic zero, that X is smooth and that L is ample.* The proof of the next theorem uses these assumptions via the technique of multiplier ideals. However, the theorem may hold more generally. See [GJKM17] for recent results in equicharacteristic p, obtained using test ideals.

Theorem 5.27. *For any DFS metric ψ on L, the psh envelope $P(\psi)$ is continuous.*

Corollary 5.28. *Properties (i)–(vi) of Proposition 5.23 are all true.*

Proof of Theorem 5.27. It suffices to consider the case when k is discretely valued. Indeed, in the trivially valued case, we consider the non-Archimedean field extension $k' = k((\varpi))/k$. If ψ is a continuous metric on L, then $\psi \circ \pi$ is continuous metric on $L' = L_{k'}$ and $P(\psi \circ \pi) = P(\phi) \circ \pi$ by Proposition 5.25 (ii), so if Theorem 5.27 holds in the discretely valued case, then $P(\phi) \circ \pi$ is continuous; hence so is $P(\phi) = P(\phi) \circ \pi \circ \sigma$.

Thus suppose k is discretely valued. We only sketch the proof, referring to the proof of Theorem 8.5 of [BFJ16a] for most of the details. However, some minor details are simpler here because of an altered order of presentation. Write $\psi = \phi_{\mathcal{L}}$, where $(\mathcal{X}, \mathcal{L})$ is an *snc* model of (X, L). For m sufficiently divisible, let $\mathfrak{a}_m$ be the base ideal of $m\mathcal{L}$; this is cosupported on the special fiber since L is ample, and satisfies $\mathfrak{a}_l \cdot \mathfrak{a}_m \subset \mathfrak{a}_{l+m}$. Set

$$\phi_m := \psi + m^{-1} \log |\mathfrak{a}_m| = m^{-1} \max\{\log |s| \mid s \in H^0(\mathcal{X}, m\mathcal{L})\}.$$

Then $\phi_m \in \mathrm{FS}(L)$ and $P(\psi) = \lim_m \sup_m \phi_m$. We claim that in fact ϕ_m converges uniformly, so that $P(\psi)$ is continuous.

To prove the uniform convergence, we use multiplier ideals. Let $\mathfrak{b}_m$ be the multiplier ideal of $\mathfrak{a}_m$. The inclusion $\mathfrak{a}_m \subset \mathfrak{b}_m$ is elementary, and we have $\mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for all m, l by the subadditivity of multiplier ideals. This implies that

$$m^{-1} \log |\mathfrak{b}_m| \geq (ml)^{-1} \log |\mathfrak{b}_{ml}| \geq (ml)^{-1} \log |\mathfrak{a}_{ml}|$$

for all sufficiently divisible m and l. Letting $l \to \infty$ shows that

$$\psi + m^{-1} \log |\mathfrak{b}_m| \geq P(\psi) \geq \psi + m^{-1} \log |\mathfrak{a}_m| \tag{5.2}$$

for all sufficiently divisible m. We may assume there exists an effective $\mathbf{Q}$-divisor E on $\mathcal{X}$ supported on the special fiber, such that $\mathcal{A} := \mathcal{L} - E$ is relatively ample. By the uniform global generation of multiplier ideals there exists $m_0 \in \mathbf{N}$ such that $\mathcal{O}_{\mathcal{X}}((m\mathcal{L} + m_0\mathcal{A})) \otimes \mathfrak{b}_m$ is globally generated for all sufficiently divisible m. This implies that

$$\log |\mathfrak{b}_m| \leq \log |\mathfrak{a}_{m+m_0}| + C$$

for a constant C independent of m. Combining this with (5.2) shows that

$$\phi_m \leq P(\psi) \leq \frac{m}{m - m_0} \phi_m + \frac{1}{m - m_0} C$$

which shows that ϕ_m converges uniformly to $P(\psi)$. □

5.6. **Psh metrics and dual complexes.** We keep the assumptions on (k, X, L) from §5.5. As in [BFJ16a], this allows us to use dual complexes of snc models and test configurations.

Recall from §4 that when k is discretely (resp. trivially) valued, $\mathrm{SNC}(X)$ denotes the poset of snc models of X (resp. snc test configurations for X). To each $\mathcal{X} \in \mathrm{SNC}(X)$ is associated a dual complex $\Delta_{\mathcal{X}} \subset X$, and we have a retraction $p_{\mathcal{X}} \colon X \to \Delta_{\mathcal{X}}$. When $\mathcal{X} \geq \mathcal{Y}$, we have $\Delta_{\mathcal{Y}} \subset \Delta_{\mathcal{X}}$ and $p_{\mathcal{Y}} = p_{\mathcal{Y}} \circ p_{\mathcal{X}}$. Further, $\lim_{\mathcal{X}} p_{\mathcal{X}} = \mathrm{id}$ pointwise on X.

When k is trivially valued, consider the non-Archimedean field extension $k' = k(\!(\varpi)\!)$. Set $X' = X_{k'}$, $L' = L_{k'}$, and write π for the canonical maps $X' \to X$ and $L' \to L$, and σ for the Gauss extensions $X \to X'$ and $L \to L'$. Given $\mathcal{X} \in \mathrm{SNC}(X)$, set $\mathcal{X}' := \mathcal{X} \times_{\mathbb{A}^1_k} \operatorname{Spec} k'^{\circ}$. Then $\mathcal{X}' \in \mathrm{SNC}(X')$, and π and σ restrict to affine homeomorphisms between $\Delta_{\mathcal{X}'}$ and $\Delta_{\mathcal{X}}$. Further, $p_{\mathcal{X}} = \pi \circ p_{\mathcal{X}'} \circ \sigma$.

Theorem 5.29. *Pick a reference metric $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$ and $\mathcal{X}_{\mathrm{ref}} \in \mathrm{SNC}(X)$ on which ϕ_{ref} is determined.*

(i) *For every $\mathcal{X} \in \mathrm{SNC}(X)$ we have that $\{\phi - \phi_{\mathrm{ref}} \mid \phi \in \mathrm{PSH}(L)\}$ is an equicontinuous family of finite-valued functions on $\Delta_{\mathcal{X}}$.*
(ii) *If $\mathcal{X} \in \mathrm{SNC}(X)$, $\mathcal{X} \geq \mathcal{X}_{\mathrm{ref}}$ and $\phi \in \mathrm{PSH}(L)$, the function $\varphi := \phi - \phi_{\mathrm{ref}}$ is convex on each face of $\Delta_{\mathcal{X}}$ and satisfies $\varphi \leq \varphi \circ p_{\mathcal{X}}$ on X.*
(iii) *For any $\phi \in \mathrm{PSH}(L)$, the net $((\phi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}})_{\mathcal{X}}$ of continuous functions on X, indexed by $\mathcal{X} \in \mathrm{SNC}(X)$ with $\mathcal{X} \geq \mathcal{X}_{\mathrm{ref}}$, is decreasing, with limit $\phi - \phi_{\mathrm{ref}}$.*

Proof. First assume that k is discretely valued. All the assertions then essentially follow from [BFJ16a], but since the definition of singular semipositive metrics in *loc. cit.* ostensibly differs from the one here, we need to argue in a somewhat roundabout way.

By Proposition 2.13, ϕ_{ref} is a model metric, and hence defines a closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ in the terminology of [BFJ16a, §4.2]. Similarly, any $\phi \in \mathrm{FS}(L)$ is a model metric associated to an semiample, hence nef, model of L, so the function $\phi - \phi_{\mathrm{ref}}$ is θ-psh in the sense of [BFJ16a, §5]. It now follows from [BFJ16a, Corollary 7.8] that $\{(\phi - \phi_{\mathrm{ref}})|_{\Delta_{\mathcal{X}}} \mid \phi \in \mathrm{FS}(L)\}$

is an equicontinuous family of finite-valued functions on $\Delta_{\mathcal{X}}$. Since the metrics in $\mathrm{PSH}(L)$ are decreasing limits of metrics in L, the full equicontinuity statement in (i) follows.

Similarly, by [BFJ16a, Proposition 7.5], the properties in (ii) hold for $\phi \in \mathrm{FS}(L)$. The same properties then carry over to $\phi \in \mathrm{PSH}(L)$. Indeed, this is clear for the second assertion, and convexity is closed under taking decreasing limits.

Finally [BFJ16a, Proposition 7.6] shows that if $\phi \in \mathrm{FS}(L)$, and $\varphi := \phi - \phi_{\mathrm{ref}}$, the net $\mathcal{X}_{\mathrm{ref}} \leq \mathcal{X} \to \varphi \circ p_{\mathcal{X}}$ is decreasing. The same property then holds for all $\phi \in \mathrm{PSH}(L)$. In particular, $\varphi(x) \leq \liminf_{\mathcal{X}} \varphi(p_{\mathcal{X}}(x))$ for any $x \in X$. But since φ is usc and $\lim_{\mathcal{X}} p_{\mathcal{X}} = \mathrm{id}$ by Theorem 4.1, we get $\varphi(x) \geq \limsup_{\mathcal{X}} \varphi(p_{\mathcal{X}}(x))$. This completes the proof of (iii).

Now assume k is trivially valued and use the notation immediately before the theorem. Set $\phi'_{\mathrm{ref}} := \phi_{\mathrm{ref}} \circ \pi \in \mathrm{DFS}(L')$. If $\phi \in \mathrm{PSH}(L)$, then $\phi' := \phi \circ \pi \in \mathrm{PSH}(L')$ by Proposition 5.12. Further, $\phi - \phi_{\mathrm{ref}} = (\phi' - \phi'_{\mathrm{ref}}) \circ \sigma$ and $\sigma\colon \Delta \to \Delta'$ is a homeomorphism, so if $\{\phi' - \phi'_{\mathrm{ref}} \mid \phi' \in \mathrm{PSH}(L')\}$ is an equicontinuous family of finite valued functions on Δ', then $\{\phi - \phi_{\mathrm{ref}} \mid \phi \in \mathrm{PSH}(L)\}$ is an equicontinuous family of finite valued functions on Δ. This takes care of (i), and (ii)–(iii) are treated in the same way, except for the last assertion of (iii), which is instead proved exactly as above, using Theorem 4.1. ☐

Corollary 5.30. *When k is trivially valued and $\phi \in \mathrm{PSH}(L)$, the function $\phi - \phi_{\mathrm{triv}}$ attains its maximum at the generic point of X.*

Proof. This follows by using $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$ and $\mathcal{X}_{\mathrm{ref}}$ the trivial test configuration for X. ☐

Corollary 5.31. *Let $\varphi \in \mathrm{QPSH}(X)$.*

(i) *For any sufficiently large $\mathcal{X} \in \mathrm{SNC}(X)$, φ is continuous on $\Delta_{\mathcal{X}}$, convex on each face, and satisfies $\varphi \leq \varphi \circ p_{\mathcal{X}}$.*
(ii) *Then φ is the limit of the eventually decreasing net $(\varphi \circ p_{\mathcal{X}})_{\mathcal{X}}$ of continuous functions on X, where $\mathcal{X}$ ranges over $\mathrm{SNC}(X)$. As a consequence, φ is determined by its values on X^{qm}, and in fact on its values on X^{Shi} (resp. X^{div}) when k is nontrivally valued (resp. trivially valued).*

Proof. Write $\varphi = \phi - \phi_{\mathrm{ref}}$, where $\phi \in \mathrm{PSH}(L)$ and $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$ for some semiample line bundle L. Pick $\mathcal{X}_{\mathrm{ref}} \in \mathrm{SNC}(X)$ such on which ϕ_{ref} is determined. Then (i) immediately follows, as does (ii), except for the very last assertion, which follows from the fact that Shilov points (resp. divisorial points) are dense on any dual complex $\Delta_{\mathcal{X}}$ when k is nontrivially valued (resp. trivially valued) and the continuity of φ on $\Delta_{\mathcal{X}}$. ☐

Corollary 5.32. *A net $(\phi_j)_j$ in $\mathrm{PSH}(L)$ converges weakly to $\phi \in \mathrm{PSH}(L)$ iff $\phi_j - \phi \to 0$ pointwise on Shilov points (resp. divisorial points) when k is nontrivially valued (resp. trivially valued).*

Proof. The direct implication is trivial since $\phi_j \to \phi$ weakly means that $\phi_j - \phi \to 0$ pointwise on X^{qm}, which contains all Shilov points (resp. divisorial points). The reverse implication follows from the fact that any point in X^{qm} belongs to some dual complex $\Delta_{\mathcal{X}}$, together with the equicontinuity statement in Theorem 5.29 and the fact that Shilov points (resp. divisorial points) are dense in $\Delta_{\mathcal{X}}$. ☐

5.7. **Compactness.** Keep the assumptions on (k, X, L) from §5.5. Our next goal is to prove that $\mathrm{PSH}(L)$ is compact modulo constants. For this we use dual complexes as well as the continuity of envelopes, see Theorem 5.27. The key is the following lemma.

Lemma 5.33. *A singular metric ϕ on L belongs to* $\mathrm{PSH}(L)$ *iff it is usc, and:*

(i) *for any $\mathcal{X} \in \mathrm{SNC}(X)$, there exists a sequence $(\phi_m)_1^\infty$ (depending on $\mathcal{X}$) in* $\mathrm{PSH}(L)$ *such that* $\lim_m(\phi_m - \phi) = 0$ *uniformly on $\Delta_\mathcal{X}$;*

(ii) *given any reference metric $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$, we have $\phi - \phi_{\mathrm{ref}} \le (\phi - \phi_{\mathrm{ref}}) \circ p_\mathcal{X}$ on X, for any sufficiently large $\mathcal{X} \in \mathrm{SNC}(X)$.*

Before proving this lemma, we record some consequences.

Corollary 5.34. *Fix a reference metric $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$. Then $\phi \mapsto ((\phi - \phi_{\mathrm{ref}})|_{\Delta_\mathcal{X}})_\mathcal{X}$ gives an embedding of* $\mathrm{PSH}(L)$ *onto a closed subset of $\prod_{\mathcal{X}\in\mathrm{SNC}(X)} C^0(\Delta_\mathcal{X})$. In particular, the topology on* $\mathrm{PSH}(L)$ *is equivalent to the topology of uniform convergence on dual complexes.*

Here the latter topology is the one for which a basis of open subsets of $\phi \in \mathrm{PSH}(L)$ is given by $\{\psi \in \mathrm{PSH}(L) \mid \sup_{\Delta_\mathcal{X}} |\psi - \phi| < \varepsilon\}$, where $\mathcal{X}$ ranges over $\mathrm{SNC}(X)$.

Proof. The map is well-defined and injective by Corollary 5.31. It is further continuous by Theorem 5.29: if $(\phi_j)_j$ is a net in $\mathrm{PSH}(L)$ converging to $\phi \in \mathrm{PSH}(L)$, then for each $\mathcal{X} \in \mathrm{SNC}(X)$, $(\phi_j - \phi)|_{\Delta_\mathcal{X}}$ is an equicontinuous net in $C^0(\Delta_\mathcal{X})$ converging to zero pointwise on a dense subset, hence must converge to zero uniformly. It now follows that the map $\mathrm{PSH}(L) \to \prod_\mathcal{X} C^0(\Delta_\mathcal{X})$ is a homeomorphism onto its image. Let us finally prove that this image is closed.[16] Thus suppose we have a net $(\phi_j)_j$ in $\mathrm{PSH}(L)$ and an element $(g_\mathcal{X})_\mathcal{X} \in \prod_\mathcal{X} C^0(\Delta_\mathcal{X})$ such that $\lim_j \varphi_j|_{\Delta_\mathcal{X}} = g_\mathcal{X}$ uniformly on $\Delta_\mathcal{X}$ for every $\mathcal{X}$, where $\varphi_j := \phi_j - \phi_{\mathrm{ref}}$. It is clear that if $\mathcal{X}' \ge \mathcal{X}$, then $g_{\mathcal{X}'} \le g_\mathcal{X} \circ r_\mathcal{X}$ on $\Delta_{\mathcal{X}'}$, with equality on $\Delta_\mathcal{X}$. For any $x \in X$, the net $(g_\mathcal{X}(r_\mathcal{X}(x)))_\mathcal{X}$ is therefore decreasing, and we set $\varphi(x) := \lim_\mathcal{X} g_\mathcal{X}(r_{\Delta_\mathcal{X}}(x))$. The singular metric $\phi := \phi_{\mathrm{ref}} + \varphi$ then satisfies the conditions of Lemma 5.33, so $\phi \in \mathrm{PSH}(L)$. This completes the proof since the image of ϕ under the map $\mathrm{PSH}(L) \to \prod_\mathcal{X} C^0(\Delta_\mathcal{X})$ above is equal to $(g_\mathcal{X})_\mathcal{X}$. □

Corollary 5.35. *For any $\phi_{\mathrm{ref}} \in \mathrm{DFS}(L)$, the map $\mathrm{PSH}(L) \ni \phi \mapsto \sup(\phi - \phi_{\mathrm{ref}})$ is continuous and proper. Hence $\mathrm{PSH}(L)/\mathbf{R}$ is compact.*

Proof. The continuity of $\phi \mapsto \sup(\phi - \phi_{\mathrm{ref}})$ was established in Corollary 5.11. To prove properness, we must prove that the set $\mathcal{F}_C := \{\phi \in \mathrm{PSH}(L) \mid |\sup(\phi - \phi_{\mathrm{ref}})| \le C\}$ is compact for any $C > 0$. By Tychonoff's theorem and Corollary 5.34, it suffices to prove that for any dual complex $\Delta \subset X$, the set of functions $(\phi - \phi_{\mathrm{ref}})|_\Delta$ for $\phi \in \mathcal{F}_C$, forms a precompact family of continuous functions on Δ. But this follows from Theorem 5.29 and the Arzelà-Ascoli theorem. □

Corollary 5.36. *If k is discretely valued of residue characteristic zero, then* $\mathrm{PSH}(L)$ *coincides with the class of singular semipositive metrics on L as defined in* [BFJ16a].

Proof. In the current language, a singular metric ϕ is semipositive in the sense of [BFJ16a, Definition 7.3] if it is usc, satisfies Lemma 5.33 (ii), and satisfies Lemma 5.33 (i) with ϕ_m a model metric associated to a nef model of L, i.e. a ϕ_m is a metric in $\mathrm{DFS}(L) \cap \mathrm{PSH}(L)$, see Proposition 5.17. It then follows from Lemma 5.33 that $\phi \in \mathrm{PSH}(L)$.

Conversely, if $\phi \in \mathrm{PSH}(L)$, then ϕ is usc and satisfies Lemma 5.33 (ii) by Theorem 5.29 (ii). Pick a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ converging to ϕ, and consider any dual complex $\Delta \subset X$. By Theorem 5.29 (i), the functions $\phi_j - \phi$ are continuous on Δ and hence converge uniformly

[16] This step was overlooked in the proof of [BFJ16a, Theorem 7.10].

to zero by Dini's theorem. We can therefore extract a sequence $(\phi_m)_m$ in $\mathrm{FS}(L)$ such that $\phi_m - \phi$ converges uniformly to 0 on Δ. By Proposition 2.13 (ii), ϕ_m is a model metric associated to a semiample (hence nef) model of L, so ϕ is semipositive in the sense of [BFJ16a]. $\square$

Proof of Lemma 5.33. Let $\mathrm{PSH}'(L)$ be the set of usc singular metrics on L satisfying (i)–(ii). Then $\mathrm{PSH}(L) \subset \mathrm{PSH}'(L)$. Indeed, if $\phi \in \mathrm{PSH}(L)$, then ϕ is usc by definition, (i) is trivial, and (ii) follows from Theorem 5.29 (ii), so $\phi \in \mathrm{PSH}'(L)$.

For the reverse inclusion, we consider the following modified envelope:

$$P'(\psi) := \sup\{\varphi \in \mathrm{PSH}'(L) \mid \varphi \le \psi\}$$

for ψ a continuous metric on L. Let us first prove that $P'(\psi) = P(\psi)$. The envelope P' has the same Lipschitz property as in Lemma 5.24, so by Corollary 2.8 we may assume that ψ is a DFS metric. Write $\varphi = P(\psi) - \phi_{\mathrm{ref}}$ and $\varphi' = P'(\psi) - \phi_{\mathrm{ref}}$. We need to prove that $\varphi(x) = \varphi'(x)$ for all $x \in X$. Clearly $\varphi \le \varphi'$, so it suffices to prove $\varphi'(x) \le \varphi(x)$.

First assume $x \in X^{\mathrm{qm}}$, and pick $\mathcal{X}$ sufficiently large so that $x \in \Delta_{\mathcal{X}}$ and $\psi - \phi_{\mathrm{ref}} = (\psi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}}$; the latter is possible since $\psi \in \mathrm{DFS}(L)$. Pick any $\varepsilon > 0$. By the definition of φ' there exists $\phi' \in \mathrm{PSH}'(L)$ such that $\phi' \le \psi$ and $(\phi' - \phi_{\mathrm{ref}})(x) \ge \varphi(x) - \varepsilon$. Since $\phi' - \varepsilon \in \mathrm{PSH}'(L)$, we may further find $\phi \in \mathrm{PSH}(L)$ such that $|\phi - (\phi' - \varepsilon)| \le \varepsilon$ on $\Delta_{\mathcal{X}}$. In fact, we may assume $\phi \in \mathrm{FS}(L)$. Indeed, there exists a decreasing net $(\phi_j)_j$ in $\mathrm{FS}(L)$ such that $\phi_j - \phi$ converges pointwise to 0 on X. By Theorem 5.29, the restriction of $\phi_j - \phi$ to $\Delta_{\mathcal{X}}$ is continuous, so Dini's theorem implies that $\phi_j - \phi$ converges uniformly to 0 on $\Delta_{\mathcal{X}}$. Then

$$\phi - \phi_{\mathrm{ref}} \le (\phi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}} \le (\phi' - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}} \le (\psi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}} = \psi - \phi_{\mathrm{ref}},$$

so that $\phi \le \psi$. On the other hand, we have

$$(\phi - \phi_{\mathrm{ref}})(x) \ge (\phi' - \phi_{\mathrm{ref}})(x) - 2\varepsilon \ge \varphi'(x) - 3\varepsilon.$$

This implies that $\varphi(x) \ge \psi'(x) - 3\varepsilon$. Hence $\varphi \ge \varphi'$ on X^{qm}, as claimed.

Now consider a general point $x \in X$. From the definition of $\mathrm{PSH}'(L)$ we have $\varphi' \le \varphi' \circ p_{\mathcal{X}}$ for every sufficiently large $\mathcal{X} \in \mathrm{SNC}(X)$. Thus

$$\varphi'(x) \le \varliminf_{\mathcal{X}} \varphi' \circ p_{\mathcal{X}}(x) = \varliminf_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}(x) = \varphi(x),$$

where the last equality follows from Corollary 5.31 since $\psi \in \mathrm{QPSH}(X)$.

Thus we know that $P'(\psi) = P(\psi)$ for every continuous metric ψ on L. Let us now show that in fact $\mathrm{PSH}'(L) = \mathrm{PSH}(L)$. Consider any singular metric $\phi' \in \mathrm{PSH}'(L)$. We need to show that ϕ' is the limit of a decreasing net of FS metrics. For this we argue exactly as in the implication (iii) $\Longrightarrow$ (i) of [BFJ16a, Lemma 8.9]. Namely, we first prove that for every $x \in X$ we have

$$(\phi' - \phi_{\mathrm{ref}})(x) = \inf\{(\phi - \phi_{\mathrm{ref}})(x) \mid \phi \in \mathrm{FS}(L), \phi \ge \phi'\}. \tag{5.3}$$

Indeed, fix $\varepsilon > 0$. Since ϕ' is usc, there exists a continuous metric u on L such that $\phi' \le u$ and $(u - \phi')(x) \le \varepsilon$. Since $\phi' \in \mathrm{PSH}'(L)$, we have $\phi' \le P'(u)$. But $P'(u) = P(u)$ by what precedes, and $P(u)$ is continuous by Corollary 5.28. The definition of $P(u)$ and Dini's theorem then yields a metric $\phi \in \mathrm{FS}(L)$ such that $P(u) \le \phi \le P(u) + \varepsilon$. Thus we have $\phi \ge \phi'$ and $(\phi - \phi')(x) \le 2\varepsilon$, which completes the proof of (5.3).

Now let J be the set of metrics $\phi \in \mathrm{FS}(L)$ such that $\phi > \phi'$. We claim that J is a directed set. Indeed, given $\phi_1, \phi_2 \in J$ there exists $\varepsilon > 0$ such that $\phi_i - \phi' \geq 3\varepsilon$ for $i = 1, 2$, since ϕ_i is continuous and ϕ' usc. This implies $\phi' \leq P'(\min\{\phi_1, \phi_2\}) + 3\varepsilon = P(\min\{\phi_1, \phi_2\}) + 3\varepsilon$. Now pick $\phi \in \mathrm{FS}(L)$ such that $|\phi - P(\min\{\phi_1, \phi_2\})| \leq \varepsilon$. Then $\phi' + \varepsilon \leq \phi - \varepsilon \leq \min\{\phi_1, \phi_2\}$, so $\phi \in J$ and $\phi \leq \phi_i$ for $i = 1, 2$. Thus J is a directed set, and the associated net $(\phi_j)_{j \in J}$ is decreasing. It now follows from (5.3) that its limit equals ϕ', which completes the proof. □

6. Metrics of finite energy

In §5 we introduced the class $\mathrm{PSH}(L)$ of psh metrics[17] on L. Here we define the subspace $\mathcal{E}^1(L) \subset \mathrm{PSH}(L)$ of metrics of *finite energy*, and extend the Monge-Ampère operator and associated functionals to this space.

To begin with, k is an arbitrary non-Archimedean field, whereas X is a geometrically integral variety and L an ample line bundle on X. However, for some of the deeper results, we need further restrictions on k and X.

6.1. **Metrics of finite energy.** We start by extending the Monge-Ampère energy functional to the space of all psh metrics. Fix a reference metric $\phi_{\mathrm{ref}} \in \mathrm{FS}(L)$.

Definition 6.1. *For any metric $\phi \in \mathrm{PSH}(L)$ we set*

$$E(\phi) = \inf\{E(\psi) \mid \psi \geq \phi,\ \psi \textit{ FS metric on } L\} \in \mathbf{R} \cup \{-\infty\}.$$

We define

$$\mathcal{E}^1(L) \subset \mathrm{PSH}(L)$$

as the set of psh metrics ϕ with $E(\phi) > -\infty$.

Note that E extends the functional earlier defined on FS metrics, since the latter is nondecreasing. In particular, $\mathrm{FS}(L) \subset \mathcal{E}^1(L)$.

Proposition 6.2. *The function $E\colon \mathrm{PSH}(L) \to [-\infty, +\infty)$ is nondecreasing, concave, and satisfies $E(\phi+c) = E(\phi)+c$ for any $c \in \mathbf{R}$. It is also upper semicontinuous, and continuous along decreasing nets.*

Proof. We follow the proof of [BFJ15, Proposition 6.2]. That E is nondecreasing is obvious. Similarly, the formula $E(\phi+c) = E(\phi)+c$ follows since it holds for $\phi \in \mathrm{FS}(L)$ and $c \in \mathbf{Q}$. To prove that E is usc, pick $\phi_0 \in \mathrm{PSH}(L)$ and $s_0 \in \mathbf{R}$ such that $E(\phi_0) < s_0$. By the definition of $E(\phi_0)$, there exists $\psi_0 \in \mathrm{FS}(L)$ and $t_0 < s_0$ such that $\psi_0 \geq \phi_0$ and $E(\psi_0) < t_0$. Since $\phi \mapsto \sup_X(\phi - \psi_0)$ is continuous, see Corollary 5.11, there exists an open neighborhood U of ϕ_0 in $\mathrm{PSH}(L)$ such that $\phi - \psi_0 < s_0 - t_0$ for $\phi \in U$. Since E is nondecreasing, this gives $E(\phi) \leq E(\psi_0 + s_0 - t_0) = E(\psi_0) + s_0 - t_0 < s_0$ for $\phi \in U$. Now that we know that E is usc, continuity along decreasing nets follows formally from E being nondecreasing. Finally we prove that E is concave. Pick $\phi, \psi \in \mathrm{PSH}(L)$ and consider decreasing nets $(\phi_i)_i$ and $(\psi_j)_j$ in $\mathrm{FS}(L)$ converging to ϕ and ψ, respectively. For $t \in \mathbf{Q} \cap [0,1]$ we know that $E((1-t)\phi_i + t\psi_j) \geq (1-t)E(\phi_i) + tE(\phi_j)$ for all i, j. Since E is continuous under decreasing nets, this implies $E((1-t)\phi + t\psi) \geq (1-t)E(\phi) + tE(\phi)$ for $t \in [0,1] \cap \mathbf{Q}$. The same inequality must then hold for all $t \in [0,1]$ since E is usc. Thus E is concave. □

Corollary 6.3. *The space $\mathcal{E}^1(L)$ has the following properties:*

(i) *$\mathcal{E}^1(L)$ does not depend on the choice of reference metric ϕ_{ref};*
(ii) *if $\phi \in \mathcal{E}^1(L)$ and $c \in \mathbf{R}$, then $\phi + c \in \mathcal{E}^1(L)$;*
(iii) *if $\phi \in \mathcal{E}^1(L)$, $\psi \in \mathrm{PSH}(L)$ and $\psi \geq \phi$, then $\psi \in \mathcal{E}^1(L)$;*
(iv) *$\mathcal{E}^1(L)$ is convex;*
(v) *any bounded psh metric is in $\mathcal{E}^1(L)$;*

[17]Recall that we refer to the elements of $\mathrm{PSH}(L)$ as metrics even though some of them are singular metrics.

Proof. Statements (ii)-(iii) follow easily from the definition, and imply (v) since $\mathrm{FS}(L) \subset \mathcal{E}^1(L)$. That $\mathcal{E}^1(L)$ is convex follows since E is concave. Finally, (i) is a consequence of (3.12). Indeed, if $E'(\phi)$ denotes the energy with respect to a different reference metric ϕ'_{ref}, then $E(\phi) - E'(\phi) = E(\phi'_{\mathrm{ref}}, \phi_{\mathrm{ref}})$. □

Corollary 6.4. *For any $C > 0$, the set*

$$\mathcal{E}^1_C(L) := \{\phi \in \mathcal{E}^1(L) \mid \phi \le \phi_{\mathrm{ref}}, E(\phi) \ge -C\} \subset \mathrm{PSH}(L)$$

is a convex and weakly closed.

Proof. The convexity of $\mathcal{E}^1_C(L)$ follows from the concavity of E. That $\mathcal{E}^1_C(L)$ is weakly closed follows since E is usc and $\phi \mapsto \sup(\phi - \phi_{\mathrm{ref}})$ is continuous on $\mathrm{PSH}(L)$, see Corollary 5.11. □

Corollary 6.5. *If k is trivially or discretely valued, of residue characteristic zero, X is smooth, and L is ample, then $\mathcal{E}^1_C(L)$ is weakly compact for any $C > 0$.*

Proof. By Corollary 6.4, $\mathcal{E}^1_C$ is a closed subset of $P_C := \{\phi \in \mathrm{PSH}(L) \mid \max(\phi - \phi_{\mathrm{ref}}) \in [-C, 0]\}$, which is compact under the hypotheses on (k, X, L), see Corollary 5.35. □

Proposition 6.6. *If $\phi, \psi \in \mathcal{E}^1(L)$, then the function $[0,1] \ni t \mapsto E((1-t)\phi + t\psi)$ is the restriction of a polynomial of degree at most $n+1$.*

Proof. Note that $\phi_t := (1-t)\phi + t\psi \in \mathcal{E}^1(L)$ for $0 \le t \le 1$ by concavity of E.

First assume $\phi, \psi \in \mathrm{FS}(L)$. Then $\phi_t \in \mathrm{FS}(L)$ for $t \in [0,1] \cap \mathbf{Q}$, and uit follows from the definition of the functional E that there exists a polynomial $p(t)$ of degree at most $n+1$ such that $E(\phi_t) = p(t)$ for $t \in \mathbf{Q} \cap [0,1]$. The same equality must then hold for all $t \in [0,1]$. Indeed, given $t \in [0,1]$, and a sequence $(t_m)_m$ in $[0,1] \cap \mathbf{Q}$ with $t_m \to t$, there exists $\varepsilon_m > 0$ with $\lim_m \varepsilon_m = 0$ such that $\phi_{t_m} + \varepsilon_m$ decreases to ϕ_t as $m \to \infty$, and then

$$E(\phi_t) = \lim_m E(\phi_{t_m} + \varepsilon_m) = \lim_m (E(\phi_{t_m}) + \varepsilon_m) = \lim_m p(t_m) + \varepsilon_m = p(t).$$

Now consider any $\phi, \psi \in \mathcal{E}^1(L)$. We can find decreasing nets $(\phi_j)_j$ and $(\psi_j)_j$ in $\mathrm{FS}(L)$, with the same index set J, converging to ϕ and ψ, respectively. Then $\phi_{j,t} := (1-t)\phi_j + t\psi_j$ decreases to ϕ_t for any $t \in [0,1]$. Thus the sequence $(p_j(t))_j$ of functions on $[0,1]$ decreases to the finite-valued function $p(t) := E(\phi_t)$. Since $p_j(t)$ is given by a polynomial of degree at most $n+1$ for all j, the same is true for $p(t)$. □

Proposition 6.7. *Let k'/k be a non-Archimedean extension, set $L' := L_{k'}$, and let $\pi\colon L' \to L$ be the canonical map.*

(i) *If $\phi \in \mathrm{PSH}(L)$, then $E(\phi \circ \pi) = E(\phi)$. Thus $\phi \circ \pi \in \mathcal{E}^1(L')$ iff $\phi \in \mathcal{E}^1(L)$.*
(ii) *If k is trivially valued, $k' := k((\varpi))$, $\sigma\colon k \to k'$ is the Gauss extension, and $\phi' \in \mathcal{E}^1(L')$, then $\phi' \circ \sigma \in \mathcal{E}^1(L)$ and $E(\phi' \circ \sigma) \ge E(\phi')$.*

Proof. To prove (i), let $(\phi_j)_j$ be a decreasing net in $\mathrm{FS}(L)$ converging to ϕ. Then $(\phi_j \circ \pi)_j$ is a decreasing net in $\mathrm{FS}(L')$ converging to $\phi \circ \pi$. For each j, we have $E(\phi_j \circ \pi) = E(\phi_j)$, see Corollary 3.10. Since E is continuous for decreasing nets, we have $E(\phi) = E(\phi \circ \pi)$.

Now assume k is trivially valued, and let $(\phi'_j)_j$ be a decreasing net in $\mathrm{FS}(L')$ converging to $\phi' \in \mathcal{E}^1(L')$. Then $(\phi'_j \circ \sigma)_j$ is a decreasing net in $\mathrm{FS}(L)$ converging to $\phi' \circ \sigma$. For each j, we have $E(\phi'_j \circ \sigma) = E(\phi'_j \circ \sigma \circ \pi) \ge E(\phi'_j)$ since $\phi'_j \le \phi'_j \circ \sigma \circ \pi$, see Proposition 2.11 (i). Thus (ii) follows, since E is continuous along decreasing nets. □

Proposition 6.8. *Assume k is trivially valued. If $\phi \in \mathcal{E}^1(L)$, then $\phi_t \in \mathcal{E}^1(L)$ and $E(\phi_t) = tE(\phi)$ for all $t \in \mathbf{R}_+^\times$.*

Proof. After adding a constant, we may assume $\phi \le \phi_{\mathrm{triv}}$. Then $t \to \phi_t$ is decreasing by Proposition 5.13, so it suffices to consider $t \in \mathbf{Q}_+^\times$. This case follows from regularization and Corollary 3.11. □

6.2. **Monge-Ampère operator.** Next we extend the Monge-Ampère operator to the space $\mathcal{E}^1(L)$. In the complex case, this is usually done in two steps: first one considers the case of bounded metrics, following the approach by Bedford-Taylor [BT82, BT87] who treated bounded plurisubharmonic functions in $\mathbf{C}^n$; then one uses the canonical approximation $\phi_t := \max\{\phi, \phi_{\mathrm{ref}} - t\}$, $t \le 0$ to treat the general case [Ceg98, GZ05]. This approach was adapted to the case of discretely valued fields of residue characteristic zero in [BFJ15].

Here we follow a different strategy that bypasses the intermediate step of bounded metrics and instead uses the estimates of §3.12 to directly treat metrics of finite energy.

Theorem 6.9. *There exists a unique operator*

$$(\phi_1, \dots, \phi_n) \mapsto \mathrm{MA}(\phi_1, \dots, \phi_n)$$

taking an n-tuple in $\mathcal{E}^1(L)$ to a Radon probability measure on X, such that

(i) *the definition is compatible with the one for FS metrics in §3.1;*
(ii) *we have $\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1, \dots, \phi_n) > -\infty$, when $\psi, \phi_1, \dots, \phi_n \in \mathcal{E}^1(L)$;*
(iii) *for any decreasing nets $\psi^j \to \psi$ and $\phi_i^j \to \phi_i$ in $\mathcal{E}^1(L)$ we have*

$$\int (\psi^j - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1^j, \dots, \phi_n^j) \longrightarrow \int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1, \dots, \phi_n).$$

Remark 6.10. *Following* [Ceg98, GZ05, BEGZ10, BBGZ13] *in the complex case and* [BFJ15] *in the non-Archimedean case, it is possible to define the Monge-Ampère operator for an even larger class than $\mathcal{E}^1(L)$, but we shall not do so here.*

We will also prove that the Monge-Ampère operator is continuous also along *increasing* nets in $\mathcal{E}^1(L)$. Note that such a net may not converge pointwise everywhere. This continuity result is useful since psh metrics can be constructed as usc envelopes, see §5.4.

Theorem 6.11. *For any increasing nets $\psi^j \to \psi$ and $\phi_i^j \to \phi_i$ in $\mathcal{E}^1(L)$ we have*

$$\int (\psi^j - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1^j, \dots, \phi_n^j) \longrightarrow \int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1, \dots, \phi_n).$$

Theorem 6.11 is new also in the setting of [BFJ16a, BFJ15]. In the original work of Bedford-Taylor [BT82], continuity of the Monge-Ampère operator along increasing sequences of locally bounded psh functions in $\mathbf{C}^n$ was proved using capacity theory. Here we take a more direct approach, using the estimates in §3.12 and their extensions to metrics of finite energy (see Corollary 6.16).

The proof of Theorems 6.9 and 6.11 are given in §6.8 below. For now, we deduce some consequences.

Corollary 6.12. *Let $(\phi_i^j)_j$, $1 \le i \le n$ be nets in $\mathcal{E}^1(L)$, all decreasing or all increasing, converging to metrics $\phi_i \in \mathcal{E}^1(L)$. Then $\lim_j \mathrm{MA}(\phi_1^j, \dots, \phi_n^j) = \mathrm{MA}(\phi_1, \dots, \phi_n)$.*

Proof. In the case of decreasing nets, this follows from Theorem 6.9 (iii) by taking $\psi^j = \psi$ to be an FS metric on L and using the fact that any continuous function on X can be uniformly approximated by differences of FS metrics on L, see Lemma 2.6 and Theorem 2.7.

The case of increasing nets is handled in the same way, using Theorem 6.11. □

Corollary 6.13. *For any metrics $\phi_i \in \mathcal{E}^1(L)$ and $c_i \in \mathbf{R}$, $1 \le i \le n$, we have*

$$\mathrm{MA}(\phi_1 + c_1, \dots, \phi_n + c_n) = \mathrm{MA}(\phi_1, \dots, \phi_n). \tag{6.1}$$

Proof. This holds when $\phi_i \in \mathrm{FS}(L)$ and $c_i \in \mathbf{Q}$; the general case follows by regularization. □

Corollary 6.14. *If $(\psi^j)_j$ and $(\phi_i^j)_j$, $1 \le i \le n$, are nets in $\mathcal{E}^1(L)$ converging* uniformly *to metrics ψ and ϕ_i, then $\psi, \phi_i \in \mathcal{E}^1(L)$ and*

$$\int \psi^j \, \mathrm{MA}(\phi_1^j, \dots \phi_n^j) \longrightarrow \int \psi \, \mathrm{MA}(\phi_1, \dots, \phi_n).$$

Proof. After adding small constants to the ϕ_i^j and ψ^j, and invoking Corollary 6.13, we may assume that all the nets are decreasing. □

Corollary 6.15. *The mapping $(\phi_1, \dots, \phi_n) \mapsto \mathrm{MA}(\phi_1, \dots, \phi_n)$ is symmetric in its arguments, and additive in the following sense:*

$$\mathrm{MA}(t\phi_1 + (1-t)\phi_1', \phi_2, \dots, \phi_n) = t \, \mathrm{MA}(\phi_1, \phi_2, \dots, \phi_n) + (1-t) \, \mathrm{MA}(\phi_1', \phi_2, \dots, \phi_n)$$

for $0 \le t \le 1$.

Proof. This holds when $\phi_i \in \mathrm{FS}(L)$ and $t \in \mathbf{Q}$; the general case follows by regularization. □

6.3. **Energy functionals.** It is straightforward to extend the energy functionals I and J to $\mathcal{E}^1(L)$. Namely, we define $I(\phi, \psi)$ and $J_\psi(\phi)$ for $\phi, \psi \in \mathcal{E}^1(L)$ using the same formulas as in §3.9. We see that if $(\phi^j)_j$ and $(\psi^j)_j$ are decreasing (or increasing) nets in $\mathcal{E}^1(L)$ converging to $\phi, \psi \in \mathcal{E}^1(L)$, then $\lim_j I(\phi^j, \psi^j) = I(\phi, \psi)$ and $J_\psi(\phi) = \lim_j J_{\psi^j}(\phi^j)$. Using regularization, we also get

Corollary 6.16. *All the estimates in §3.12 for FS metrics on L extend to metrics in $\mathcal{E}^1(L)$.*

The estimate in Lemma 3.14 also holds for all real $t \in [0, 1]$. Later we will need a similar estimate, proved in Appendix A.

Lemma 6.17. [Din88, Remark 2]. *For $\phi, \psi \in \mathcal{E}^1(L)$ and $t \in [0, 1]$ we have*

$$J_\psi(t\phi + (1-t)\psi) \le t^{1+\frac{1}{n}} J_\psi(\phi).$$

In the rest of this subsection, we deduce consequences of the estimates in §3.12. First we consider continuity properties. For a general Radon measure μ on X (e.g. for $\mu = \delta_x$, $x \in X^{\mathrm{rig}}$), the functional $\phi \mapsto \int (\phi - \phi_{\mathrm{ref}})\mu$ may fail to be continuous. However, we have

Proposition 6.18. *For any metric $\psi \in \mathcal{E}^1(L)$, the functional $\phi \mapsto \int (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\psi)$ is weakly continuous on $\mathcal{E}^1_C(L)$ for every $C > -\infty$.*

Proof. Let $(\phi_i)_i$ be a net in $\mathcal{E}^1_C(L)$ converging weakly to $\phi \in \mathcal{E}^1_C(L)$. We must prove that $\lim_i \int (\phi_i - \phi) \, \mathrm{MA}(\psi) = 0$. Pick any $\varepsilon > 0$. Let $(\psi_j)_j$ be a decreasing net of FS metrics converging to ψ. Then $\lim_j I(\psi_j, \psi) = 0$, whereas $\sup_i I(\phi_i, \phi) < \infty$, $\sup_i J(\phi_i) < \infty$

and $\sup_j J(\psi_j) < \infty$, see Corollary 3.17. It therefore follows from Corollary 3.20 that $\lim_j \sup_i |\int(\phi_i - \phi)(\mathrm{MA}(\psi_j) - \mathrm{MA}(\psi))| = 0$. Since, for every j, we have $\lim_i \int(\phi_i - \phi)\,\mathrm{MA}(\psi_j) = 0$, see Lemma 6.19 below, this uniform estimate completes the proof. □

Lemma 6.19. *Let $\phi_1, \dots, \phi_n$ be DFS metrics on line bundles $L_1, \dots, L_n$. Then the map*

$$\mathrm{QPSH}(X) \ni \varphi \to \int \varphi dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$$

is continuous.

Proof. This follows by the definition of the topology on $\mathrm{QPSH}(X)$, since the measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is a finite signed atomic measure supported on quasimonomial valuations. □

6.4. **Ground field extension and scaling.** Consider a non-Archimedean field extension k'/k and set $X' = X_{k'}$, $L' = L_{k'}$. Write $\pi\colon X' \to X$ and $\pi\colon L' \to L$ for the canonical maps. Use $\phi'_{\mathrm{ref}} := \phi_{\mathrm{ref}} \circ \pi$ as reference metric on L'.

Corollary 6.20. *Consider metrics $\psi, \phi_1, \dots, \phi_n \in \mathcal{E}^1(L)$.*

(i) *If we set $\phi'_i = \phi_i \circ \pi$ for $1 \le i \le n$ and $\psi' = \psi \circ \pi$, then*

$$\int_{X'} (\psi' - \phi'_{\mathrm{ref}})\,\mathrm{MA}(\phi'_1, \dots, \phi'_n) = \int_X (\psi - \phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_1, \dots, \phi_n).$$

(ii) *If k is trivially valued, then, for any $t \in \mathbf{R}_+^\times$ we have*

$$\int (\psi_t - \phi_{\mathrm{triv}})\,\mathrm{MA}(\phi_{1,t} \dots, \phi_{n,t}) = t \int (\psi - \phi_{\mathrm{triv}})\,\mathrm{MA}(\phi_1, \dots, \phi_n).$$

Proof. The equality in (i) follow directly from Proposition 3.2 and regularization. We similarly obtain (ii) when $t \in \mathbf{Q}_+^\times$. To treat the general case we may assume that $\psi \le \phi_{\mathrm{triv}}$ and $\phi_i \le \phi_{\mathrm{triv}}$ for $1 \le i \le n$. In this case, $t \mapsto \psi_t$ and $t \mapsto \phi_{i,t}$ are decreasing, so the result follows by approximating any $t \in \mathbf{R}_+^\times$ by a increasing sequence in $\mathbf{Q}_+^\times$. □

Corollary 6.21. *The functionals E, I and J are invariant under ground field extension, in the sense that all the assertions Corollary 3.10 hold for $\phi, \psi \in \mathcal{E}^1(L)$. Further, the homogeneity properties of Corollary 3.11 hold for all $t \in \mathbf{R}_+^\times$.*

6.5. **Countable regularization.** We have the following *countable regularization* result.

Corollary 6.22. *If L is ample, then every metric $\phi \in \mathcal{E}^1(L)$ is the limit of a decreasing* sequence *in* $\mathrm{FS}(L)$.

Proof. Let $(\phi_j)_{j \in J}$ be a decreasing *net* in $\mathrm{FS}(L)$ converging to ϕ. Then $\lim_j I(\phi_j, \phi) = 0$ and $\lim_j \int(\phi_j - \phi)\,\mathrm{MA}(\phi_{\mathrm{ref}}) = 0$. We can therefore pick an increasing sequence $(j_m)_m$ in I such that $I(\phi_{j(m)}, \phi) \le m^{-1}$ and $0 \le \int(\phi_{j(m)} - \phi)\,\mathrm{MA}(\phi_{\mathrm{ref}}) \le m^{-1}$ for all m. To simplify notation, write $\phi_m := \phi_{j(m)}$. We claim that $\lim_{m\to\infty} \phi_m = \phi$ in $\mathrm{PSH}(L)$, i.e. $\lim_m(\phi_m - \phi)(x) = 0$ for every $x \in X^{\mathrm{qm}}$.

In fact, it suffices to consider the case $x \in X^{\mathrm{Shi}}$. Indeed, there exists a non-Archimedean field extension k'/k and a point $x' \in X^{\mathrm{Shi}}_{k'}$ such that $\pi(x') = x$. Here π denotes the canonical map $X_{k'} \to X$ as well as the corresponding map $L_{k'} \to L$. Write $\phi'_m = \phi_m \circ \pi$ and $\phi' = \phi \circ \pi$. Then $I(\phi'_m, \phi') \le m^{-1}$ and $0 \le \int(\phi'_m - \phi')\,\mathrm{MA}(\phi'_{\mathrm{ref}}) \le m^{-1}$, where $\phi'_{\mathrm{ref}} = \phi_{\mathrm{ref}} \circ \pi$. Further, $X_{k'}$ is geometrically integral and $L_{k'}$ ample. If we can prove that $\lim_m(\phi'_m - \phi')(x') = 0$, it will then follow that $\lim_m(\phi_m - \phi)(x) = 0$.

Thus suppose $x \in X^{\mathrm{Shi}}$. Pick a sufficiently large model $\mathcal{X}$ of X such that x corresponds to an irreducible component of $\mathcal{X}_0$, and such that L admits an ample model $\mathcal{L}$ on $\mathcal{X}$. Let $\psi \in \mathrm{FS}(L)$ be the metric induced by $\mathcal{L}$. By Corollary 3.20 (or rather, its version for metrics in $\mathcal{E}^1(L)$, see Corollary 6.16), we have $\lim_{m\to\infty} \int (\phi_m - \phi)(\mathrm{MA}(\psi) - \mathrm{MA}(\phi_{\mathrm{ref}})) = 0$, and hence $\lim_{m\to\infty} \int (\phi_m - \phi)\,\mathrm{MA}(\psi) = 0$. But $\phi_m - \phi \geq 0$, and $\mathrm{MA}(\psi)$ is a probability measure putting mass on x, so we conclude that $\lim_{m\to\infty}(\phi_m - \phi)(x) = 0$. ☐

Remark 6.23. *Suppose k is discretely or trivially valued and of residue characteristic zero, X is smooth, and L ample. Then we have a countable regularization result also in* $\mathrm{PSH}(L)$*. In the discretely valued case, this was proved in* [BFJ15] *using capacity theory. The trivially valued case follows using ground field extension.*

6.6. **The strong topology.** In this subsection, we assume for simplicity that *k is discretely or trivially valued, of residue characteristic zero and X is smooth*. These assumptions are not necessary for the basic definitions, but we need them for the proofs of the main results.

Following [BBGZ13, §2] we define the *strong topology* on $\mathcal{E}^1(L)$ as the coarsest refinement of the weak topology inhereted from $\mathrm{PSH}(L)$ such that $E\colon \mathcal{E}^1(L) \to \mathbf{R}$ is continuous. Thus a net $(\phi_j)_j$ in $\mathcal{E}^1(L)$ converges strongly to $\phi \in \mathcal{E}^1(L)$ iff $\lim_j(\phi_j - \phi) = 0$ on X^{qm} and $\lim_j E(\phi_j) = E(\phi)$. Note that this notion does not depend on the choice of reference metric $\phi_{\mathrm{ref}} \in \mathrm{FS}(L)$. Also note that if ϕ_j is a decreasing net in $\mathrm{PSH}(L)$ and $\phi := \lim_j \phi_j \in \mathcal{E}^1(L)$, then ϕ_j converges strongly to ϕ.

Lemma 6.24. *Every strongly convergent net in $\mathcal{E}^1(L)$ is eventually contained in $\mathcal{E}^1_C(L)$ for some $C > 0$.*

Proof. Let $(\phi_j)_j$ be a strongly convergent net, and set $\phi = \lim_j \phi_j$. Then $\lim_j \sup(\phi_j - \phi_{\mathrm{ref}}) = \sup(\phi - \phi_{\mathrm{ref}})$ by Corollary 5.11, and $\lim_j E(\phi_j) = E(\phi)$ by definition, so the result follows. ☐

The strong topology on $\mathcal{E}^1(L)/\mathbf{R}$ is defined as the quotient topology induced by the strong topology on $\mathcal{E}^1(L)$. If $(\phi_j)_j$ is a net in $\mathcal{E}^1(L)$ and $\phi \in \mathcal{E}^1(L)$, we slightly abusively say that $\phi_j \to \phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$ if there exist $c_j, c \in \mathbf{R}$ such that $\lim_j \phi_j + c_j = \phi + c$ strongly in $\mathcal{E}^1(L)$. We can then pick $c_j = \max(\phi_j - \phi_{\mathrm{ref}})$ and $c = \max(\phi - \phi_{\mathrm{ref}})$.

Proposition 6.25. *Assume k is trivially valued and set $k' = k(\!(\varpi)\!)$, $L' := L_{k'}$. Let $\pi\colon L' \to L$ be the canonical map. Then the map $\mathcal{E}^1(L) \ni \phi \mapsto \phi \circ \pi \in \mathcal{E}^1(L')$ is strongly continuous. Further, if $(\phi_j)_j$ is a net in $\mathcal{E}^1(L)$ and $\phi \in \mathcal{E}^1(L)$, then $\lim \phi_j = \phi$ strongly iff $\lim_j \phi_j \circ \pi = \phi \circ \pi$ strongly.*

Proof. This follows from $\phi \mapsto \phi\circ\pi$ being weakly continuous and from the formula $E(\phi\circ\pi) = E(\phi)$ for $\phi \in \mathcal{E}^1(L)$, where we use the reference metric $\phi_{\mathrm{ref}} \circ \pi$ on L'. ☐

Proposition 6.26. *If $(\phi_j)_j$ is a net in $\mathcal{E}^1(L)$ and $\phi \in \mathcal{E}^1(L)$, then the following conditions are equivalent:*

(i) $\phi_j \to \phi$ *strongly in* $\mathrm{PSH}(L)/\mathbf{R}$*;*
(ii) $\lim_j I(\phi_j, \phi) = 0$*;*
(iii) $\lim_j J_\phi(\phi_j) = 0$.

Proof. The case when k is trivially valued easily reduces to the discretely valued case using the field extension $k' = k(\!(\varpi)\!)$, so we may assume that k is nontrivially valued.

The equivalence of (ii) and (iii) follows from (3.19), wo we only need to prove that (i) and (iii) are equivalent. For this, we use the formula

$$J_\phi(\phi_j) = E(\phi) - E(\phi_j) + \int (\phi_j - \phi)\,\mathrm{MA}(\phi). \tag{6.2}$$

First assume that $\phi_j \to \phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$. After normalizing ϕ_j and ϕ by $\max(\phi_j - \phi_{\mathrm{ref}}) = \max(\phi - \phi_{\mathrm{ref}}) = 0$, this simply means $\phi_j \to \phi$ strongly in $\mathcal{E}^1(L)$. In particular, $E(\phi_j) \to E(\phi)$, and there exists $C > 0$ such that $\phi_j \in \mathcal{E}^1_C(L)$ for all j. Since $\phi_j \to \phi$ weakly, it follows from Proposition 6.18 that $\int(\phi_j - \phi)\,\mathrm{MA}(\phi) \to 0$. Thus $J_\phi(\phi_j) \to 0$ by (6.2).

Conversely suppose that $\lim_j J_\phi(\phi_j) \to 0$. We must prove that $\phi_j \to \phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$. In this case, we normalize ϕ_j and ϕ by $\int(\phi_j - \phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}}) = 0$ and $\int(\phi - \phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}}) = 0$, respectively. We then claim that $\phi_j \to \phi$ weakly. Granted this, we argue as follows. From Lemma 3.16 and (3.21) we have $\sup_j J(\phi_j) < \infty$, so there exists $C > 0$ such that $\phi_j \in \mathcal{E}^1_C$ for all C, and then $E(\phi_j) \to E(\phi)$ follows from (6.2) and Proposition 6.18.

Proving that $\phi_j \to \phi$ weakly amounts to showing $\phi_j - \phi \to 0$ pointwise on X^{qm}. In fact, it suffices to prove $\phi_j - \phi \to 0$ pointwise on X^{Shi}, see Corollary 5.32. Set $f_j := \phi_j - \phi$ and view f_j as a function on X^{Shi}. By the Izumi inequality (Theorem 2.21), there exists, for any $x \in X^{\mathrm{Shi}}$ a constant $C(x) \ge 0$ such that $|f_i(x)| \le C(x)$ for all i. By Tychonoff's theorem, we may, after replacing f by a subnet, assume that f_j converges pointwise to a function $f\colon X^{\mathrm{Shi}} \to \mathbf{R}$ such that $|f(x)| \le C(x)$ for all $x \in X^{\mathrm{Shi}}$. We must then prove $f \equiv 0$ on X^{Shi}.

Given metrics $\psi_i \in \mathrm{FS}(L)$, $1 \le i \le n$, we have

$$\begin{aligned}\int f dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_n &= \lim_i \int f_i dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_n \\ &= \lim_i \left(\int f_i dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_n - \int f_i (dd^c\phi_{\mathrm{ref}})^n \right) = 0,\end{aligned}$$

where the first equality holds since $dd^c\psi_1 \wedge \cdots \wedge dd^c\psi_n$ is a finite atomic measures supported on X^{Shi}, the second by the normalizations of ϕ and the ϕ_i, and the third by Corollary 3.19 and the assumption $\lim_i I(\phi_i, \phi) = 0$. In view of Proposition 3.8, this implies $f \equiv 0$. ☐

Remark 6.27. *If Conjecture 3.7 is true, Proposition 6.26 holds for any non-Archimedean field k, any geometrically integral projective variety X, and any ample line bundle L on X. Indeed, we only used the assumptions on k and X to prove that $\lim_j I(\phi_j, \phi) = 0$ implies that $\phi_j \to \phi$ weakly,* i.e. *pointwise on X^{qm}. Using Lemma 2.23, it suffices to prove that $(\phi_j - \phi) \to$ pointwise on X^{Shi}, and the argument above shows that this follows from Conjecture 3.7.*

Since the topology on $\mathrm{PSH}(L)/\mathbf{R}$ is Hausdorff, the strong topology on $\mathcal{E}^1(L)/\mathbf{R}$ must also be Hausdorff. We have the following slightly more precise statement.

Corollary 6.28. *If $\phi_1, \phi_2 \in \mathcal{E}^1(L)$, the following statements are equivalent:*

(i) *$I(\phi_1, \phi_2) = 0$;*
(ii) *$\mathrm{MA}(\phi_1) = \mathrm{MA}(\phi_2)$;*
(iii) *the function $\phi_1 - \phi_2$ is constant on X.*

By (3.18)–(3.19), these conditions are also equivalent to $J_{\phi_1}(\phi_2) = 0$ and to $J_{\phi_2}(\phi_1) = 0$.

Proof. It is clear that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). Hence it suffices to show that (i) implies (iii). But this follows from Proposition 6.18 applied to a constant net. ☐

The assignment $(\phi_1, \phi_2) \to I(\phi_1, \phi_2)$ (presumably) does not define a metric on $\mathcal{E}^1(L)/\mathbf{R}$, but it does satisfy the quasi-triangle inequality $I(\phi_1, \phi_3) \le C_n \max\{I(\phi_1, \phi_2), I(\phi_2, \phi_3)\}$, see Lemma 3.16. It hence defines a *uniform structure* on $\mathcal{E}^1(L)/\mathbf{R}$. The next result says that this structure is complete.

Proposition 6.29. *If $(\phi_j)_{j\in J}$ is a net in $\mathcal{E}^1(L)$ with $\lim_{i,j} I(\phi_i, \phi_j) = 0$, then there exists $\phi \in \mathcal{E}^1(L)$, unique up to an additive constant, such that $\phi_j \to \phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$.*

Proof. We may assume $\sup(\phi_j - \phi_{\mathrm{ref}}) = 0$ for all j. It follows from Lemma 3.16 that $\sup_j J(\phi_j) < \infty$. By weak compactness of $\mathrm{PSH}(L)/\mathbf{R}$, there exists an accumulation point ϕ of the net ϕ_j, and it satisfies $\sup(\phi - \phi_{\mathrm{ref}}) = 0$. We claim that $\lim_j I(\phi_j, \phi) = 0$. Pick a subnet $(\phi_{j(\alpha)})_{\alpha\in A}$ converging to ϕ in $\mathrm{PSH}(L)$. Since E is usc, we have $E(\phi) \ge \limsup_\alpha E(\phi_{j(\alpha)})$. Fix an index $i \in J$. For any $\alpha \in A$ we have

$$J_{\phi_i}(\phi_{j(\alpha)}) = E(\phi_i) - E(\phi_{j(\alpha)}) + \int (\phi_{j(\alpha)} - \phi_i)\, \mathrm{MA}(\phi_i).$$

Now let $\alpha \to \infty$. The last integral converges to $\int (\phi - \phi_i)\, \mathrm{MA}(\phi_i)$ by Proposition 6.18, so by what precedes, we get

$$J_{\phi_i}(\phi) = E(\phi_i) - E(\phi) + \int (\phi - \phi_i)\, \mathrm{MA}(\phi_i) \le \varliminf_\alpha J_{\phi_i}(\phi_j(\alpha)).$$

Since $J_{\phi_i}(\phi) \ge 0$, this shows that $\lim_i J_{\phi_i}(\phi) = 0$, and hence $\lim_i I(\phi_i, \phi) = 0$ by (3.19). Using Proposition 6.26, we see that $\lim_i \phi_i = \phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$. □

6.7. **Orthogonality and differentiability.** As in §6.6 we assume that k is discretely or trivially valued, of residue characteristic zero and that X is smooth.

Recall from §5.5 that the psh envelope $P(\psi)$ of a continuous metric ψ on L is a continuous psh metric on L. In particular, $P(\psi) \in \mathcal{E}^1(L)$. In this subsection we state two key facts about the interaction of the operator P and the Monge-Ampère operator.

Theorem 6.30. *The following two properties hold:*

(i) *For any continuous metric ψ on L, we have $\int (\psi - P(\psi))\, \mathrm{MA}(P(\psi)) = 0$.*
(ii) *The composition $E \circ P$ is Gateaux differentiable on the space of continuous metrics on L, with derivative given by $(E \circ P)' = \mathrm{MA} \circ P$. Equivalently, we have*

$$E(P(\psi + f)) = E(P(\psi)) + \int_0^1 dt \int f\, \mathrm{MA}(P(\psi + tf))$$

for every continuous metric ψ on L and every $f \in C^0(X)$.

Following [BFJ15] we refer to (i) as the *orthogonality property* for L, whereas (ii) is the *differentiability property.*

Proof. For k discretely valued, the orthogonality property was proved in [BFJ15] under an additional technical assumption; the general case appears in [BGJKM16, Theorem 6.3.2]. Further, differentiability was shown in [BFJ15, Theorem 7.2] to follow from orthogonality. Thus Theorem 6.30 holds in the discretely valued case.

Now assume k is trivially valued. Set $k' = k(\!(\varpi)\!)$ and consider the base changes $X' = X_{k'}$ and $L' = L_{k'}$. Write π for the canonical maps $X' \to X$ and $L' \to L$. By Proposition 5.25, we have $P(\psi \circ \pi) = P(\psi) \circ \pi$. Corollary 6.20 now yields

$$\int_X (\psi - P(\psi))\, \mathrm{MA}(P(\psi)) = \int_{X'} (\psi - P(\psi)) \circ \pi\ \mathrm{MA}(P(\psi) \circ \pi)$$
$$= \int_{X'} (\psi \circ \pi - P(\psi \circ \pi))\ \mathrm{MA}(P(\psi \circ \pi)) = 0,$$

where the last equality follows from the orthogonality property in the discretely valued case. This proves (i). As for (ii), write $\psi' := \psi \circ \pi$, $f' = f \circ \pi$. For $t \in \mathbf{R}$ we then have $P(\psi' + tf') = P(\psi + tf) \circ \pi$ by Proposition 5.25, and hence $E(P(\psi' + tf')) = E(P(\psi + tf))$ by Proposition 6.7. Thus

$$E(P(\psi + f)) = E(P(\psi' + f'))$$
$$= E(P(\psi')) + \int_0^1 dt \int_{X'} f'\ \mathrm{MA}(P(\psi' + tf'))$$
$$= E(P(\psi)) + \int_0^1 dt \int_X f\ \mathrm{MA}(P(\psi + tf)),$$

where the second equality follows from the differentiability in the discretely valued case, and the third equality from Corollary 6.20 (i). This completes the proof. □

When solving the Monge-Ampère equation in §7.3 we need a generalized version of the differentiability property. Given a usc metric ψ on L that dominates some psh metric, set

$$P(\psi) := \sup\{\phi \in \mathrm{PSH}(L) \mid \phi \leq \psi\}.$$

Lemma 6.31. *We have $P(\psi) \in \mathrm{PSH}(L)$. If ψ is continuous, then so is $P(\psi)$.*

Proof. We saw in Theorem 5.27 that $P(\psi)$ is psh and continuous when ψ is continuous. By Corollary 5.28, all the properties in Proposition 5.23 hold. Property (v) applied to the family of psh metrics bounded above by ψ shows that the usc regularization $P(\psi)^*$ is psh. Now u is usc and $P(\psi) \leq u$, so $P(\psi)^* \leq u$. Thus $P(\psi)^*$ is a candidate in the definition of $P(\psi)$, so $P(\psi)^* \leq P(\psi)$, and hence $P(\psi) = P(\psi)^*$ is psh. □

Corollary 6.32. *If $\phi \in \mathcal{E}^1(L)$ and $f \in C^0(X)$, then $P(\phi + tf) \in \mathcal{E}^1(L)$ for all $t \in \mathbf{R}$. Further, the function $t \to E(P(\phi + tf))$ is differentiable, and $\frac{d}{dt} E(P(\phi + tf))|_{t=0} = \mathrm{MA}(\phi)$.*

Proof. We repeat the proof of [BFJ15, Corollary 7.3]. First note that $P(\phi + tf) \geq \phi - t \inf f$, so $P(\phi + tf) \in \mathcal{E}^1(L)$ for all t. It now suffices to prove that $E(P(\phi + f)) - E(\phi) = \int_0^1 dt \int f\, \mathrm{MA}(P(\phi + tf))$ for all $f \in C^0(X)$. For this, we use Corollary 6.22, which yields a decreasing *sequence* $(\phi_m)_1^\infty$ in $\mathrm{FS}(L)$ converging to ϕ. For each m we have

$$E(P(\phi_m + f)) - E(\phi_m) = \int_0^1 dt \int f\, \mathrm{MA}(P(\phi_m + tf)) \tag{6.3}$$

It is easy to see that $P(\phi_m + tf)$ decreases to $P(\phi + tf)$ for all t. Hence the left hand side of (6.3) converges to $E(P(\phi + f)) - E(\phi)$. Further, $\int f\ \mathrm{MA}(P(\phi_m + tf))$ converges weakly to $\int f\ \mathrm{MA}(P(\phi + tf)$ by Theorem 6.9. We conclude using the dominated convergence theorem, since $\int f\ \mathrm{MA}(P(\phi_m + tf))$ is uniformly bounded in m and $t \in [0, 1]$. □

6.8. **Proof of Theorem 6.9.** We follow [BFJ16a]. The proof below works in both the discretely and trivially valued case. Fix $0 \le p \le n$ and consider the following statement.

Assertion A(p). To any metrics $\phi_1, \dots, \phi_p$ in $\mathcal{E}^1(L)$ and $\phi'_{p+1}, \dots, \phi'_n$ in $\mathrm{FS}(L)$ is associated a Radon probability measure

$$\mathrm{M}_p(\phi_1, \dots, \phi_p) = \mathrm{M}_p(\phi_1, \dots, \phi_p; \phi'_{p+1}, \dots, \phi'_n)$$

such that:

(i) if $\phi_1, \dots, \phi_p$ are FS metrics, then

$$\mathrm{M}_p(\phi_1, \dots, \phi_p; \phi'_{p+1}, \dots, \phi'_n) = \mathrm{MA}(\phi_1, \dots, \phi_p, \phi'_{p+1}, \dots, \phi'_n) \tag{6.4}$$

as defined in (3.3);

(ii) we have $\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p) > -\infty$, when $\psi, \phi_1, \dots, \phi_n \in \mathcal{E}^1(L)$;

(iii) the mapping

$$(\psi, \phi_1, \dots, \phi_p) \mapsto \int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p)$$

is continuous along decreasing nets in $\mathcal{E}^1(L)$.

We shall prove $\mathrm{A}(p)$ by induction on p. For $p = n$, this proves Theorem 6.9.

The assertion $\mathrm{A}(0)$ is clear using Proposition 5.2, since $\mathrm{M}_0 = \mathrm{MA}(\phi'_1, \dots, \phi'_n)$ is a finite sum of Dirac masses at points in X^{qm} in view of Proposition 3.2.

Now assume $1 \le p \le n$, that $\mathrm{A}(p-1)$ holds, and let $\phi'_{p+1}, \dots, \phi'_n \in \mathrm{FS}(L)$. Given metrics $\phi_1, \dots, \phi_p \in \mathcal{E}^1(L)$, we define $\mathrm{M}_p(\phi_1, \dots, \phi_p)$ by forcing the integration by parts formula

$$\begin{aligned}\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p) := & \int (\phi_p - \phi_{\mathrm{ref}}) \, \mathrm{M}_{p-1}(\phi_1, \dots, \phi_{p-1}; \psi, \phi'_{p+1}, \dots, \phi'_n) \\ & + \int (\psi - \phi_p) \, \mathrm{M}_{p-1}(\phi_1, \dots, \phi_{p_1}; \phi_{\mathrm{ref}}, \phi'_{p+1}, \dots, \phi'_n)\end{aligned} \tag{6.5}$$

for every FS metric ψ on L.

By $\mathrm{A}(p-1)$, the right-hand side of (6.5) is well-defined and continuous along decreasing nets as a function of $(\phi_1, \dots, \phi_p)$. Further if the ϕ_i are all FS metrics, then equality holds in (6.5) when $\mathrm{M}_p(\phi_1, \dots, \phi_p)$ is replaced by $\mathrm{MA}(\phi_1, \dots, \phi_p, \phi'_{p+1}, \dots, \phi'_n)$. It then follows by regularization that the right-hand side is linear in ψ, and non-negative when $\psi \ge \phi_{\mathrm{ref}}$.

Since the space of differences of FS metrics on L is dense in $C^0(X)$, it follows that $\mathrm{M}_p(\phi_1, \dots, \phi_p)$ is well-defined as a Radon probability measure on X, and is continuous along decreasing nets as a function of $(\phi_1, \dots, \phi_p)$.

Next we prove (ii). We may assume $\psi, \phi_i \le \phi_{\mathrm{ref}}$. Set

$$B := \max\{-E(\psi), -E(\phi_1), \dots, -E(\phi_p), -E(\phi'_{p+1}), \dots, -E(\phi'_n)\} < \infty.$$

Let $(\psi^j)_j$ and $(\phi_i^j)_j$, $1 \le i \le p$, be decreasing nets of FS metrics converging to ψ and ϕ_i, respectively. Then $J(\psi^j) \le B$, $J(\phi_i^j) \le B$ for $1 \le i \le p$, and $J(\phi'_i) \le B$ for $p < i \le n$. Fix any index j_0. For $j \geq j_0$ we have

$$\int (\psi^{j_0} - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1^j, \dots, \phi_p^j) \geq \int (\psi^j - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1^j, \dots, \phi_p^j)$$
$$= \int (\psi^j - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_1^j, \dots, \phi_p^j; \phi'_{p+1}, \dots, \phi'_n)$$
$$\geq \int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - C_n(\mathrm{D}_{\mathrm{ref}} + B) \geq -B - C_n(\mathrm{D}_{\mathrm{ref}} + B) =: B',$$

where the second inequality follows from Lemma 3.22 and the last inequality from $\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) \geq E(\psi) \geq -B$. Letting $j \to \infty$ we get $\int (\psi^{j_0} - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p) \geq B' > -\infty$ by the continuity of $(\phi_1, \dots, \phi_p) \to \mathrm{M}_p(\phi_1, \dots, \phi_p)$ along decreasing nets. Since $\mathrm{M}_p(\phi_1, \dots, \phi_p)$ is a Radon probability measure and $(\psi^j)_j$ is a decreasing net of usc metrics, it follows from [Fol99, 7.12] that

$$\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p) = \lim_{j_0} \int (\psi^{j_0} - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p) \geq B > -\infty,$$

which completes the proof of (ii).

Finally we prove (iii). Let $(\phi_i^j)_j$, $i = 1, \dots, p$ and $(\psi^j)_j$ be decreasing nets in $\mathcal{E}^1(L)$ converging, respectively, to metrics ϕ_i and ψ in $\mathcal{E}^1(L)$. Set $\mu^j := \mathrm{M}_p(\phi_1^j, \dots, \phi_p^j)$. Then $(\mu^j)_j$ is a net of Radon measures converging weakly to $\mu := \mathrm{M}_p(\phi_1, \dots, \phi_p)$. Further, $(\psi^j - \phi_{\mathrm{ref}})_j$ is a decreasing net of usc functions converging pointwise to $\psi - \phi_{\mathrm{ref}}$, so it follows from general integration theory (see [BFJ15, Corollary 2.25]) that

$$\varlimsup_j \int (\psi^j - \phi_{\mathrm{ref}}) \mu_j \leq \int (\psi - \phi_{\mathrm{ref}}) \mu. \tag{6.6}$$

For the reverse estimate, we rely on the following approximate monotonicity property:

Lemma 6.33. *Let ψ and $\chi_i \geq \phi_i$, $i = 1, \dots, p$ be metrics in $\mathcal{E}^1(L)$. Then*

$$\int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\chi_1, \dots, \chi_p) \geq \int (\psi - \phi_{\mathrm{ref}}) \, \mathrm{M}_p(\phi_1, \dots, \phi_p)$$
$$+ \sum_{i=1}^p \int (\phi_i - \chi_i) \, \mathrm{M}_p(\phi_1, \dots, \phi_{i-1}, \phi_{\mathrm{ref}}, \chi_{i+1}, \dots, \chi_p).$$

The lemma implies that, for each j:

$$\int (\psi^j - \phi_{\mathrm{ref}}) \mu^j \geq \int (\psi - \phi_{\mathrm{ref}}) \mu^j$$
$$\geq \int (\psi - \phi_{\mathrm{ref}}) \mu + \sum_{i=1}^p \int (\phi_i - \phi_i^j) \, \mathrm{M}_p(\phi_1, \dots, \phi_{i-1}, \phi_{\mathrm{ref}}, \phi_{i+1}^j, \dots, \phi_p^j).$$

By the inductive hypothesis, the sum in the right-hand side tends to 0 as $j \to \infty$. Thus we obtain $\varliminf_j \int (\psi^j - \phi_{\mathrm{ref}}) \mu^j \geq \int (\psi - \phi_{\mathrm{ref}}) \mu$. Together with (6.6), this completes the proof of (iii) and hence the proof of Theorem 6.9.

Proof of Lemma 6.33. Since $\mathrm{M}_p(\chi_1, \dots, \chi_p)$ and $\mathrm{M}_p(\phi_1, \dots, \phi_p)$ are Radon probability measures, we may after regularization assume that ψ is an FS metric. Further, since we already know that $(\phi_1,\dots,\phi_p)\mapsto \mathrm{M}_p(\phi_1,\dots,\phi_p)$ is continuous along decreasing nets, we may by regularization assume that all ϕ_i and χ_i are FS metrics. Integration by parts (Proposition 3.4) then yields

$$\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\chi_1,\chi_2,\dots,\chi_p)-\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\phi_1,\chi_2,\dots,\chi_p)= \\ =\int(\chi_1-\phi_1)\,\mathrm{M}(\psi,\chi_2,\dots,\chi_p)-\int(\chi_1-\phi_1)\,\mathrm{M}(\phi_{\mathrm{ref}},\chi_2,\dots,\chi_p);$$

hence

$$\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\chi_1,\dots,\chi_p)\ge\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\phi_1,\chi_2,\dots,\chi_p) \\ +\int(\phi_1-\chi_1)\,\mathrm{M}_p(\phi_{\mathrm{ref}},\chi_2,\dots,\chi_p).$$

We similarly have

$$\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\phi_1,\chi_2,\chi_3,\dots,\chi_p)\ge\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{M}_p(\phi_1,\phi_2,\chi_3,\dots,\chi_p) \\ +\int(\phi_2-\chi_2)\,\mathrm{M}_p(\phi_1,\phi_{\mathrm{ref}},\chi_3,\dots,\chi_p).$$

Iterating this argument and summing up then yields the desired result. □

6.9. **Proof of Theorem 6.11.** Consider increasing nets $\psi^j\to\psi$ and $\phi_i^j\to\phi_i$ in $\mathcal{E}^1(L)$. We must show that

$$\int(\psi^j-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_1^j,\dots,\phi_n^j)\longrightarrow\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_1,\dots,\phi_n).$$

We shall prove this by induction on q, where q is the largest integer such that $\phi_p^j=\phi_p$ for all j and $p>q$.

First consider the case $q=0$. Thus $\phi_p^j=\phi_p$ for $1\le p\le n$. For each p, pick a *decreasing* net $(\phi_{p,l})_p$ of FS metrics on L converging to ϕ_p.

$$\int(\psi-\psi^j)\,\mathrm{MA}(\phi_1,\dots,\phi_n)=\int(\psi-\psi^j)\,\mathrm{MA}(\phi_{1,l},\dots,\phi_{n,l}) \\ +\left(\int(\psi-\psi^j)\,\mathrm{MA}(\phi_1,\dots,\phi_n)-\int(\psi-\psi^j)\,\mathrm{MA}(\phi_{1,l},\dots,\phi_{n,l})\right).$$

Pick any $\varepsilon>0$. By Corollary 3.19 we can find l such that the last line is bounded by ε for all j. Since $dd^c\phi_{1,l}\wedge\cdots\wedge dd^c\phi_{n,l}$ is a finite atomic measure supported on quasimonomial points, we then have $0\le\int(\psi-\psi^j)\,\mathrm{MA}(\phi_{1,l},\dots,\phi_{n,l})\le\varepsilon$ for $j\gg0$. Thus $0\le\int(\psi-\psi^j)\,\mathrm{MA}(\phi_1,\dots,\phi_n)\le2\varepsilon$.

Now assume $0<q\le n$. For $0\le p\le n$ set

$$\mu_p^j:=dd^c\phi_1^j\wedge\cdots\wedge dd^c\phi_p^j\wedge dd^c\phi_{p+1}\wedge\cdots\wedge dd^c\phi_n.$$

Note that $\mu_0^j=\mu:=dd^c\phi_1\wedge\cdots\wedge dd^c\phi_n$ and that $\mu_q^j=\cdots=\mu_n^j$. With this notation, we must prove that

$$\lim_j\int(\psi^j-\phi_{\mathrm{ref}})\mu_q^j=\int(\psi-\phi_{\mathrm{ref}})\mu. \tag{6.7}$$

To this end, we first claim that

$$\lim_j \int (\chi - \phi_{\mathrm{ref}})\mu_p^j = \int (\chi - \phi_{\mathrm{ref}})\mu \tag{6.8}$$

for $0 \le p \le q$ and every $\chi \in \mathcal{E}^1(L)$. Indeed, this is clear for $p = 0$, since $\mu_0^j = \mu$, and for $1 \le p \le q$ we have

$$\begin{aligned}
&\int (\chi - \phi_{\mathrm{ref}})\mu_p^j - \int (\chi - \phi_{\mathrm{ref}})\mu_{p-1}^j \\
&\quad = \int (\chi - \phi_{\mathrm{ref}}) dd^c\phi_1^j \wedge \cdots \wedge dd^c\phi_{p-1}^j \wedge dd^c(\phi_p^j - \phi_p) \wedge dd^c\phi_{p+1} \wedge \cdots \wedge dd^c\phi_n \\
&\qquad = \int (\phi_p^j - \phi_p) dd^c\phi_1^j \wedge \cdots \wedge dd^c\phi_{p-1}^j \wedge dd^c(\chi - \phi_{\mathrm{ref}}) \wedge dd^c\phi_{p+1} \wedge \cdots \wedge dd^c\phi_n,
\end{aligned}$$

which tends to zero by the inductive assumption.

Now we turn to (6.7). On the one hand, the fact that $\psi^j \le \psi$ implies that

$$\varlimsup_j \int (\psi^j - \phi_{\mathrm{ref}})\mu_q^j \le \varlimsup_j \int (\psi - \phi_{\mathrm{ref}})\mu_q^j = \int (\psi - \phi_{\mathrm{ref}})\mu$$

where we have used (6.8). On the other hand, for every index j' we have $\psi^j \ge \psi^{j'}$ for $j \ge j'$, and hence

$$\varliminf_j \int (\psi^j - \phi_{\mathrm{ref}})\mu_q^j \ge \varlimsup_{j'} \varliminf_j \int (\psi^{j'} - \phi_{\mathrm{ref}})\mu_q^j = \varlimsup_{j'} \int (\psi^{j'} - \phi_{\mathrm{ref}})\mu = \int (\psi - \phi_{\mathrm{ref}})\mu,$$

where the second equality follows from (6.8) and the last equality follows from the inductive assumption. This completes the proof.

7. The Calabi-Yau theorem

In this section we study the Calabi-Yau problem, by which we mean finding a solution $\phi \in \mathcal{E}^1(L)$ to the the non-Archimedean Monge-Ampère equation

$$\mathrm{MA}(\phi) = \mu, \tag{7.1}$$

for a suitable measure μ. This problem makes sense in the setting of an ample line bundle L on a projective variety X over any non-Archimedean field k, and we shall work in this setting as far as possible. However, we will only prove the existence of solutions when the ground field k is discretely or trivially valued, of residue characteristic zero. and X is smooth.

In the discretely valued case, our result is more general than the ones in [BFJ15, BGJKM16]. In the trivially valued case, it is completely new. The analogous Archimedean result is [BBGZ13, Theorem A]. See also [GJKM17] for recent (contitional) results in equicharacteristic p.

Fix a reference metric $\phi_{\mathrm{ref}} \in \mathrm{FS}(L)$. When k is trivially valued, we assume that $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$ is the trivial metric on L.

7.1. **The Calabi-Yau theorem.**

The strategy to solve (7.1) is to consider the functional

$$F_\mu \colon \mathcal{E}^1(L) \to \mathbf{R} \cup \{+\infty\}$$

defined by

$$F_\mu(\phi) := E(\phi) - \int (\phi - \phi_{\mathrm{ref}})\mu. \tag{7.2}$$

At least formally, we have $F'_\mu(\phi) = \mathrm{MA}(\phi) - \mu$, so a metric maximizing F_μ should be a solution to (7.1). However, it is nontrivial that a maximum ϕ exists, and even if one does exist, it is not obvious that it satisfies (7.1). At any rate, for this strategy to work, the functional F_μ must be bounded from above on $\mathcal{E}^1(L)$. This leads to the following definition.

Definition 7.1. *The* energy *of a Radon probability measure μ on X is*

$$E^*(\mu) := \sup_{\phi \in \mathcal{E}^1(L)} F_\mu(\phi) = \sup_{\phi \in \mathcal{E}^1(L)} \left(E(\phi) - \int (\phi - \phi_{\mathrm{ref}})\mu\right) \in \mathbf{R} \cup \{+\infty\}.$$

We say that μ has finite energy *if $E^*(\mu) < \infty$, and write $\mathcal{M}^1(X)$ for the set of probability measures of finite energy.*

Changing the reference metric ϕ_{ref} only changes F_μ by an additive constant. Hence the same is true for $E^*(\mu)$, so the space $\mathcal{M}^1(X)$ is independent of the choice of reference metric.

Let us give two examples of measures of finite energy. First, the Izumi inequality in Theorem 2.21 shows that any finite atomic measure supported on quasimonomial points has finite energy. Second. Monge-Ampère measures of finite energy metrics are of finite energy:

Proposition 7.2. *If $\phi \in \mathcal{E}^1(L)$ and $\mu := \mathrm{MA}(\phi)$, then $E^*(\mu) = (I - J)(\phi) < \infty$.*

Proof. For any $\psi \in \mathcal{E}^1(L)$ we see, by unwinding the formulas in §3.9 that

$$E(\psi) - \int (\psi - \phi_{\mathrm{ref}})\mu - (I - J)(\phi) = E(\psi) - E(\phi) - \int (\psi - \phi)\,\mathrm{MA}(\phi) = J_\phi(\psi) \geq 0,$$

with equality if $\psi = \phi$, so taking the supremum over ψ shows that $E^*(\mu) = (I - J)(\phi)$. $\square$

We can now state the Calabi–Yau theorem.

Theorem 7.3. *Assume k is discretely or trivially valued, of residue characteristic zero, X is smooth, and L is ample. Then the Monge-Ampère operator defines a bijection*

$$\mathrm{MA}\colon \mathcal{E}^1(L)/\mathbf{R} \xrightarrow{\sim} \mathcal{M}^1(X). \tag{7.3}$$

Further, given a measure $\mu \in \mathcal{M}^1(X)$ and a metric $\phi \in \mathcal{E}^1(L)$, the following two conditions are equivalent:

(i) $\mathrm{MA}(\phi) = \mu$;

(ii) ϕ *maximizes the functional* F_μ.

When these conditions hold, we have $E^*(\mu) = F_\mu(\phi) = (I - J)(\phi)$.

Note that $E^*(\mu) \geq 0$ for all $\mu \in \mathcal{M}^1(X)$, since $F_\mu(\phi_{\mathrm{ref}}) = 0$.

Corollary 7.4. *Under the assumptions of Theorem 7.3, we have $E^*(\mu) = 0$ iff $\mu = \mathrm{MA}(\phi_{\mathrm{ref}})$. When k is trivially valued and $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$ is the trivial metric, this means μ is a Dirac mass at the generic point of X.*

Proof. If $E^*(\mu) = 0$, then F_μ is maximized at ϕ_{ref}. □

The proof of Theorem 7.3 will be presented in the rest of this section. We start by making some preliminary remarks. Proposition 7.2, together with the formula $\mathrm{MA}(\phi + c) = \mathrm{MA}(\phi)$ implies that the map in (7.3) is well-defined. It follows from Corollary 6.28 that it is injective, under the assumptions on k and X. We shall prove surjectivity shortly.

7.2. **Measures of finite energy.** Before proving the surjectivity of the the map in (7.3), we record some further properties of the energy operator E^* and the space $\mathcal{M}^1(X)$ of measures of finite energy. In this subsection, k is arbitrary and X is an arbitrary variety. We first prove that the energies E and E^* are Legendre dual, in the following sense.

Proposition 7.5. *For $\phi \in \mathcal{E}^1(L)$ we have*

$$E(\phi) = \inf_{\mu \in \mathcal{M}^1(X)} \left(E^*(\mu) + \int (\phi - \phi_{\mathrm{ref}})\mu \right).$$

Proof. The inequality $E(\phi) \leq \inf_\mu \left(E^*(\mu) + \int(\phi - \phi_{\mathrm{ref}})\mu\right)$ is definitional, and by Proposition 7.2, the reverse inequality follows by choosing $\mu = \mathrm{MA}(\phi)$. □

Lemma 7.6. *The function $\mu \mapsto E^*(\mu)$ on the space of Radon probability measures on X is lsc and convex in the sense that $E^*(t_1\mu_1 + t_2\mu_2) \leq t_1 E^*(\mu_1) + t_2 E^*(\mu_2)$ for $t_i \geq 0$, $t_1 + t_2 = 1$.*

Proof. Convexity is clear since $E^*(\mu) = \sup_{\phi \in \mathcal{E}^1(L)} F_\mu(\phi)$ and $\mu \to F_\mu(\phi)$ is linear for every $\phi \in \mathcal{E}^1(L)$. To prove that E^* is lsc, it suffices, for the same reason, to prove that $\mu \mapsto \int(\phi - \phi_{\mathrm{ref}})\mu$ is usc for each $\phi \in \mathcal{E}^1(L)$. But this follows from $\phi - \phi_{\mathrm{ref}}$ being usc. □

Corollary 7.7. *The set $\mathcal{M}^1(X)$ is convex.*

The following characterization of measures of finite energy, closely related to [BBGZ13, Proposition 3.4], is quite useful. Recall that $\mathcal{E}^1_C(L)$ denotes the set of metrics $\phi \in \mathrm{PSH}(L)$ with $\sup(\phi - \phi_{\mathrm{ref}}) = 0$ and $E(\phi) \geq -C$.

Proposition 7.8. *For a Radon probability measure μ on X, the following assertions are equivalent:*

(i) $E^*(\mu) < +\infty$;

(ii) $\inf\{\int(\phi-\phi_{\mathrm{ref}})\mu \mid \phi\in\mathcal{E}^1_C(L)\}>-\infty$ *for every* $C\in\mathbf{R}$;
(iii) $\int(\phi-\phi_{\mathrm{ref}})\mu>-\infty$ *for every* $\phi\in\mathcal{E}^1(L)$.
(iv) *there exists* $A,B>0$ *such that* $\int(\phi-\phi_{\mathrm{ref}})\mu\ge -A-BJ(\phi)^{1/2}$ *for every* $\phi\in\mathcal{E}^1(L)$ *with* $\sup(\phi-\phi_{\mathrm{ref}})=0$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial. Further, (iv) implies (i) since in the definition of $E^*(\mu)$, it suffices to consider ϕ normalized by $\sup(\phi-\phi_{\mathrm{ref}})=0$, and for such ϕ we have $J(\phi)\le -E(\phi)$ and $E(\phi)\le 0$.

We will use contradiction arguments to prove (iii) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iv). To prove (iii) $\Longrightarrow$ (ii), we may assume there exist $C\in\mathbf{R}$ and a sequence $(\phi_j)_1^\infty$ in $\mathcal{E}^1_C(L)$ such that $\int(\phi_j-\phi_{\mathrm{ref}})\mu\le -2^j$ for all j. Set $\psi_m:=2^{-m}\phi_{\mathrm{ref}}+\sum_{j=1}^m 2^{-j}\phi_j$ for $m\ge 1$. Then ψ_m is a decreasing sequence in $\mathrm{PSH}(L)$ converging to $\psi:=\sum_{j=1}^\infty 2^{-j}\phi_j$. By concavity of E we have $E(\psi_m)\ge -C$ for all m; hence $E(\psi)=\lim_{m\to\infty}E(\psi_m)\ge -C$. On the other hand, monotone convergence gives

$$\int(\psi-\phi_{\mathrm{ref}})\mu=\lim_{m\to\infty}\int(\psi_m-\phi_{\mathrm{ref}})\mu=\lim_{m\to\infty}\sum_{j=1}^m 2^{-j}\int(\phi_j-\phi_{\mathrm{ref}})=-\infty,$$

contradicting (iii).

To prove (ii) $\Longrightarrow$ (iv), we similarly assume that there exists a sequence $(\phi_j)_j$ in $\mathcal{E}^1(L)$ such that $\sup(\phi_j-\phi_{\mathrm{ref}})=0$, $t_j:=J(\phi_j)^{1/2}\to 0$, and $t_j\int(\phi_j-\phi_{\mathrm{ref}})\mu\to-\infty$. Set $\tilde\phi_j:=t_j\phi_j+(1-t_j)\phi_{\mathrm{ref}}$. Then

$$\int(\tilde\phi_j-\phi_{\mathrm{ref}})\mu=t_j\int(\phi_j-\phi_{\mathrm{ref}})\mu\to-\infty.$$

On the other hand, Lemma 3.14 gives $I(\tilde\phi_j)\le nt_j^2I(\phi_j)$; hence $J(\tilde\phi_j)$ is uniformly bounded in view of (3.21). Since $E(\tilde\phi_j)=\int(\tilde\phi_j-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_{\mathrm{ref}})-J(\tilde\phi_j)$, and the first term is uniformly bounded in j, there exists $C>0$ such that $\tilde\phi_j\in\mathcal{E}^1_C$ for all j, contradicting (ii). ☐

Corollary 7.9. *If* $\mu\in\mathcal{M}^1(X)$, *then there exist* $A,B>0$ *such that*

$$F_\mu(\phi)\le A-J(\phi)-BJ(\phi)^{1/2} \tag{7.4}$$

for all $\phi\in\mathcal{E}^1(L)$. *Hence there exists* $C>0$ *such that* $E^*(\mu)=\sup_{\phi\in\mathcal{E}^1_C(L)}F_\mu(\phi)$.

Proof. To prove (7.4), we may assume ϕ is normalized by $\sup(\phi-\phi_{\mathrm{ref}})=0$. In this case, we have $E(\phi)\le -J(\phi)$, so (7.4) follows from Proposition 7.8 (iv).

By (7.4), $E^*(\mu)$ equals the supremum of $F_\mu(\phi)$ normalized as above and such that $J(\phi)\le C'$, for $C'>0$ large enough. But if ϕ is normalized, then $E(\phi)\ge -J(\phi)-\mathrm{D}_{\mathrm{ref}}$, where $\mathrm{D}_{\mathrm{ref}}\ge 0$ only depends on ϕ_{ref}, so the last assertion of the Corollary holds for $C=C+\mathrm{D}_{\mathrm{ref}}$. ☐

While the energy $E^*(\mu)$ of a probability measure μ depends on the line bundle L, we have the following result, due to Di Nezza [DiN16, Proposition 4.1] in the Archimedean case.

Corollary 7.10. *The set* $\mathcal{M}^1(X)$ *does not depend on the choice of ample line bundle* L.

Proof. Let L_1 and L_2 be ample line bundles on X, and μ a Radon probability measure on X. By symmetry, and by Proposition 7.8 it suffices to prove that if $\int\varphi_1\mu>-\infty$ for every L_1-psh function φ_1 then $\int\varphi_2\mu>-\infty$ for every L_2-psh function φ_2. But there exists a constant $C > 0$ such that $CL_2 - L_1$ is ample. This implies that $C\varphi_2$ is L_1-psh for every L_2-psh function φ_2. Hence $\int \varphi_2 \mu > -\infty$. □

Proposition 7.11. *Consider a non-Archimedean field extension k'/k. Set $X' := X_{k'}$, $L' := L_{k'}$, and let $\pi \colon X' \to X$ be the canonical map.*

(i) *If μ' is a Radon probability measure on X', then $E^*(\pi_*\mu') \le E^*(\mu')$.*
(ii) *if k is trivially valued, $k' = k((\varpi))$ and $\sigma \colon X \to X'$ is the Gauss extension, then $E^*(\sigma_*\mu) = E^*(\mu)$ for any Radon probability measure μ on X.*

Here the energies are computed with respect to L and L', using reference metrics ϕ_{ref} and $\phi'_{\mathrm{ref}} := \phi_{\mathrm{ref}} \circ \pi$, respectively. In (ii) we use $\phi_{\mathrm{ref}} = \phi_{\mathrm{triv}}$.

Proof. To prove (i), note that Proposition 6.7 (i) shows that

$$E^*(\pi_*\mu') = \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi) - \int (\phi - \phi_{\mathrm{ref}})\pi_*\mu') = \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi \circ \pi) - \int (\phi \circ \pi - \phi'_{\mathrm{ref}})\mu')$$
$$\le \sup_{\phi' \in \mathcal{E}^1(L')} (E(\phi') - \int (\phi' - \phi'_{\mathrm{ref}})\mu') = E^*(\mu'),$$

which proves (i). Similarly, Proposition 6.7 (ii) together with $\phi'_{\mathrm{triv}} \circ \sigma = \phi_{\mathrm{triv}}$ gives

$$E^*(\sigma_*\mu) = \sup_{\phi' \in \mathcal{E}^1(L')} (E(\phi') - \int (\phi' - \phi'_{\mathrm{triv}})\sigma_*\mu) \le \sup_{\phi' \in \mathcal{E}^1(L')} (E(\phi' \circ \sigma) - \int (\phi' \circ \sigma - \phi_{\mathrm{triv}})\mu)$$
$$\le \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi) - \int (\phi - \phi_{\mathrm{triv}})\mu) = E^*(\mu),$$

Combining these two inequalities, and using $\pi_*\sigma_* = \mathrm{id}$, we get $E^*(\sigma_*\mu) = E^*(\mu)$. □

Proposition 7.12. *If k is trivially valued, and μ is a Radon probability measure on X, then $E^*(t_*\mu) = tE^*(\mu)$ for any $t \in \mathbf{R}_+^\times$.*

Proof. Note that $\phi \mapsto \phi_t$ is a bijection of $\mathcal{E}^1(L)$ (with inverse $\phi \mapsto \phi_{t^{-1}}$). Proposition 6.7 (iii) and (2.1) then yield

$$E^*(t_*\mu) = \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi) - \int (\phi - \phi_{\mathrm{triv}})t_*\mu) = \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi) - \int t^*(\phi - \phi_{\mathrm{triv}})\mu)$$
$$= \sup_{\phi \in \mathcal{E}^1(L)} (E(\phi_t) - \int t^*(\phi_t - \phi_{\mathrm{triv}})\mu) = \sup_{\phi \in \mathcal{E}^1(L)} t(E(\phi) - (\phi - \phi_{\mathrm{triv}})\mu) = tE^*(\mu),$$

which completes the proof. □

7.3. **Proof of Theorem 7.3: general remarks.** We now continue the proof of Theorem 7.3. *For the rest of this section, k is discretely or trivially valued and X is smooth.* We must show that the equation (7.3) has a solution for every $\mu \in \mathcal{M}^1(X)$. We start by proving that such solutions are exactly minimizers of the functional F_μ. For this we use the differentiability property in in §6.7.

Lemma 7.13. *Given $\mu \in \mathcal{M}^1(X)$ and $\phi \in \mathcal{E}^1(L)$, the following two conditions are equivalent:*

(i) $\mathrm{MA}(\phi) = \mu$;

(ii) *ϕ maximizes the functional F_μ.*

When these conditions hold, we have $E^*(\mu) = F_\mu(\phi) = (I - J)(\phi)$.

Proof. The implication (i) $\implies$ (ii) and the last assertion of the lemma both follow from Proposition 7.2. It remains to prove (ii) $\implies$ (i). This is done as in [BBGZ13, Theorem 4.1] or [BFJ15, §8.2]. Namely, given $f \in C^0(X)$, define a function g on $\mathbf{R}$ by

$$g(t) = E(P(\phi + tf)) - \int (\phi - \phi_{\mathrm{ref}})\mu - t \int f\,\mu.$$

By Corollary 6.32, g is differentiable at $t = 0$, and $g'(0) = \int f\ \mathrm{MA}(\phi) - \int f\,\mu$. Now g has a maximum at $t = 0$, since $F_\mu(P(\phi + tf)) \le F_\mu(\phi)$ and $P(\phi + tf) \le \phi + tf$ for all t. Thus $g'(0) = 0$, and hence $\mathrm{MA}(\phi) = \mu$, since f was arbitrary. □

Corollary 7.14. *Assume k is trivially valued and use notation as in Proposition 7.11. Let $\mu \in \mathcal{M}^1(X)$, set $\mu' := \sigma_*\mu$, and assume there exists $\phi' \in \mathcal{E}^1(L')$ such that $\mathrm{MA}(\phi') = \mu'$. Then $\phi := \phi' \circ \sigma \in \mathcal{E}^1(L)$, and we have $\phi' = \phi \circ \pi$ and $\mathrm{MA}(\phi) = \mu$.*

Proof. By Proposition 6.7 we have $\phi \in \mathcal{E}^1(L)$ and $E(\phi) \ge E(\phi')$. Further, $\psi' := \phi \circ \pi \in \mathcal{E}^1(L')$, and $E(\psi') = E(\phi) \ge E(\phi')$. $\psi' \circ \sigma = \phi = \phi' \circ \sigma$, so

$$\int (\phi' - \phi'_{\mathrm{triv}})\mu' = \int (\phi' - \phi'_{\mathrm{triv}}) \circ \sigma\,\mu = \int (\psi' - \phi'_{\mathrm{triv}}) \circ \sigma\,\mu = \int (\psi' - \phi'_{\mathrm{triv}})\mu'.$$

This implies $F_{\mu'}(\psi') \ge F_\mu(\phi')$, so Lemma 7.13 gives $\mathrm{MA}(\psi') = \mathrm{MA}(\phi') = \mu'$. Since also $\sup(\phi' - \phi'_{\mathrm{triv}}) = \sup(\phi - \phi_{\mathrm{triv}}) = \sup(\psi' - \phi'_{\mathrm{triv}})$, Corollary 6.28 now shows that $\phi' = \psi' = \phi \circ \pi$, as desired. □

Corollary 7.15. *Given $\mu \in \mathcal{M}^1(X)$, the following conditions are equivalent:*

(i) *there exists $\phi \in \mathcal{E}^1(L)$ such that $\mathrm{MA}(\phi) = \mu$;*

(ii) *the functional $\phi \mapsto \int(\phi - \phi_{\mathrm{ref}})\mu$ is continuous on $\mathcal{E}^1_C(L)$ for all $C > 0$;*

(iii) *the functional F_μ is usc on $\mathcal{E}^1_C(L)$ for all $C > 0$.*

Proof. Since E is usc, (ii) implies (iii). Further, (i) implies (ii) by Proposition 6.18. It remains to see that (iii) implies (i). But (iii) and Corollary 7.9 imply that F_μ attains its supremum on $\mathcal{E}^1(L)$, since $\mathcal{E}^1_C(L)$ is compact for all $C > 0$, and this implies (i) in view of Lemma 7.13. □

7.4. **Proof of Theorem 7.3: dual complexes.** To solve (7.3), we first treat the case when μ is supported on a dual complex, then treat the general case by approximation.

Corollary 7.16. *If μ is a Radon probability measure supported on some dual complex $\Delta \subset X$, then $\mu \in \mathcal{M}^1(X)$.*

Proof. For any $\phi \in \mathcal{E}^1(L) \subset \mathrm{PSH}(L)$, the function $\phi - \phi_{\mathrm{ref}}$ is finite-valued and continuous on Δ, see Theorem 5.29. The result therefore follows from Proposition 7.8 (iii). □

Corollary 7.17. *If $\mu \in \mathcal{M}^1(X)$ is supported on a dual complex, then $\mu = \mathrm{MA}(\phi)$ for some $\phi \in \mathcal{E}^1(L)$.*

Proof. By Lemma 7.13 it suffices to find a minimizer of F_μ on $\mathcal{E}^1(L)$. It follows from Proposition 7.8 that there exists $C > 0$ such that $\sup_{\mathcal{E}^1_C(L)} F_\mu = \sup_{\mathcal{E}^1(L)} F_\mu$. It therefore suffices to prove that F_μ attains its supremum on $\mathcal{E}^1_C(L)$. But E is usc on $\mathcal{E}^1(L)$ and the equicontinuity assertion in Theorem 5.29 implies that $\phi \to \int(\phi - \phi_{\mathrm{ref}})\mu$ is continuous on $\mathcal{E}^1(L)$. Since $\mathcal{E}^1_C(L)$ is weakly compact, the supremum of F_μ is attained. □

In the discretely valued case, Corollary 7.17 was proved in [BFJ15] under an additional algebraicity assumption on X, that was removed in [BGJKM16]. The proof is essentially the same as the one given here.

Theorem 7.18. *Under the assumptions of Corollary 7.17, the metric ϕ is continuous.*

Proof. In the discretely valued case, this was proved in [BFJ15, §8.3] using capacity theory. We shall not revisit this proof here. In the trivially valued case, the continuity of the solution follows from Corollary 7.14. □

To prove Theorem 7.3 in general, we approximate the measure $\mu \in \mathcal{M}^1(X)$ by measures supported on dual complexes. The proof below works in both the discretely and trivially valued case. Recall that $\mathrm{SNC}(X)$ is the directed set of snc models for X (resp. snc test configurations for X). For any $\mathcal{X} \in \mathrm{SNC}(X)$, set

$$\mu_{\mathcal{X}} := (p_{\mathcal{X}})_*\mu,$$

where $p_{\mathcal{X}} : X \to \Delta_{\mathcal{X}}$ is the retraction. The operation $\mu \to \mu_{\mathcal{X}}$ should be thought of as a smoothing procedure.

Lemma 7.19. *For any $\phi \in \mathcal{E}^1(L)$, the net $(\int(\phi - \phi_{\mathrm{ref}})\mu_{\mathcal{X}})_{\mathcal{X}}$ is eventually decreasing and converges to $\int(\phi - \phi_{\mathrm{ref}})\mu$.*

Proof. The net $((\phi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}})_{\mathcal{X}}$ of continuous functions is eventually decreasing, see Corollary 5.31. Since μ is a Radon measure and $\int(\phi - \phi_{\mathrm{ref}})\mu_{\mathcal{X}} = \int(\phi - \phi_{\mathrm{ref}}) \circ p_{\mathcal{X}}\mu$, the result therefore follows from [Fol99, 7.12]. □

Corollary 7.20. *The net $(E^*(\mu_{\mathcal{X}}))_{\mathcal{X}}$ is eventually increasing, and $\lim_{\mathcal{X}} E^*(\mu_{\mathcal{X}}) = E^*(\mu)$.*

Pick an increasing sequence $(\mathcal{X}_j)_1^\infty$ in $\mathrm{SNC}(X)$. such that $E^*(\mu_j) \to E^*(\mu)$, where $\mu_j := \mu_{\mathcal{X}_j}$. For each j there exists a unique (continuous) metric $\phi_j \in \mathcal{E}^1(L)$ such that $\sup(\phi_j - \phi_{\mathrm{ref}}) = 0$ and $\mathrm{MA}(\phi_j) = \mu_j$. We shall prove that ϕ_j converges to a metric $\phi \in \mathcal{E}^1(L)$ such that $\mathrm{MA}(\phi) = \mu$. To simplify notation, write $\phi_j = \phi_{\mathrm{ref}} + \varphi_j$, where $\varphi_j \in \mathrm{QPSH}(X)$.

Lemma 7.21. *For every j we have $J(\phi_j) \le nE^*(\mu) < \infty$.*

Proof. We have $J(\phi_j) \le n(I - J)(\phi_j) = nE^*(\mu_j) \le nE^*(\mu)$. □

Lemma 7.22. *For $j \le l$ we have $I(\phi_j, \phi_l) \le (n+1)(E^*(\mu_l) - E^*(\mu_j))$.*

Proof. We have $I(\phi_j, \phi_l) \le (n+1)J_{\phi_l}(\phi_j)$. Using Lemma 7.19 we also have

$$\begin{aligned} J_{\phi_l}(\phi_j) &= E(\phi_l) - E(\phi_j) + \int \varphi_j \mu_l - \int \varphi_l \mu_l \\ &\le E(\phi_l) - E(\phi_j) + \int \varphi_j \mu_j - \int \varphi_l \mu_l \\ &= E^*(\mu_l) - E^*(\mu_j), \end{aligned}$$

completing the proof. □

Lemma 7.23. *The sequence $(\phi_j)_j$ converges strongly to a metric $\phi \in \mathcal{E}^1(L)$ with $\sup(\phi - \phi_{\mathrm{ref}}) = 0$. Further, we have $J(\phi) \le nE^*(\mu)$ and $I(\phi_j, \phi) \le (n+1)^2(E^*(\mu) - E^*(\mu_j)) \to 0$.*

Proof. By Proposition 6.29, ϕ_j converges strongly in $\mathcal{E}^1(L)$ to a metric $\phi \in \mathcal{E}^1(L)$ with $\sup(\phi - \phi_{\mathrm{ref}}) = 0$. To get the estimate in the proposition, note that

$$E(\phi_j) - E(\phi_l) - \int \varphi_j \mu_j + \int \varphi_l \mu_j = J_{\phi_j}(\phi_l) \le I(\phi_j, \phi_l) \le (n+1)(E^*(\mu) - E^*(\mu_j))$$

for $j \le l$. Now let $l \to \infty$. Since E is usc we have $E(\phi) \ge \overline{\lim}_l E(\phi_l)$, and since μ_j is supported on a dual complex, we have $\int \varphi \mu_j = \lim_l \int \varphi_l \mu_j$, where $\varphi = \phi - \phi_{\mathrm{ref}}$. This yields

$$J_{\phi_j}(\phi) = E(\phi_j) - E(\phi) - \int \varphi_j \mu_j + \int \varphi \mu_j \le (n+1)(E^*(\mu) - E^*(\mu_j)),$$

and hence $I(\phi_j, \phi) \le (n+1) J_{\phi_j}(\phi) \le (n+1)^2 (E^*(\mu) - E^*(\mu_l))$. □

Proposition 7.24. *We have* $\mathrm{MA}(\phi) = \mu$.

Proof. It suffices to show that $\int (\psi - \phi_{\mathrm{ref}}) \mathrm{MA}(\phi) = \int (\psi - \phi_{\mathrm{ref}}) \mu$ for every FS metric ψ on L such that $\sup(\psi - \phi_{\mathrm{ref}}) = 0$. By Lemma 3.23 we have

$$\left| \int (\psi - \phi_{\mathrm{ref}}) \mathrm{MA}(\phi) - \int (\psi - \phi_{\mathrm{ref}}) \mathrm{MA}(\phi_j) \right| \le C \sqrt{I(\phi_j, \phi)} \le C(n+1) \sqrt{E^*(\mu) - E^*(\mu_j)}$$

for some constant C independent of j. Hence

$$\int (\psi - \phi_{\mathrm{ref}}) \mathrm{MA}(\phi) = \lim_{j \to \infty} \int (\psi - \phi_{\mathrm{ref}}) \mathrm{MA}(\phi_j) = \lim_{j \to \infty} \int (\psi - \phi_{\mathrm{ref}}) \mu_j = \int (\psi - \phi_{\mathrm{ref}}) \mu,$$

which completes the proof. □

7.5. **The strong topology.** In analogy with the strong topology on $\mathcal{E}^1(L)$ introduced in §6.6, we define the *strong topology* on $\mathcal{M}^1(X)$ as the coarsest refinement of the weak topology such that $E^* \colon \mathcal{M}^1(X) \to \mathbf{R}$ is continuous. Thus a net $(\mu_j)_j$ in $\mathcal{M}^1(X)$ converges strongly to $\mu \in \mathcal{M}^1(X)$ iff $\mu_j \to \mu$ weakly and $\lim_j E^*(\mu_j) = E^*(\mu)$. The Calabi-Yau theorem can now be supplemented as follows.

Theorem 7.25. *The map*

$$\mathrm{MA} \colon \mathcal{E}^1(L)/\mathbf{R} \to \mathcal{M}^1(X)$$

is homeomorphism in the strong topology.

In the proof we shall use the pairing $\mathcal{E}^1(L) \times \mathcal{M}^1(X) \to \mathbf{R}$ given by $(\psi, \mu) \to \int (\psi - \phi_{\mathrm{ref}}) \mu$. In this way, every measure $\mu \in \mathcal{M}^1(X)$ defines a function on $\mathcal{E}^1(L)$, and every metric $\phi \in \mathcal{E}^1(L)$ a function on $\mathcal{M}^1(X)$.

Lemma 7.26. *For $C > 0$, set $\mathcal{M}_C(X) := \{\mu \in \mathcal{M}^1(X) \mid E^*(\mu) \le C\}$. Then:*

(i) *every $\mu \in \mathcal{M}^1(X)$ defines a continuous function on $\mathcal{E}^1_C(L)$ for any $C > 0$;*
(ii) *every $\psi \in \mathcal{E}^1(L)$ defines a continuous function on $\mathcal{M}_C(X)$ for any $C > 0$.*

Here continuity is in the weak topology on $\mathcal{E}^1_C(L)$ and $\mathcal{M}_C(X)$, respectively.

Proof. The statement in (i) is clear when $\mu = \mathrm{MA}(\phi)$ for $\phi \in \mathrm{FS}(L)$, since μ is then a finite atomic measure supported on quasimonomial points. Similarly, (ii) is clear when $\psi \in \mathrm{FS}(L)$, since $\psi - \phi_{\mathrm{ref}}$ is then a continuous function on X. To treat the general case in (i) and (ii) we use the Calabi-Yau theorem together with the estimates in §3.12.

Specifically, to prove (i) in general, write $\mu = \mathrm{MA}(\phi)$ where $\phi \in \mathcal{E}^1(L)$, and pick a decreasing net $(\phi_l)_l$ in $\mathrm{FS}(L)$ converging to ϕ. Then $\phi_l \to \phi$ strongly, so $I(\phi_l, \phi) \to 0$.

Lemma 3.23 therefore shows that $\lim_l \int(\psi-\phi_{\mathrm{ref}})(\mathrm{MA}(\phi_l)-\mu)=0$ uniformly in $\psi\in\mathcal{E}^1_C(L)$, Since $\psi\to\int(\psi-\phi_{\mathrm{ref}})\,\mathrm{MA}(\phi_l)$ is continuous on $\mathcal{E}^1_C(L)$, so is $\psi\to\int(\psi-\phi_{\mathrm{ref}})\mu$.

Similarly, to prove (ii), let (ψ_j) be a decreasing net in $\mathrm{FS}(L)$ converging to ψ. Then $\lim_j I(\psi_j,\psi)=0$ and $\lim_j\int(\psi_j-\psi)\,\mathrm{MA}(\phi_{\mathrm{ref}})=0$. Further, any $\mu\in\mathcal{M}_C(X)$ is of the form $\mu=\mathrm{MA}(\phi)$, where $\phi\in\mathcal{E}^1(L)$ and $J(\phi)\le n(I-J)(\phi)=nE^*(\mu)\le nC$. Corollary 3.20 now shows that $\lim_j\int(\psi_j-\psi)(\mu-\mathrm{MA}(\phi_{\mathrm{ref}}))=0$ uniformly in $\mu\in\mathcal{M}_C(X)$; hence $\mu\mapsto\int(\psi-\phi_{\mathrm{ref}})\mu$ is continuous on $\mathcal{M}_C(X)$. □

Proof of Theorem 7.25. We already know that the map MA is a bijection. To prove that it is continuous, consider a net $(\phi_j)_j$ in $\mathcal{E}^1(L)/\mathbf{R}$ converging strongly to $\phi\in\mathcal{E}^1(L)$. We must prove that $\mu_j:=\mathrm{MA}(\phi_j)$ converges weakly to $\mu:=\mathrm{MA}(\phi)$, and that $E^*(\mu_j)\to E^*(\mu)$. The latter assertion follows from Corollary 3.21 since $E^*(\mu_j)=(I-J)(\phi_j)$, $E^*(\mu)=(I-J)(\phi)$ and $I(\phi_j,\phi)\to 0$. To prove that μ_j converges weakly to μ, it suffices to prove that $\lim_j\int(\psi-\phi_{\mathrm{ref}})\mu_j=\int(\psi-\phi_{\mathrm{ref}})\mu$ for every $\psi\in\mathrm{FS}(L)$, since differences of functions of the form $\psi-\phi_{\mathrm{ref}}$ are dense in $C^0(X)$ by Theorem 2.7. But this equality follows from Lemma 3.23.

It remains to prove that the inverse of MA is continuous. Thus consider a net $(\phi_j)_j$ in $\mathcal{E}^1(L)$ and assume that the measures $\mu_j:=\mathrm{MA}(\phi_j)$ converge strongly to $\mu\in\mathcal{M}^1(X)$. We must prove that $\phi_j\to\phi$ strongly in $\mathcal{E}^1(L)$, where $\mathrm{MA}(\phi)=\mu$. Pick C such that $\mu_j,\mu\in\mathcal{M}_C(X)$ for all j. Lemma 7.26 (ii) implies that $\lim_j\int(\phi-\phi_{\mathrm{ref}})\mu_j=\int(\phi-\phi_{\mathrm{ref}})\mu$, Next, note that $E^*(\mu)=(I-J)(\phi)=E(\phi)-\int(\phi-\phi_{\mathrm{ref}})\mu$, and similarly $E^*(\mu_j)=E(\phi_j)-\int(\phi_j-\phi_{\mathrm{ref}})\mu_j$ for all j. Since $\lim_j E^*(\mu_j)=E^*(\mu)$, we conclude that

$$J_{\phi_j}(\phi)=E(\phi_j)-E(\phi)-\int(\phi_j-\phi)\mu_j\to 0,$$

and hence $I(\phi_j,\phi)$ by (3.19), so that $\phi_j\to\phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$. □

For completeness, we also prove the following result.

Proposition 7.27. *Consider a measure $\mu\in\mathcal{M}^1(X)$ and let $(\phi_j)_j$ be a net in $\mathcal{E}^1(L)/\mathbf{R}$ such that $F_\mu(\phi_j)\to E^*(\mu)$. Then $\lim_j\phi_j=\phi$ strongly in $\mathcal{E}^1(L)/\mathbf{R}$, where $\phi\in\mathcal{E}^1(L)/\mathbf{R}$ is the unique solution to $\mathrm{MA}(\phi)=\mu$.*

Proof. We may assume that the ϕ_j are normalized by $\sup(\phi_j-\phi_{\mathrm{ref}})=0$. By Corollary 7.9 there exists $C>0$ such that $\phi_j\in\mathcal{E}^1_C(L)$ for large j. By weak compactness of $\mathcal{E}^1_C(L)$, we can therefore find a subnet $(\phi_{j(\alpha)})_\alpha$ converging to some $\phi\in\mathcal{E}^1_C(L)$. Since F_μ is usc, we have $F_\mu(\phi)\ge\varlimsup_\alpha F_\mu(\phi_{j(\alpha)})=E^*(\mu)$. Hence $\mathrm{MA}(\phi)=\mu$. □

Remark 7.28. *In the Archimedean case, the space $\mathcal{M}^1(X)$ enjoy certain compactness properties in the strong topology, see* [BBEGZ16, 2.4]. *These are not valid in the non-Archimedean setting. For example, let k be trivially valued, $X=\mathbf{P}^1_k$, and $L=\mathcal{O}(1)$. Let $(\xi_n)_1^\infty$ be a sequence of pairwise distinct closed points in X^{sch}, and set $x_n=r^{\mathrm{ord}_{\xi_n}}$, for some $r\in(0,1)$. If $\mu_n=\delta_{x_n}$, then $\mu_n\in\mathcal{M}^1(X)$ and $E^*(\mu_n)>0$ is independent of n. Now μ_n converges weakly to the measure μ which is the DIrac mass at the generic point of X, but $E^*(\mu)=0$, so $\mu_n\not\to\mu$ strongly.*

Appendix A. Proof of estimates

In this section we prove the estimates in §3.12.

Proof of Lemma 3.12. The first inequality is trivial. Since $\mathrm{MA}(\phi_{\mathrm{ref}})$ is supported on a finite set of quasimonomial points, the second inequality follows from the Izumi estimate in Theorem 2.21. When k is trivially valued, $\mathrm{MA}(\phi_{\mathrm{triv}})$ is a Dirac mass at the generic point of X, and $\phi-\phi_{\mathrm{triv}}$ attains its supremum at this point by Lemma 2.20, so we can pick $\mathrm{D}_{\mathrm{ref}}=0$. □

Proof of Corollary 3.13. This follows immediately from (3.20) and Lemma 3.12. □

Proof of Lemma 3.14. Adding a constant to ϕ, we may assume $\int(\phi-\psi)(dd^c\psi)^n=0$. Set $\phi_t:=t\phi+(1-t)\psi$. Then $\phi_t-\psi=t(\phi-\psi)$, so $\int(\phi_t-\psi)(dd^c\psi)^n=0$, and

$$\begin{aligned}I(\phi_t,\psi)&=-tV^{-1}\int(\phi-\psi)(dd^c\phi_t)^n\\&=-tV^{-1}\sum_{j=0}^{n}\binom{n}{j}t^j(1-t)^{n-j}\int(\phi-\psi)(dd^c\phi)^j\wedge(dd^c\psi)^{n-j}.\end{aligned}$$

Here the integral vanishes for $j=0$ and is bounded below by $\int(\phi-\psi)(dd^c\phi)^n$ for $j>0$, so

$$I(\phi_t,\psi)\le -t(1-(1-t)^n)V^{-1}\int(\phi-\psi)(dd^c\phi)^n\le nt^2I(\phi,\psi),$$

by the concavity of $t\to(1-(1-t)^n)$. □

Proof of Lemma 3.15. Set $\phi:=\frac{1}{2}(\phi_1+\phi_2)$, $u:=\phi_1-\phi_2$, and, for $0\le p<n$,

$$b_p:=-V^{-1}\int udd^cu\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-1-p}.$$

We aim to bound b_{n-1}. Set

$$A:=I(\phi_1,\phi_2)\quad\text{and}\quad B:=\max_{i=1,2}I(\phi_i,\psi).$$

Note that $b_0\le A$. To get a preliminary bound on b_{n-1} we write $u=(\phi_1-\psi)+(\psi-\phi_2)$ and use the triangle inequality resulting from Cauchy-Schwartz:

$$b_{n-1}\le\left(\sum_{i=1}^{2}\left(-V^{-1}\int(\phi_i-\psi)dd^c(\phi_i-\psi)\wedge(dd^c\psi)^{n-1}\right)^{\frac{1}{2}}\right)^2\le 4B. \tag{A.1}$$

For $0\le p\le n-2$ we have

$$\begin{aligned}b_{p+1}-b_p&=-V^{-1}\int udd^cu\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}\wedge dd^c(\psi-\phi)\\&=-V^{-1}\int udd^c(\psi-\phi)\wedge dd^c\phi_1\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}\\&\quad+V^{-1}\int udd^c(\psi-\phi)\wedge dd^c\phi_2\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}.\end{aligned}$$

Here we can use Cauchy-Schwartz to bound the each of the last two terms:

$$\left(-V^{-1}\int udd^c(\psi-\phi)\wedge dd^c\phi_i\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}\right)^2$$
$$\leq\left(-V^{-1}\int udd^cu\wedge dd^c\phi_i\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}\right)$$
$$\cdot\left(-V^{-1}\int(\psi-\phi)dd^c(\psi-\phi)\wedge dd^c\phi_i\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-2-p}\right)$$

Using $dd^c\phi_i\leq 2dd^c\phi$ we can bound the first factor by $2b_p$, and the second factor by

$$-2V^{-1}\int(\psi-\phi)dd^c(\psi-\phi)\wedge(dd^c\psi)^p\wedge(dd^c\phi)^{n-1-p}\leq 2I(\phi,\psi).$$

Hence $b_{p+1}-b_p\leq 4\sqrt{b_pI(\phi,\psi)}$. Now

$$I(\phi,\psi)\leq(n+1)J_\psi(\phi)\leq(n+1)\max_{i=1,2}J_\psi(\phi_i)\leq(n+1)\max_{i=1,2}I(\phi_i,\psi)=(n+1)B$$

by (3.19) and the convexity of $\phi\to J_\psi(\phi)$. This implies

$$b_{p+1}\leq b_p+4\sqrt{(n+1)Bb_p} \tag{A.2}$$

for $0\leq p\leq n-2$.

Now we consider two cases. First assume that $A\geq 2^{-2^{n-1}}B$. In this case, (A.1) yields

$$b_{n-1}\leq 4B\leq 8A^{\frac{1}{2^{n-1}}}B^{1-\frac{1}{2^{n-1}}}.$$

Now assume $A\leq 2^{-2^{n-1}}B$. In this case, we prove by induction that

$$b_p\leq C_{n,p}A^{\frac{1}{2^p}}B^{1-\frac{1}{2^p}} \tag{A.3}$$

for $0\leq p\leq n-1$. Setting $p=n-1$ will complete the proof.

We already know that $b_0\leq A$, so (A.3) holds for $p=0$ with $C_{n,0}=1$. When $0\leq p\leq n-2$, we get from (A.2) and the induction hypothesis that

$$\begin{aligned}b_{p+1}&\leq b_p+4\sqrt{(n+1)Bb_p}\\&\leq C_{n,p}A^{\frac{1}{2^p}}B^{1-\frac{1}{2^p}}+4\sqrt{(n+1)C_{n,p}}A^{\frac{1}{2^{p+1}}}B^{1-\frac{1}{2^{p+1}}}\\&=A^{\frac{1}{2^{p+1}}}B^{1-\frac{1}{2^{p+1}}}\left(C_{n,p}\left(\frac{A}{B}\right)^{\frac{1}{2^{p+1}}}+4\sqrt{(n+1)C_{n,p}}\right).\end{aligned}$$

By assumption, $A\leq 2^{-2^{n-1}}B\leq B$, so $b_{p+1}\leq C_{n,p+1}A^{\frac{1}{2^{p+1}}}B^{1-\frac{1}{2^{p+1}}}$, where $C_{n,p+1}=C_{n,p}+4\sqrt{(n+1)C_{n,p}}$. The proof is complete. □

Proof of Lemma 3.16. For $\psi\in\mathrm{FS}(L)$ and $u\in\mathrm{DFS}(X)$, we set

$$\|u\|_\psi:=(-V^{-1}\int udd^cu\wedge(dd^c\psi)^{n-1})^{1/2}.$$

By Cauchy-Schwartz, this defines a seminorm on $\mathrm{DFS}(X)$.

Now set $\phi:=(\phi_1+\phi_2)/2$. It follows from (3.16) that

$$\|\phi_1-\phi_2\|_\phi^2\leq I(\phi_1,\phi_2)\leq 2^{n-1}\|\phi_1-\phi_2\|_\phi^2.$$

Together with the triangle inequality for $\|\cdot\|_\phi$, this yields

$$I(\phi_1,\phi_2)\le 2^{n-1}\|\phi_1-\phi_2\|_\phi^2\le 2^{n-1}(2\max_{i=1,2}\|\phi_1-\phi_3\|_\phi)^2=2^{n+1}\max_{i=1,2}\|\phi_1-\phi_3\|_\phi^2.$$

We can now use Lemma 3.15 to get, for $i=1,2$,

$$\|\phi_i-\phi_3\|_\phi^2\le C_n I(\phi_i,\phi_3)^{\frac{1}{2^{n-1}}}\max\{I(\phi_i,\phi),I(\phi_3,\phi)\}^{1-\frac{1}{2^{n-1}}}.$$

By (2.2) and the convexity of $\phi\mapsto J_\psi(\phi)$, we have

$$I(\phi_i,\phi)\le(n+1)J_{\phi_i}(\phi)\le(n+1)\max\{J_{\phi_i}(\phi_1),J_{\phi_i}(\phi_2)\}\le(n+1)I(\phi_1,\phi_2)$$

and

$$I(\phi_3,\phi)\le(n+1)J_{\phi_3}(\phi)\le(n+1)\max_{i=1,2}J_{\phi_3}(\phi_i)\le(n+1)\max_{i=1,2}I(\phi_i,\phi_3)$$

Altogether, this yields

$$I(\phi_1,\phi_2)\le C'_n\max_{i=1,2}I(\phi_i,\phi_3)^{\frac{1}{2^{n-1}}}\max\{I(\phi_1,\phi_2),\max_{i=1,2}I(\phi_i,\phi_3)\}^{1-\frac{1}{2^{n-1}}},\tag{A.4}$$

where $C'_n=2^{n+1}(n+1)^{1-\frac{1}{2^{n-1}}}C_n$.

If $I(\phi_1,\phi_2)\ge\max_{i=1,2}I(\phi_i,\phi_3)$, then (A.4) yields $I(\phi_1,\phi_2)\le(C'_n)^{2^{n-1}}\max_{i=1,2}I(\phi_i,\phi_3)$. On the other hand, since $C'_n\ge 1$, which we may assume, then the same inequality trivially holds also when $I(\phi_1,\phi_2)\le\max_{i=1,2}I(\phi_i,\phi_3)$. This completes the proof. □

Proof of Corollary 3.17. This follows from Lemma 3.16 applied to $\phi_1=\phi$, $\phi_2=\phi_{\mathrm{ref}}$ and $\phi_3=\phi$, and using the inequalities $I(\phi)\le(n+1)J(\phi)$, $I(\psi)\le(n+1)J(\psi)$. □

Proof of Corollary 3.18. Note that $\psi:=\frac{1}{n-1}\sum_{j=1}^{n-1}\psi_j\in\mathrm{FS}(L)$. Expanding $(dd^c\psi)^{n-1}$ yields

$$-\int(\phi_1-\phi_2)dd^c(\phi_1-\phi_2)\wedge(dd^c\psi)^{n-1}$$
$$\ge-\frac{(n-1)!}{(n-1)^{n-1}}\int(\phi_1-\phi_2)dd^c(\phi_1-\phi_2)\wedge dd^c\psi_1\wedge\cdots\wedge dd^c\psi_{n-1}.$$

Here the left hand side is bounded above by $C_nI(\phi_1,\phi_2)^{\frac{1}{2^{n-1}}}\max_{i=1,2}I(\phi_i,\psi)^{1-\frac{1}{2^{n-1}}}$ in view of Lemma 3.15. Now $I(\phi_i,\psi)\le C_n\max\{J(\phi_i),J(\psi)\}$ by Corollary 3.17, and $J(\psi)\le\max_i J(\psi_i)$ by convexity. This completes the proof. □

Proof of Corollary 3.19. We proceed as in the proof of Lemma 3.15. For $0\le p\le n$, set

$$a_p:=V^{-1}\int(\psi_1-\psi_2)dd^c\phi_1\wedge\cdots\wedge dd^c\phi_p\wedge dd^c\phi'_{p+1}\wedge\cdots\wedge dd^c\phi'_n.$$

We want to estimate $|a_n-a_0|$. Cauchy-Schwartz gives $|a_p-a_{p-1}|^2\le A_pB_p$, where

$$A_p=-V^{-1}\int(\psi_1-\psi_2)dd^c(\psi_1-\psi_2)\wedge dd^c\phi_1\wedge\cdots\wedge dd^c\phi_{p-1}\wedge dd^c\phi'_{p+1}\wedge\cdots\wedge dd^c\phi'_n$$

and

$$B_p=-V^{-1}\int(\phi_p-\phi'_p)dd^c(\phi_p-\phi'_p)\wedge dd^c\phi_1\wedge\cdots\wedge dd^c\phi_{p-1}\wedge dd^c\phi'_{p+1}\wedge\cdots\wedge dd^c\phi'_n.$$

By Corollary 3.18 we have

$$A_p\le C_nI(\psi_1,\psi_2)^{\frac{1}{2^{n-1}}}M^{1-\frac{1}{2^{n-1}}}\quad\text{and}\quad B_p\le C_nI(\phi_p,\phi'_p)^{\frac{1}{2^{n-1}}}M^{1-\frac{1}{2^{n-1}}};$$

hence

$$|a_p - a_{p-1}| \le C_n I(\psi_1, \psi_2)^{\frac{1}{2^n}} I(\phi_p, \phi'_p)^{\frac{1}{2^n}} M^{1-\frac{1}{2^{n-1}}},$$

and the desired bound on $|a_n - a_0|$ follows from the triangle inequality. □

Proof of Corollary 3.20. This is an immediate consequence of Corollary 3.19. □

Proof of Corollary 3.21. We have

$$\begin{aligned} I(\phi) - I(\psi) = V^{-1} \int (\phi - \psi)((dd^c\phi_{\mathrm{ref}})^n - (dd^c\psi)^n) \\ + V^{-1} \int (\phi - \phi_{\mathrm{ref}})((dd^c\psi)^n - (dd^c\phi)^n), \end{aligned}$$

and each of the two integrals is easily estimated using Corollary 3.20. Similarly,

$$J(\phi) - J(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int (\phi - \psi)((dd^c\phi_{\mathrm{ref}})^n - (dd^c\phi)^j \wedge (dd^c\psi)^{n-j}),$$

and each term can be bounded using Corollary 3.19. □

Proof of Lemma 3.22. We may assume $\max(\phi_i - \phi_{\mathrm{ref}}) = 0$ for all i. To simplify notation, write $\varphi_i := \phi_i - \phi_{\mathrm{ref}}$ for $0 \le i \le n$ and $\phi = \frac{1}{n+1}\sum_0^n \phi_i$, $\varphi = \frac{1}{n+1}\sum_0^n \varphi_i$. By the concavity of E we have

$$\begin{aligned} E(\phi) \ge \frac{1}{n+1}\sum_{i=0}^{n} E(\phi_i) \ge \min_i E(\phi_i) \\ = \min_i \{ \int (\phi_i - \phi_{\mathrm{ref}}) \,\mathrm{MA}(\phi_{\mathrm{ref}}) - J(\phi_i)\} \ge -\mathrm{D}_{\mathrm{ref}} - \max_i J(\phi_i), \end{aligned}$$

where the last inequality follows from Lemma 3.12. Now expand the left-hand side:

$$\begin{aligned} E(\phi) = \frac{1}{n+1}\sum_{p=0}^{n} V^{-1} \int \varphi (dd^c\phi)^p \wedge (dd^c\phi_{\mathrm{ref}})^{n-p} \le \frac{1}{n+1} V^{-1} \int \varphi (dd^c\phi)^n \\ \le \frac{1}{(n+1)^2} V^{-1} \int \varphi_0 (dd^c\phi)^n \le \frac{n!}{(n+1)^{n+2}} \int \varphi_0 \,\mathrm{MA}(\phi_1, \dots, \phi_n), \end{aligned}$$

which completes the proof. □

Proof of Lemma 3.23. We may assume $\max(\psi - \phi_{\mathrm{ref}}) = \max(\phi_i - \phi_{\mathrm{ref}}) = 0$. For $p = 0, \dots, n$ set

$$a_p := V^{-1} \int (\psi - \phi_{\mathrm{ref}})(dd^c\phi_1)^p \wedge (dd^c\phi_2)^{n-p}.$$

Our goal is to estimate $|a_n - a_0|$. Note that

$$a_{p+1} - a_p = V^{-1} \int (\psi - \phi_{\mathrm{ref}}) dd^c(\phi_1 - \phi_2) \wedge (dd^c\phi_1)^p \wedge (dd^c\phi_2)^{n-p-1},$$

so the Cauchy-Schwarz inequality yields $|a_{p+1} - a_p|^2 \leq A_p B_p$, where

$$\begin{aligned}A_p &= -V^{-1} \int (\psi - \phi_{\mathrm{ref}}) dd^c(\psi - \phi_{\mathrm{ref}}) \wedge (dd^c\phi_1)^p \wedge (dd^c\phi_2)^{n-p-1} \\ &\leq -V^{-1} \int (\psi - \phi_{\mathrm{ref}}) dd^c\psi \wedge (dd^c\phi_1)^p \wedge (dd^c\phi_2)^{n-p-1} \\ &\leq C_n(\mathrm{D}_{\mathrm{ref}} + \max\{J(\psi), J(\phi_1), J(\phi_2)\}),\end{aligned}$$

the last inequality following from Lemma 3.22 and Lemma 3.12, and where

$$B_p = -V^{-1} \int (\phi_1 - \phi_2) dd^c(\phi_1 - \phi_2) \wedge (dd^c\phi_1)^p \wedge (dd^c\phi_2)^{n-p-1} \leq I(\phi_1, \phi_2).$$

We thus obtain the desired estimate by invoking the triangle inequality. □

Proof of Lemma 6.17. Set $\phi_t := t\phi + (1-t)\psi$. Then $\phi_t - \psi = t(\phi - \psi)$ and

$$J_\psi(\phi_t) = E(\psi) - E(\phi_t) + t \int (\phi - \psi) \,\mathrm{MA}(\psi)$$

Differentiating this with respect to t, gives

$$\frac{d}{dt} J_\psi(\phi_t) = -\int (\phi - \psi) \,\mathrm{MA}(\phi_t) + \int (\phi - \psi) \,\mathrm{MA}(\psi) = t^{-1} I(\phi_t, \psi) \geq \frac{n+1}{n} t^{-1} J_\psi(\phi_t)$$

for $0 < t \leq 1$, where the last inequality follows from (3.21). The desired estimate follows. □

References

[BPR11] M. Baker, S. Payne, and J. Rabinoff. *Non-archimedean geometry, tropicalization, and metrics on curves*. Algebr. Geom. **3** (2016), 63–105. 24

[BR10] M. Baker and R. Rumely. *Potential theory on the Berkovich projective line*. Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010. 24

[BT82] E. Bedford and B. A. Taylor. *A new capacity for plurisubharmonic functions.* Acta Math. **149** (1982), 1–40. 53

[BT87] E. Bedford and B. A. Taylor. *Fine topology, Shilov boundary and* $(dd^c)^n$. J. Funct. Anal. **72** (1987), 225–251. 53

[Berk90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields.* Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990. 2, 7, 10, 14, 24, 25

[Berm13] R. J. Berman. *A thermodynamical formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kaähler-Einstein metrics.* Adv. Math. **248** (2013), 1254–1297. 5

[BBGZ13] R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi. *A variational approach to complex Monge-Ampère equations*. Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245. 4, 6, 33, 34, 53, 56, 64, 65, 68

[BBEGZ16] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi. *Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties*. J. Reine Angew. Math. Published Online: 2016-09-14. 6, 33, 71

[BBJ15] R. J. Berman, S. Boucksom and M. Jonsson. *A variational approach to the Yau-Tian-Donaldson conjecture* `arXiv:1509.04561`. 5

[BBJ17] R. J. Berman, S. Boucksom and M. Jonsson. *A variational approach to the Yau-Tian-Donaldson conjecture*. Updated version of [BBJ15], in preparation. 2, 5

[BGS95] S. Bloch, H. Gillet and C. Soulé. *Non-Archimedean Arakelov Geometry.* J. Algebraic Geom. **4** (1995), 427–485.

[BłK07] Z. Błocki and S. Kołodziej. *On regularization of plurisubharmonic functions on manifolds.* Proc. Amer. Math. Soc. **135** (2007), 2089–2093. 3

[BlJ17] H. Blum and M. Jonsson. *Thresholds, valuations, and K-stability.* `arXiv:1706.04548`. 5

[BGR] S. Bosch, U. Güntzer and R. Remmert. *Non-Archimedean Analysis.* Springer-Verlag, Berlin, Heidelberg, 1994. 27

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi. *Monge-Ampère equations in big cohomology classes.* Acta Math. **205** (2010), 199–262. 53

[BE17] S. Boucksom, D. Eriksson. *Determinant of cohomology and Deligne pairings in non-Archimedean geometry.* In preparation. 17, 23, 26, 32, 44

[BFJ08b] S. Boucksom, C. Favre and M. Jonsson. *Valuations and plurisubharmonic singularities*. Publ. Res. Inst. Math. Sci. **44** (2008), 449–494. 37

[BFJ14] S. Boucksom, C. Favre and M. Jonsson. *A refinement of Izumi's Theorem.* Valuation Theory in Interaction, 55–81. EMS Series of Congress Reports, vol 10. European Mathematical Society, Zürich, 2014. 24

[BFJ15] S. Boucksom, C. Favre and M. Jonsson. *Solution to a non-Archimedean Monge-Ampère equation.* J. Amer. Math. Soc. **28** (2015), 617–667. 2, 4, 29, 38, 51, 53, 56, 58, 59, 61, 64, 68, 69

[BFJ16a] S. Boucksom, C. Favre and M. Jonsson. *Singular semipositive metrics in non-Archimedean geometry.* J. Algebraic Geom. **25** (2016), 77–139. 2, 3, 4, 6, 8, 22, 24, 35, 36, 38, 42, 45, 46, 47, 48, 49, 53, 60

[BFJ16b] S. Boucksom, C. Favre and M. Jonsson. *The Non-Archimedean Monge-Ampère Equation.* In *Tropical and Non-Archimedean Geometry*, 195–285. Simons Symposia. Springer International Publishing, 2016. 2

[BHJ17] S. Boucksom, T. Hisamoto and M. Jonsson. *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs.* Ann. Inst. Fourier **67** (2017), 743–841. 2, 3, 5, 12, 21, 22, 23, 28, 29, 32

[BHJ16] S. Boucksom, T. Hisamoto and M. Jonsson. *Uniform K-stability and asymptotics of energy functionals in Kähler geometry.* `arXiv:1603.01026`. To appear in J. Eur. Math. Soc. 32

[BoJ17] S. Boucksom and M. Jonsson. *A non-Archimedean approach to K-stability*. In preparation. 2, 5

[BKMS16] S. Boucksom, A. Küronya, C. Maclean and T. Szemberg. *Vanishing sequences and Okounkov bodies*. Math. Ann. **361** (2015), 811–834. 24

[BPS11] J. I. Burgos Gil, P. Philippon and M. Sombra. *Arithmetic geometry of toric varieties. Metrics, measures, and heights*. Astérisque No. 360 (2014).

[BGJKM16] J. I. Burgos Gil, W. Gubler, P. Jell, K. Künnemann and F. Martin *Differentiability of non-archimedean volumes and non-archimedean Monge-Ampre equations (with an appendix by Robert Lazarsfeld)*. `arXiv:1608.01919`. 2, 4, 58, 64, 69

[Ceg98] U. Cegrell. *Pluricomplex energy*. Acta Math. **180** (1998), 187–217. 3, 4, 53

[CL06] A. Chambert-Loir. *Mesures et équidistribution sur les espaces de Berkovich*. J. Reine Angew. Math. **595** (2006), 215–235. 18, 26, 27

[CD12] A. Chambert-Loir and A. Ducros. *Formes différentielles réelles et courants sur les espaces de Berkovich*. `arXiv:1204.6277`. 3, 6, 14, 18, 26, 27, 29, 38

[CDS15] X.X. Chen, S. K. Donaldson and S. Sun. *Kähler-Einstein metrics on Fano manifolds, I-III*. J. Amer. Math. Soc. **28** (2015), 183–197, 199–234, 235–278. 5

[Dem] J.-P. Demailly. *Complex analytic and differential geometry*. Book available at the author's web page `www-fourier.ujf-grenoble.fr/~demailly`

[Dem92] J.-P. Demailly. *Regularization of closed positive currents and intersection theory*. J. Algebraic Geom. **1** (1992), 361–409. 3

[Der15] R. Dervan. *Uniform stability of twisted constant scalar curvature Kähler metrics*. Int. Math. Res. Notices. Published online: October 14, 2015 5

[Din88] W.-Y. Ding. *Remarks on the existence problem of positive Kähler-Einstein metrics*. Math. Ann. **282** (1988), 463–471. 54

[DiN16] E. Di Nezza. *Finite pluricomplex energy measures*. Potential Anal. 44 (2016), 155–167. 66

[Don02] S.K. Donaldson. *Scalar curvature and stability of toric varieties.* J. Differential Geom. **62** (2002), no. 2, 289–349. 12

[ELS03] L. Ein, R. Lazarsfeld and K. Smith. *Uniform approximation of Abhyankar valuation ideals in smooth function fields*. Amer. J. Math. **125** (2003), 409–440. 36

[FJ04] C. Favre and M. Jonsson. *The valuative tree*. Lecture Notes in Mathematics, 1853. Springer-Verlag, Berlin, 2004. 37

[FJ05a] C. Favre and M. Jonsson. *Valuative analysis of planar plurisubharmonic functions*. Invent. Math. **162** (2005), 271–311. 37

[FJ05b] C. Favre and M. Jonsson. *Valuations and multiplier ideals*. J. Amer. Math. Soc, **18** (2005), 655–684. 37

[FJ07] C. Favre and M. Jonsson. *Eigenvaluations*. Ann. Sci. École Norm. Sup. **40** (2007), 309–349. 37

[FJ11] C. Favre and M. Jonsson. *Dynamical compactifications of* $\mathbf{C}^2$. Ann. of Math. **173** (2011), 211–249. 37

[Fol99] G. B. Folland. *Real analysis: modern techniques and their applications, second edition*. Pure and applied mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. 61, 69

[Fuj16b] K. Fujita. *A valuative criterion for uniform K-stability of* $\mathbf{Q}$*-Fano varieties*. `arXiv:1602.00901`. To appear in J. Reine Angew. Math. 5

[FO16] K. Fujita and Y. Odaka. *On the K-stability of Fano varieties and anticanonical divisors*. `arXiv:1602.01305v2`. To appear in Tohoku Math. J. 5

[Ful93] W. Fulton. *Introduction to toric varieties*. Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993.

[Gub98] W. Gubler. *Local heights of subvarieties over non-Archimedean fields*. J. Reine Angew. Math. **498** (1998), 61–113. 3, 18, 27, 42

[Gub08] W. Gubler. *Equidistribution over function fields*. Manuscripta Math. **127** (2008), 485–510. 26, 27

[GJKM17] W. Gubler, P. Jell, K. Künnemann and M. Sombra. *Continuity of plurisubharmonic envelopes in non-Archimedean geometry and test ideals (with an appendix by José Ignacio Burgos Gil and Martín Sombra)*. `arXiv:1712.00980`. 4, 6, 45, 64

[GK17] W. Gubler and K. Künnemann. *A tropical approach to non-archimedean Arakelov theory*. Algebra Number Theory **11** (2017), 77–180. 6, 26

[GM16] W. Gubler and F. Martin. *On Zhang's semipositive metrics*. arXiv:1608.08030. 4, 8, 21, 42, 43

[GZ05] V. Guedj and A. Zeriahi. *Intrinsic capacities on compact Kähler manifolds*. J. Geom. Anal. **15** (2005), 607–639. 53

[GZ07] V. Guedj and A. Zeriahi. *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*. J. Funct. Anal. **250** (2007), 442–482. 3, 4

[Jon12] M. Jonsson. *Dynamics on Berkovich spaces in low dimensions*. Berkovich spaces and applications, 205–366. Lecture Notes in Mathematics, 2119. Springer, 2015. 24

[JM12] M. Jonsson and M. Mustaţă. *Valuations and asymptotic invariants for sequences of ideals*. Ann. Inst. Fourier **62** (2012), 2145–2209. 5, 36, 37

[Koł98] S. Kołodziej. *The complex Monge-Ampère equation*. Acta Math. **180** (1998), 69–117. 4

[KT00] M. Kontsevich and Y. Tschinkel. *Non-Archimedean Kähler geometry*. Unpublished. 4

[Li15b] C. Li. *K-semistability is equivariant volume minimization*. arXiv:1512.07205v4. 5

[Mum70] D.. Mumford. *Abelian varieties*. Corrected reprint of the second (1974) edition. Tata Institute of Fundamental Research Studies in Mathematics, 5. 25

[Oda13] Y. Odaka. *A generalization of the Ross-Thomas slope theory*. Osaka J. Math. **50** (2013), no. 1, 171–185. 12

[Poi13] J. Poineau. *Les espaces de Berkovich sont angéliques*. Bull. Soc. Math. France **141** (2013), 267–297. 25

[Tem16] M. Temkin. *Metrization of differential pluriforms on Berkovich analytic spaces*. In *Tropical and Non-Archimedean Geometry*, 195–285. Simons Symposia. Springer International Publishing, 2016. 8

[Thu05] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications á la théorie d'Arakelov*. Ph.D. Thesis (2005), Université de Rennes I, available at http://tel.archives-ouvertes.fr/docs/00/04/87/50/PDF/tel-00010990.pdf. 24

[Tia15] G. Tian. *K-stability and Kähler-Einstein metrics*. Comm. Pure Appl. Math. **68** (2015), 1085–1156. 5

[Vaq00] M. Vaquié. *Valuations*. Resolution of singularities (Obergurgl, 1997), 539–590 Progr. Math., 181. Birkhaüser, Basel, 2000. 8

[Yau78] S. T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*. Comm. Pure Appl. Math. **31** (1978), 339–411. 4

[YZ16] X. Yuan and S.-W. Zhang. *The arithmetic Hodge Theorem for adelic line bundles*. Math. Ann. Published online: May 12, 2016. 4, 29

[Zha95] S.-W Zhang. *Positive line bundles on arithmetic varieties*. J. Amer. Math. Soc. **8** (1995), 187–221. 3, 18, 42, 43

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