ADDENDUM TO THE ARTICLE 'GLOBAL PLURIPOTENTIAL THEORY OVER A TRIVIALLY VALUED FIELD'

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Abstract. This note is an addendum to the paper 'Global pluripotential theory over a trivially valued field' by the present authors, in which we prove two results. Let X be an irreducible projective variety over an algebraically closed field field k, and assume that k has characteristic zero, or that X has dimension at most two. We first prove that when X is smooth, the envelope property holds for any numerical class on X. Then we prove that for X possibly singular and for an ample numerical class, the Monge–Ampère energy of a bounded function is equal to the energy of its usc regularized plurisubharmonic envelope.

Introduction

The purpose of this note is to strengthen two results in the article [BJ22a], where we developed global pluripotential on the Berkovich analytification over a trivially valued field. The results here are used in [BJ22b, BJ22c]. One should view the current note as an addendum to [BJ22a], rather than a stand-alone paper.

Let k be an algebraically closed field, and X an irreducible projective variety over k. To any numerical class $\theta \in \mathrm{N}^1(X)$ we associate a class $\mathrm{PSH}(\theta)$ of θ-psh functions; these are upper semicontinuous functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}\cup\{-\infty\}$ on the Berkovich analytification of X with respect to the trivial absolute value on k. We say that θ has the *envelope property* if for any bounded-above family $(\varphi_\alpha)_\alpha$ in $\mathrm{PSH}(\theta)$, the function $\sup^\star_\alpha \varphi_\alpha$ is θ-psh.

Theorem A. *Assume that X is smooth, and that* $\operatorname{char} k = 0$ *or* $\dim X \le 2$*. Then any numerical class $\theta \in \mathrm{N}^1(X)$ has the envelope property.*

In [BJ22a, Theorem 5.20], this was established for nef classes θ following [BFJ16], and the proof here is not so different.

For the second result we allow X to be singular, but work with an *ample* class $\omega \in \mathrm{N}^1(X)$. The ω-psh envelope $\mathrm{P}_\omega(\varphi)$ of a bounded function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}$ is defined as the supremum of all functions $\psi \in \mathrm{PSH}(\omega)$ with $\psi \le \varphi$, and the envelope property for ω is equivalent to *continuity of envelopes* in the sense of $\mathrm{P}_\omega(\varphi)$ being continuous whenever φ is continuous. It is also equivalent to the usc envelope $\mathrm{P}^\star_\omega(\varphi)$ being ω-psh for any bounded function φ.

In [BJ22a] we also defined the *Monge–Ampère energy* $\mathrm{E}_\omega(\varphi) \in \mathbb{R}\cup\{-\infty\}$ of any bounded-above function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}\cup\{-\infty\}$. We did this first for ω-psh functions in terms of an energy pairing ultimately deriving from intersection numbers on compactified test configurations, see §1.4 below, then for general bounded-above functions φ, setting

$$\mathrm{E}_\omega(\varphi) := \sup\{\mathrm{E}_\omega(\psi) \mid \psi \in \mathrm{PSH}(\omega), \psi \le \varphi\}.$$

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We say that (X,ω) satisfies the *weak envelope property* if there exists a projective birational morphism $\pi\colon \tilde{X}\to X$ and an ample class $\tilde{\omega}\in \mathrm{N}^1(\tilde{X})$ such that $(\tilde{X},\tilde{\omega})$ has the envelope property and $\tilde{\omega}\ge\pi^\star\omega$ (by which we mean $\tilde{\omega}-\pi^\star\omega$ is nef). It follows from [BJ22a, Theorem 5.20] that the weak envelope property holds when $\operatorname{char} k=0$ or $\dim X\le 2$.

Theorem B. *Assume that $\omega\in\mathrm{N}^1(X)$ is an ample class, and that the weak envelope property holds for (X,ω). Then, for any bounded function $\varphi\colon X^{\mathrm{an}}\to\mathbb{R}$, we have*

$$\mathrm{E}_\omega(\varphi)=\mathrm{E}_\omega(\mathrm{P}_\omega(\varphi))=\mathrm{E}_\omega(\mathrm{P}^\star_\omega(\varphi)).$$

The first equality is definitional, see [BJ22a, (8.2)], and the second equality follows from [BJ22a, Proposition 8.3] if ω has the envelope property. The main content of Theorem B is thus the second equality when the envelope property is unknown or even fails (for example, when X is not unibranch).

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1. Preliminaries

Throughout the paper, X is an irreducible projective variety over an algebraically closed field k.

1.1. The θ-psh envelope.

Fix any numerical class $\theta\in\mathrm{N}^1(X)$. We refer to [BJ22a, §4] for the definition of the class $\mathrm{PSH}(\theta)$ of θ-psh functions. We have that $\mathrm{PSH}(\theta)$ is nonempty only if θ is psef, whereas $\mathrm{PSH}(\theta)$ contains the constant functions iff θ is nef.

Definition 1.1. *The θ-*psh envelope *of a function $\varphi\colon X^{\mathrm{an}}\to\mathbb{R}\cup\{\pm\infty\}$ is the function $\mathrm{P}_\theta(\varphi)\colon X^{\mathrm{an}}\to\mathbb{R}\cup\{\pm\infty\}$ defined as the pointwise supremum*

$$\mathrm{P}_\theta(\varphi):=\sup\{\psi\in\mathrm{PSH}(\theta)\mid\psi\le\varphi\}\,.$$

Thus $\mathrm{P}_\theta(\varphi)\equiv-\infty$ iff there is no $\psi\in\mathrm{PSH}(\theta)$ with $\psi\le\varphi$. When $\theta=c_1(L)$ for a $\mathbb{Q}$-line bundle L, we write $\mathrm{P}_L:=\mathrm{P}_\theta$. Despite the name, $\mathrm{P}_\theta(\varphi)$ is not always θ-psh (and indeed not even usc in general). However, it is clear that

- $\varphi\mapsto\mathrm{P}_\theta(\varphi)$ is increasing;
- $\mathrm{P}_\theta(\varphi+c)=\mathrm{P}_\theta(\varphi)+c$ for all $c\in\mathbb{R}$.

The envelope operator is also continuous along increasing nets of lsc functions:

Lemma 1.2. *If $\varphi\colon X^{\mathrm{an}}\to\mathbb{R}\cup\{+\infty\}$ is the pointwise limit of an increasing net (φ_j) of bounded-below, lsc functions, then $\mathrm{P}_\theta(\varphi_j)\nearrow\mathrm{P}_\theta(\varphi)$ pointwise on X^{an}.*

Proof. We trivially have $\lim_j\mathrm{P}_\theta(\varphi_j)=\sup_j\mathrm{P}_\theta(\varphi_j)\le\mathrm{P}_\theta(\varphi)$. Pick $\varepsilon>0$ and $\psi\in\mathrm{PSH}(\theta)$ such that $\psi\le\varphi$, and hence $\psi<\varphi+\varepsilon$. Since ψ is usc and the φ_j is lsc, a simple variant of Dini's lemma shows that $\psi<\varphi_j+\varepsilon$ for all j large enough, and hence $\psi\le\mathrm{P}_\theta(\varphi_j)+\varepsilon$. Taking the supremum over ψ yields $\mathrm{P}_\theta(\varphi)\le\sup_j\mathrm{P}_\theta(\varphi_j)$, and we are done. □

As in [BE21, Lemma 7.30], the envelope property admits the following useful reformulation.

Lemma 1.3. *If $\mathrm{PSH}(\theta)\neq\emptyset$, then the following statements are equivalent:*

(i) *θ has the envelope property;*

(ii) *for any function* $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$*, we have*

$$\mathrm{P}_\theta(\varphi) \equiv -\infty,\ \mathrm{P}_\theta(\varphi)^\star \equiv +\infty,\ \textit{or}\ \mathrm{P}_\theta(\varphi)^\star \in \mathrm{PSH}(\theta);$$

(iii) $\varphi \in \mathrm{PL}(X) \Longrightarrow \mathrm{P}_\theta(\varphi) \in \mathrm{PSH}(\theta)$.

Proof. First assume (i). Pick any $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$, and suppose that the set $\mathcal{F} := \{\psi \in \mathrm{PSH}(\theta) \mid \psi \le \varphi\}$ is nonempty, so that $\mathrm{P}_\theta(\varphi) \not\equiv -\infty$. If the functions in $\mathcal{F}$ are uniformly bounded above, then $\mathrm{P}_\theta(\varphi)^\star \in \mathrm{PSH}(\theta)$, by (i). If not, choose $\omega \in \mathrm{Amp}(X)$ with $\omega \ge \theta$, and hence $\mathcal{F} \subset \mathrm{PSH}(\omega)$. By the definition of the Alexander–Taylor capacity, see [BJ22a, §4.6], we then have

$$\mathrm{P}_\theta(\varphi)(v) = \sup\{\psi(v) \mid \psi \in \mathcal{F}\} \ge \sup\{\sup\psi \mid \psi \in \mathcal{F}\} - \mathrm{T}_\omega(v) = +\infty$$

for all $v \in X^{\mathrm{div}}$, and hence $\mathrm{P}_\theta(\varphi)^\star \equiv +\infty$, by density of X^{div}. This proves (i)⇒(ii).

Next we prove (ii)⇒(iii), so pick $\varphi \in \mathrm{PL}(X)$. Since φ is bounded and $\mathrm{PSH}(\theta)$ is nonempty and invariant under addition of constants, we have $\mathrm{P}_\theta(\varphi) \not\equiv -\infty$. Now $\mathrm{P}_\theta(\varphi) \le \varphi$ implies $\mathrm{P}_\theta(\varphi)^\star \le \varphi$ since φ is usc. In particular, $\mathrm{P}_\theta(\varphi)^\star \not\equiv +\infty$, so $\mathrm{P}_\theta(\varphi)^\star \in \mathrm{PSH}(\theta)$ by (ii). Thus $\mathrm{P}_\theta(\varphi)^\star$ is a competitor in the definition of $\mathrm{P}_\theta(\varphi)$, so $\mathrm{P}_\theta(\varphi) = \mathrm{P}_\theta(\varphi)^\star$ is θ-psh.

Finally, we prove (iii)⇒(i), following [BE21, Lemma 7.30]. Let (φ_i) be a bounded-above family in $\mathrm{PSH}(\theta)$, and set $\varphi := \sup_i^\star \varphi_i$. Since φ is usc and X^{an} is compact, we can find a decreasing net (ψ_j) in $\mathrm{C}^0(X)$ such that $\psi_j \to \varphi$. By density of $\mathrm{PL}(X)$ in $\mathrm{C}^0(X)$ wrt uniform convergence (see [BJ22a, Theorem 2.2]), we can in fact assume $\psi_j \in \mathrm{PL}(X)$, and hence $\mathrm{P}_\theta(\psi_j) \in \mathrm{PSH}(\theta)$, by (iii). For all i, j, we have $\varphi_i \le \psi_j$, and hence $\varphi_i \le \mathrm{P}_\theta(\psi_j)$, which in turn yields $\varphi \le \mathrm{P}_\theta(\psi_j) \le \psi_j$. We have thus written φ as the limit of the decreasing net of θ-psh functions $\mathrm{P}_\theta(\psi_j)$, which shows that φ is θ-psh. □

Corollary 1.4. *Assume that* θ *has the envelope property, and consider a usc function* $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}\cup\{-\infty\}$*. Then:*

(i) $\mathrm{P}_\theta(\varphi)$ *is* θ*-psh, or* $\mathrm{P}_\theta(\varphi) \equiv -\infty$*;*

(ii) *if* φ *is the limit of a decreasing net* (φ_j) *of bounded-above, usc functions, then* $\mathrm{P}_\theta(\varphi_j) \searrow \mathrm{P}_\theta(\varphi)$.

Proof. By Lemma 1.3, either $\psi := \mathrm{P}_\theta(\varphi)^\star$ is θ-psh, or $\mathrm{P}_\theta(\varphi) \equiv -\infty$ (the latter being automatic if $\mathrm{PSH}(\theta) = \emptyset$). Since $\mathrm{P}_\theta(\varphi) \le \varphi$ and φ is usc, we also have $\psi \le \varphi$. If ψ is θ-psh, then $\psi \le \mathrm{P}_\theta(\varphi)$, which proves (i).

To see (ii), note that $\rho := \lim_j \mathrm{P}_\theta(\varphi_j)$ satisfies either $\rho \in \mathrm{PSH}(\theta)$ or $\rho \equiv -\infty$, by [BJ22a, Theorem 4.7]. Furthermore, $\mathrm{P}_\theta(\varphi_j) \le \varphi_j$ yields, in the limit, $\rho \le \varphi$, and hence $\rho \le \mathrm{P}_\theta(\varphi)$ (by definition of $\mathrm{P}_\theta(\varphi)$ if $\rho \in \mathrm{PSH}(\theta)$, and trivially if $\rho \equiv -\infty$). Thus $\lim_j \mathrm{P}_\theta(\varphi_j) = \rho = \mathrm{P}_\theta(\varphi)$. On the other hand, $\mathrm{P}_\theta(\varphi_j) \ge \mathrm{P}_\theta(\varphi)$ implies $\rho \ge \mathrm{P}_\theta(\varphi)$, which completes the proof of (ii). □

1.2. **The Fubini–Study envelope.** Now consider a big $\mathbb{Q}$-line bundle L. Recall [BJ22a, §2.4] that for any subgroup $\Lambda \subset \mathbb{R}$, $\mathcal{H}^{\mathrm{gf}}_\Lambda(L)$ denotes the set of functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}$ of the form

$$\varphi = m^{-1}\max_j\{\log|s_j| + \lambda_j\},$$

where $m \in \mathbb{Z}_{>0}$ is such that mL is an honest line bundle, $(s_j)_j$ is a finite set of nonzero global sections of mL, and $\lambda_j \in \Lambda$.

We define the *Fubini–Study envelope* of a bounded function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}$ as

$$\mathrm{Q}_L(\varphi) := \sup\left\{\psi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{R}}(L) \mid \psi \le \varphi\right\}. \tag{1.1}$$

By approximation, $\mathcal{H}^{\mathrm{gf}}_{\mathbb{R}}(L)$ can be replaced by $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L) = \mathcal{H}^{\mathrm{gf}}_{\mathbb{Z}}(L)$ in this definition, see [BJ22a, (2.10)]. Note also that $\mathrm{Q}_L(\varphi)\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ is bounded above and lsc.

Recall that the *augmented base locus* of L can be described as

$$\mathbb{B}_+(L) := \bigcap \{\operatorname{supp} E \mid E \text{ effective } \mathbb{Q}\text{-Cartier divisor, } L - E \text{ ample}\},$$

a strict Zariski closed subset of X, see [ELMNP06].

Lemma 1.5. *Suppose $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}$ is bounded, with lsc regularization $\varphi_\star\colon X^{\mathrm{an}} \to \mathbb{R}$. Then $\mathrm{Q}_L(\varphi) = \mathrm{Q}_L(\varphi_\star) \le \mathrm{P}_L(\varphi_\star)$, and equality holds outside $\mathbb{B}_+(L)$.*

In particular, $\mathrm{Q}_L(\varphi) = \mathrm{P}_L(\varphi_\star)$ when L is ample. In this case, Q_L coincides with the envelope $\mathrm{Q}_{c_1(L)}$ in [BJ22a, §5.3].

Proof. Since any function $\psi \in \mathcal{H}^{\mathrm{gf}}(L)$ is continuous, it satisfies $\psi \le \varphi$ iff $\psi \le \varphi_\star$. Thus $\mathrm{Q}_L(\varphi) = \mathrm{Q}_L(\varphi_\star)$, and we may therefore assume wlog that φ is lsc. Since $\mathcal{H}^{\mathrm{gf}}(L) \subset \mathrm{PSH}(L)$, we trivially have $\mathrm{Q}_L(\varphi) \le \mathrm{P}_L(\varphi)$. Conversely, pick $\psi \in \mathrm{PSH}(L)$ such that $\psi \le \varphi$. Let E be an effective $\mathbb{Q}$-Cartier divisor such that $A := L - E$ is ample. By [BJ22a, Theorem 4.15], we can write ψ as the pointwise limit of a decreasing net (ψ_j) in $\mathcal{H}^{\mathrm{gf}}(L + \varepsilon_j A)$ with $\varepsilon_j \to 0$. Pick $\varepsilon > 0$, so that $\psi < \varphi + \varepsilon$. As in the proof of Lemma 1.2, since ψ_j is usc and φ is lsc, a simple variant of Dini's lemma shows that $\psi_j < \varphi + \varepsilon$ for all j large enough. Since $\log |s_E| \le 0$ lies in $\mathcal{H}^{\mathrm{gf}}(E)$, it follows that $\tau_j := (1 + \varepsilon_j)^{-1}(\psi_j + \varepsilon_j \log |s_E|)$ lies in $\mathcal{H}^{\mathrm{gf}}(L)$. Further,

$$\tau_j \le (1 + \varepsilon_j)^{-1}(\varphi + \varepsilon) \le \varphi + \varepsilon + C\varepsilon_j$$

for some uniform $C > 0$, since φ is bounded, and hence

$$\tau_j \le \mathrm{Q}_L(\varphi + \varepsilon + C\varepsilon_j) = \mathrm{Q}_L(\varphi) + \varepsilon + C\varepsilon_j.$$

We have thus proved $\psi_j + \varepsilon_j \log |s_E| \le (1 + \varepsilon_j)(\mathrm{Q}_L(\varphi) + \varepsilon + C\varepsilon_j)$; at any point of

$$(X - E)^{\mathrm{an}} = \{\log |s_E| > -\infty\},$$

this yields $\psi \le \mathrm{Q}_L(\varphi)$, and hence $\mathrm{P}_L(\varphi) \le \mathrm{Q}_L(\varphi)$, which proves the result. ☐

1.3. **Envelopes from test configurations.** Let L be a big line bundle. Any test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) defines a function $\varphi_{\mathcal{L}} \in \mathrm{PL}$, and we seek to compute the Fubini–Study envelope $\mathrm{Q}_L(\varphi_{\mathcal{L}})$.

To this end, we introduce a slight generalization of the definitions in [BJ22a, §2.1]. To any $\mathbb{G}_{\mathrm{m}}$-invariant ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}$, we attach a function $\varphi_{\mathfrak{a}}\colon X^{\mathrm{an}} \to [-\infty, 0]$ by setting $\varphi_{\mathfrak{a}}(v) := -\sigma(v)(\mathfrak{a})$, where $\sigma = \sigma_{\mathcal{X}}$ denotes Gauss extension (see [BJ22a, Remark 1.9]). In terms of the weight decomposition $\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}_{\ge 0}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ with $\mathfrak{a}_\lambda \subset \mathcal{O}_X$, we have $\varphi_{\mathfrak{a}} = \max_\lambda \{\log |\mathfrak{a}_\lambda| + \lambda\}$. If $\mathcal{L}$ is an honest line bundle such that $\mathcal{L} \otimes \mathfrak{a}$ is globally generated, one easily checks as in [BJ22a, Proposition 2.25] that $\varphi_{\mathcal{L}} + \varphi_{\mathfrak{a}}$ lies in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$.

Lemma 1.6. *Let L be a big line bundle on X, and $(\mathcal{X}, \mathcal{L})$ an integrally closed test configuration for (X, L). For each sufficiently divisible $m \in \mathbb{Z}_{>0}$, denote by $\mathfrak{a}_m \subset \mathcal{O}_{\mathcal{X}}$ the base ideal of $m\mathcal{L}$, and set $\varphi_m := \varphi_{\mathcal{L}} + m^{-1}\varphi_{\mathfrak{a}_m}$. Then $\varphi_m \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ and $(\varphi_m)_m$ forms an increasing net of functions on X^{an} converging pointwise to $\mathrm{Q}_L(\varphi_{\mathcal{L}})$.*

Here we consider $(\varphi_m)_m$ as a net indexed by the set $m_0\mathbb{Z}_{>0}$ for some sufficiently divisible m_0, and partially ordered by divisibility.

To prove the lemma, recall [BJ22a, §1.2] that if $\mathcal{L}$ (and hence L) is an honest line bundle, then $\mathrm{H}^0(\mathcal{X},\mathcal{L})$ lies as a $k[\varpi]$-submodule of $\mathrm{H}^0(X,L)_{k[\varpi^{\pm 1}]}$. The next result provides a valuative characterization of this submodule in terms of $\varphi_{\mathcal{L}}$.

Lemma 1.7. *Assume $\mathcal{L}$ is an honest line bundle, pick $s \in \mathrm{H}^0(X,L)_{k[\varpi^{-\pm 1}]}$, and write $s = \sum_{\lambda\in\mathbb{Z}} s_\lambda \varpi^{-\lambda}$ with $s_\lambda \in \mathrm{H}^0(X,L)$. Then $s \in \mathrm{H}^0(\mathcal{X},\mathcal{L})$ iff $\max_\lambda\{\log|s_\lambda| + \lambda\} \le \varphi_{\mathcal{L}}$ on X^{an}.*

Proof. By $\mathbb{G}_{\mathrm{m}}$-invariance, we have $s \in \mathrm{H}^0(\mathcal{X},\mathcal{L}) \Leftrightarrow s_\lambda\varpi^{-\lambda} \in \mathrm{H}^0(\mathcal{X},\mathcal{L})$ for all $\lambda \in \mathbb{Z}$, and we may thus assume $s = s_\lambda\varpi^{-\lambda}$ for some $\lambda \in \mathbb{Z}$.

Since $\mathcal{X}$ is integrally closed, we have $\rho_\star\mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$, and hence $\mathrm{H}^0(\mathcal{X}',\rho^\star\mathcal{L}) = \mathrm{H}^0(\mathcal{X},\mathcal{L})$, for any higher test configuration $\rho\colon \mathcal{X}' \to \mathcal{X}$ (see the proof of [BJ22a, Proposition 2.30]). After pulling back $\mathcal{L}$ to a higher test configuration, we may thus assume that $\mathcal{X}$ dominates the trivial test configuration via $\mu\colon \mathcal{X} \to \mathcal{X}_{\mathrm{triv}}$. Set $D := \mathcal{L} - \mu^\star\mathcal{L}_{\mathrm{triv}}$, so that $\varphi_{\mathcal{L}} = \varphi_D$. Viewed as a rational section of $\mathcal{L}$, s is regular outside $\mathcal{X}_0$. For any $v \in X^{\mathrm{an}}$ with Gauss extension $w = \sigma(v)$, we further have

$$w(s) = v(s_\lambda) - \lambda + w(D) = -\log|s_\lambda|(v) - \lambda + \varphi_D(v).$$

If s is a regular section, then $w(s) \ge 0$, and hence $\log|s_\lambda|(v) + \lambda \le \varphi_D(v)$ for any $v \in X^{\mathrm{an}}$. Conversely, the latter condition implies $b_E^{-1}\operatorname{ord}_E(s) = -\log|s_\lambda|(v_E) - \lambda + \varphi_D(v_E) \ge 0$ for each irreducible component E of $\mathcal{X}_0$, since $\sigma(v_E) = b_E^{-1}\operatorname{ord}_E$; this yields, as desired, $s \in \mathrm{H}^0(\mathcal{X},\mathcal{L})$ (compare [BJ22a, Lemma 1.23]). □

Proof of Lemma 1.7. Replacing L and $\mathcal{L}$ by sufficiently divisible multiples, we may assume that L and $\mathcal{L}$ are honest line bundles.

We have $\mathfrak{a}_m\cdot\mathfrak{a}_{m'} \subset \mathfrak{a}_{m+m'}$ for all $m, m' \in \mathbb{N}$. This implies that the net $(\varphi_m)_m$ is increasing.

By definition of $\mathfrak{a}_m$, $m\mathcal{L}\otimes\mathfrak{a}_m$ is globally generated. As noted above, this implies $\varphi_{\mathcal{L}}+\varphi_{\mathfrak{a}_m} \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(mL)$, and hence $\varphi_m \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$. Since $\varphi_{\mathfrak{a}_m} \le 0$, we further have $\varphi_m \le \varphi_{\mathcal{L}}$, and hence $\varphi_m \le \mathrm{Q}_L(\varphi_{\mathcal{L}})$, see (1.1).

Conversely, pick $\psi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ such that $\psi \le \varphi_{\mathcal{L}}$, and write $\psi = \frac{1}{m}\max_i\{\log|s_i| + \lambda_i\}$ for a finite set of nonzero sections $s_i \in \mathrm{H}^0(X,mL)$ and $\lambda_i \in \mathbb{Z}$. For each i, we then have $\log|s_i| + \lambda_i \le m\varphi_{\mathcal{L}} = \varphi_{m\mathcal{L}}$, and hence $s_i\varpi^{-\lambda_i} \in \mathrm{H}^0(\mathcal{X},m\mathcal{L})$, see Lemma 1.7. Since $\mathfrak{a}_m$ is locally generated by $\mathrm{H}^0(\mathcal{X},m\mathcal{L})$, this implies in turn $\log|s_i| + \lambda_i \le \varphi_{m\mathcal{L}} + \varphi_{\mathfrak{a}_m}$, and hence $\psi \le \varphi_m$. Taking the supremum over ψ, we conclude, as desired, $\mathrm{Q}_L(\varphi_{\mathcal{L}}) \le \sup_m \varphi_m$. □

1.4. **The energy pairing.** Various incarnations of the energy pairing play a key role in [BJ22a]. First of all, when $\theta_0,\dots,\theta_n \in \mathrm{N}^1(X)$ are arbitrary numerical classes and $\varphi_0,\dots,\varphi_n \in \mathrm{PL}(X)_{\mathbb{R}}$ are ($\mathbb{R}$-linear combinations of) PL functions, then

$$(\theta_0,\varphi_0)\cdot\ldots\cdot(\theta_n,\varphi_n) \in \mathbb{R}$$

is defined as an intersection number on a compactified test configuration for X, see [BJ22a, §3.2]. The following result would naturally belong to [BJ22a, Proposition 3.14].

Lemma 1.8. *Let $\pi\colon Y \to X$ be a projective birational morphism, $\theta_0,\dots,\theta_n \in \mathrm{N}^1(X)$ numerical classes, and $\varphi_0,\dots,\varphi_n \in \mathrm{PL}(X)$ PL functions. Then*

$$(\theta_0,\varphi_0)\cdot\ldots\cdot(\theta_n,\varphi_n) = (\pi^\star\theta_0,\pi^\star\varphi_0)\cdot\ldots\cdot(\pi^\star\theta_n,\pi^\star\varphi_n).$$

Remark 1.9. *While we are assuming that X and Y are irreducible, the result holds even without this assumption, as in* [BJ22a, Proposition 3.14].

Proof. There exists a test configuration $\mathcal{X}$ for X that dominates $\mathcal{X}_{\rm triv} = X \times \mathbb{A}^1$, and vertical $\mathbb{Q}$-Cartier divisor $D_i \in \mathrm{VCar}(\mathcal{X})_{\mathbb{Q}}$ that determine the functions φ_i, $0 \le i \le n$. Then

$$(\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n) = (\theta_{0,\bar{\mathcal{X}}} + D_0) \cdot \ldots \cdot (\theta_{n,\bar{\mathcal{X}}} + D_n),$$

where the intersection number is computed on the canonical compactification $\bar{\mathcal{X}} \to \mathbb{P}^1$ and $\theta_{i,\mathcal{X}} \in \mathrm{N}^1(\bar{\mathcal{X}})$ denotes the pullback of θ_i. The canonical birational map $\mathcal{Y}_{\rm triv} = Y \times \mathbb{A}^1 \dashrightarrow \mathcal{X}$ being $\mathbb{G}_{\rm m}$-equivariant, we can choose a test configuration $\mathcal{Y}$ for Y that dominates $\mathcal{Y}_{\rm triv}$ such that $\pi \colon Y \to X$ extends to a $\mathbb{G}_{\rm m}$-equivariant morphism $\pi \colon \mathcal{Y} \to \mathcal{X}$. Then $\pi^\star \varphi_{D_i} = \varphi_{\pi^\star D_i}$ for all i, and we have

$$\begin{aligned}(\pi^\star\theta_0, \pi^\star\varphi_0) \cdot \ldots \cdot (\pi^\star\theta_n, \pi^\star\varphi_n) &= (\pi^\star\theta_{0,\bar{\mathcal{X}}} + \pi^\star D_0) \cdot \ldots \cdot (\pi^\star\theta_{n,\bar{\mathcal{X}}} + \pi^\star D_n) \\ &= (\theta_{0,\bar{\mathcal{X}}} + D_0) \cdot \ldots \cdot (\theta_{n,\bar{\mathcal{X}}} + D_n) = (\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n),\end{aligned}$$

where the second equality follows from the projection formula. ☐

In [BJ22a, §7], the energy pairing was extended in various ways. First, one can define

$$(\omega_0, \varphi_0) \cdot \ldots \cdot (\omega_n, \varphi_n) \in \mathbb{R} \cup \{-\infty\}$$

for $\omega_i \in \mathrm{Amp}(X)$ and $\varphi_i \in \mathrm{PSH}(\omega_i)$ by approximation from above by functions in $\mathrm{PSH}(\omega_i) \cap \mathrm{PL}(X)$. Given $\omega \in \mathrm{Amp}(X)$, a function $\varphi \in \mathrm{PSH}(\omega)$ has *finite energy* if $(\omega, \varphi)^{n+1} > -\infty$, and the set of such functions is denoted by $\mathcal{E}^1(\omega)$. If $\varphi \in \mathrm{PSH}(\omega)$, we set

$$\mathrm{E}_\omega(\varphi) := \frac{(\omega, \varphi)^{n+1}}{(n+1)(\omega^n)}.$$

The functional E_ω is increasing and satisfies $\mathrm{E}_\omega(\varphi + c) = \mathrm{E}_\omega(\varphi) + c$ for any $\varphi \in \mathrm{PSH}(\omega)$ and $c \in \mathbb{R}$. We have $(\omega_0, \varphi_0) \cdot \ldots \cdot (\omega_n, \varphi_n) > -\infty$ for any $\omega_i \in \mathrm{Amp}(X)$ and $\varphi_i \in \mathcal{E}^1(\omega_i)$.

For a general bounded-above function $\varphi \colon X^{\rm an} \to \mathbb{R} \cup \{-\infty\}$ we set

$$\mathrm{E}_\omega(\varphi) := \sup\{\mathrm{E}_\omega(\psi) \mid \psi \in \mathrm{PSH}(\omega), \psi \le \varphi\}.$$

Then $\mathrm{E}_\omega(\varphi) = \mathrm{E}_\omega(\mathrm{P}_\omega(\varphi))$ for any bounded-above function φ.

A function $\varphi \colon X^{\rm lin} \to \mathbb{R}$ is said to be of finite energy if it is of the form $\varphi = \varphi^+ - \varphi^-$, where $\varphi^\pm \in \mathcal{E}^1(\omega)$ for some $\omega \in \mathrm{Amp}(X)$. The energy pairing then extends as a (finite) multilinear pairing $(\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n)$ for arbitrary numerical classes $\theta_i \in \mathrm{N}^1(X)$ and functions φ_i of finite energy.

2. Theorem A

We now prove Theorem A and derive some consequences.

2.1. **Proof of Theorem A.** The result is trivial if θ is not pseudoeffective, as $\mathrm{PSH}(\theta)$ is then empty. Otherwise, we can write $\theta = \lim_i c_1(L_i)$ for a sequence of big $\mathbb{Q}$-line bundles L_i with $c_1(L_i) \ge \theta$; by [BJ22a, Lemma 5.9], we may thus assume that $\theta = c_1(L)$ for a big $\mathbb{Q}$-line bundle L. Pick $\varphi \in \mathrm{PL}(X)$. By Lemma 1.3, we need to show that $\mathrm{P}_L(\varphi)$ is L-psh. By [BJ22a, Theorem 2.31], we have $\varphi = \varphi_{\mathcal{L}}$ for some integrally closed test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L). After replacing L with a multiple, we may further assume that L and $\mathcal{L}$ are honest line bundles.

Since we assume that $\operatorname{char} k = 0$ or $\dim X \le 2$ (and hence $\dim \mathcal{X} \le 3$), we can rely on resolution of singularities and assume that $\mathcal{X}$ is smooth and $\mathcal{X}_0$ has simple normal crossings support. Assume first that $\operatorname{char} k = 0$, and let $\mathfrak{b}_m$ be the multiplier ideal of the graded sequence $\mathfrak{a}_\bullet^m$. The inclusion $\mathfrak{a}_m \subset \mathfrak{b}_m$ is elementary, and we have $\mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for all m, l by the subadditivity property of multiplier ideals. This implies that

$$(ml)^{-1}\varphi_{\mathfrak{a}_{ml}} \le (ml)^{-1}\varphi_{\mathfrak{b}_{ml}} \le m^{-1}\varphi_{\mathfrak{b}_m}$$

for all m and l. Letting $l \to \infty$ shows that

$$\varphi_m \le \mathrm{Q}_L(\varphi_{\mathcal{L}}) \le \psi_m := \varphi_{\mathcal{L}} + m^{-1}\varphi_{\mathfrak{b}_m} \tag{2.1}$$

for all m, by Lemma 1.6. By the uniform global generation property of multiplier ideals, we can find a $\mathbb{G}_{\mathrm{m}}$-equivariant ample line bundle $\mathcal{A}$ on $\mathcal{X}$ such that $\mathcal{O}_{\mathcal{X}}(m\mathcal{L}+\mathcal{A}) \otimes \mathfrak{b}_m$ is globally generated for all m. As noted before Lemma 1.6, this implies $\varphi_{m\mathcal{L}+\mathcal{A}} + \varphi_{\mathfrak{b}_m} \in \mathcal{H}^{\mathrm{gf}}(mL+A)$, with $A \in \mathrm{Pic}(X)$ the restriction of $\mathcal{A}$, and hence

$$\psi'_m := \psi_m + \tfrac{1}{m}\varphi_{\mathcal{A}} \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L + \tfrac{1}{m}A).$$

After adding to $\mathcal{A}$ a multiple of $\mathcal{X}_0$, we may further assume $\varphi_{\mathcal{A}} \ge 0$, which guarantees that the net (ψ'_m) is decreasing with respect to the divisibility order, and hence that $\psi := \inf_m \psi'_m$ is either L-psh or identically $-\infty$ (see [BJ22a, Theorem 4.5]). By (2.1), we have

$$\mathrm{Q}_L(\varphi_{\mathcal{L}}) \le \psi'_m \le \varphi_{\mathcal{L}} + \tfrac{1}{m}\varphi_{\mathcal{A}},$$

and hence $\mathrm{Q}_L(\varphi_{\mathcal{L}}) \le \psi \le \varphi_{\mathcal{L}}$. In particular, $\psi \not\equiv -\infty$, so $\psi \in \mathrm{PSH}(L)$, and hence $\psi \le \mathrm{P}_L(\varphi_{\mathcal{L}})$. Finally, pick $\tau \in \mathrm{PSH}(L)$ such that $\tau \le \varphi_{\mathcal{L}}$. By Lemma 1.5, we have $\tau \le \mathrm{P}_L(\varphi_{\mathcal{L}}) = \mathrm{Q}_L(\varphi_{\mathcal{L}}) \le \psi$ on a Zariski open subset of X^{an}, and hence on X^{div}. Since τ and ψ are L-psh, it follows from [BJ22a, Theorem 4.22] that $\tau \le \psi$ on X^{an}. Taking the sup over τ yields $\mathrm{P}_L(\varphi_{\mathcal{L}}) \le \psi$, and we conclude, as desired, that $\mathrm{P}_L(\varphi_{\mathcal{L}}) = \psi$ is L-psh.

When $\operatorname{char} k > 0$, the very same argument applies with test ideals in place of multiplier ideals, see [GJKM19] for details.

2.2. **Consequences.** We now list some consequences of Theorem A. First, we can characterize psef classes, similarly to the complex analytic case.

Corollary 2.1. *Assume that X satisfies the assumptions in Theorem A. Then, for any $\theta \in \mathrm{N}^1(X)$, we have $\mathrm{PSH}(\theta) \ne \emptyset$ iff θ is psef. Moreover, in this case, the function*

$$V_\theta := \mathrm{P}_\theta(0)$$

is θ-psh.

Proof. It follows from [BJ22a, Definition 4.1] that $\mathrm{PSH}(\theta) \ne \emptyset$ only if θ is psef. First suppose θ is big. By Theorem A, $V_\theta := \mathrm{P}_\theta(0)$ is θ-psh. Note that $V_\theta(v_{\mathrm{triv}}) = \sup V_\theta = 0$, where v_{triv} is the trivial valuation on X.

Now suppose θ is merely psef, and pick a sequence $(\theta_m)_1^\infty$ of big classes converging to θ, such that $\theta \le \theta_{m+1} \le \theta_m$ for all m. As $\mathrm{PSH}(\theta_{m+1}) \subset \mathrm{PSH}(\theta_m)$ for all m, the sequence $(V_{\theta_m})_m$ is pointwise decreasing on X^{an}. Let φ be its limit. We have $\sup\varphi = \varphi(v_{\mathrm{triv}}) = 0$, and $\varphi \in \mathrm{PSH}(\theta_m)$ for every m. It now follows from [BJ22a, Theorem 4.5] that $\varphi \in \mathrm{PSH}(\theta)$. Finally, it is easy to see that $\varphi = \mathrm{P}_\theta(0)$. Indeed, $\varphi \le 0$, and if $\psi \in \mathrm{PSH}(\theta)$ satisfies $\psi \le 0$, then $\psi \in \mathrm{PSH}(\theta_m)$ for all m, so $\psi \le V_{\theta_m}$, and hence $\psi \le \varphi$. □

By [BJ22a, Theorem 5.11], Theorem A now implies the following compactness result.

Corollary 2.2. *Under the assumptions on X of Theorem A, the set*

$$\mathrm{PSH}_{\sup}(\theta) = \{\varphi \in \mathrm{PSH}(\theta) \mid \sup\varphi = 0\}$$

is compact for any psef class $\theta \in \mathrm{N}^1(X)$.

Finally, as an immediate consequence of Theorem A and [BJ22a, Theorem 6.31], we have the following version of Siu's decomposition theorem.

Corollary 2.3. *Suppose that X satisfies the assumptions of Theorem A. Pick $\theta \in \mathrm{N}^1(X)$ and an effective $\mathbb{Q}$-Cartier divisor E. Then, for any $\varphi \in \mathrm{PSH}(\theta)$, we have:*

$$\varphi \le \log|s_E| + O(1) \Longleftrightarrow \varphi - \log|s_E| \in \mathrm{PSH}(\theta - E).$$

Here $\log|s_E| = m^{-1}\log|s_{mE}|$, where s_{mE} is the canonical global section of $\mathcal{O}_X(mE)$ for any $m \ge 1$ such that mE is integral.

3. Proof of Theorem B

We start by proving:

Lemma 3.1. *Let $\pi\colon \tilde{X} \to X$ be a projective birational morphism, and pick a bounded ω-psh function ψ. Then $(\omega,\psi)^{n+1} = (\pi^\star\omega, \pi^\star\psi)^{n+1}$.*

Here $\pi^\star\omega$ may not be ample, but the right hand side is well-defined, as $\pi^\star\psi$ is a function of finite energy. In fact $\pi^\star\psi \in \mathcal{E}^1(\tilde{\omega})$ for any ample class $\tilde{\omega} \ge \pi^\star\omega$.

Proof. The case when $\psi \in \mathrm{PL}(X)$ follows from Lemma 1.8. In the general case, write ψ as the pointwise limit of a decreasing net (ψ_j) in $\mathrm{PL} \cap \mathrm{PSH}(\omega)$, and pick $\tilde{\omega} \in \mathrm{Amp}(\tilde{X})$ such that $\tilde{\omega} \ge \pi^\star\omega$. Then $\pi^\star\psi_j$ decreases to $\pi^\star\psi$ pointwise on $\tilde{X}^{\mathrm{an}}$. Moreover, $\pi^\star\psi_j$ and $\pi^\star\psi$ are $\tilde{\omega}$-psh, and hence lie in $\mathcal{E}^1(\tilde{\omega})$ as they are bounded. By [BJ22a, Theorem 7.14 (iii)] we have $(\omega,\psi_j)^{n+1} \to (\omega,\psi)^{n+1}$ and $(\pi^\star\omega,\pi^\star\psi_j)^{n+1} \to (\pi^\star\omega,\pi^\star\psi)^{n+1}$. Now $(\pi^\star\omega,\pi^\star\psi_j)^{n+1} = (\omega,\psi_j)^{n+1}$ for all j by the PL case, and the result follows. □

As stated in the introduction, we introduce:

Definition 3.2. *Let X be a projective variety, and $\omega \in \mathrm{N}^1(X)$ an ample class. We say that (X,ω) has the weak envelope property if there exists a projective birational morphism $\pi\colon \tilde{X} \to X$, and an ample class $\tilde{\omega} \in \mathrm{N}^1(\tilde{X})$, such that $\tilde{\omega} \ge \pi^\star\omega$ and $(\tilde{X},\tilde{\omega})$ has the envelope property.*

Lemma 3.3. *If $\mathrm{char}\, k = 0$ or $\dim X \le 2$, then any ample class $\omega \in \mathrm{N}^1(X)$ has the weak envelope property.*

Proof. In both cases, we can pick $\pi\colon \tilde{X} \to X$ as a resolution of singularities, and then pick any ample class $\tilde{\omega} \ge \pi^\star\omega$. By [BJ22a, Theorem 5.20] (or Theorem A), the envelope property holds for $(\tilde{X},\tilde{\omega})$, and we are done. □

Proof of Theorem B. Set $\tau := \mathrm{P}_\omega(\varphi)$. For any $\psi \in \mathrm{PSH}(\omega)$, we have $\psi \le \varphi \Longleftrightarrow \psi \le \tau$, and hence $\mathrm{E}_\omega(\varphi) = \mathrm{E}_\omega(\tau) \le \mathrm{E}_\omega(\tau^\star)$. Since τ is the pointwise supremum of the family $\mathcal{F} = \{\psi \in \mathrm{PSH}(\omega) \mid \psi \le \varphi\}$, and since $\mathcal{F}$ is stable under finite max, we can find an increasing net (ψ_i) of ω-psh functions such that $\sup_i \psi_i = \tau$ pointwise on X^{an}. Replacing ψ_i with $\max\{\psi_i, \inf\psi\}$, we can further assume that ψ_i is bounded.

By assumption, we can find a projective birational morphism $\pi\colon \tilde{X} \to X$, and an ample class $\tilde{\omega} \in \mathrm{N}^1(\tilde{X})$ such that $\tilde{\omega} \ge \pi^\star\omega$ and $(\tilde{X},\tilde{\omega})$ has the envelope property. Now $\tilde{\tau} := \pi^\star\tau = \sup_i \pi^\star\psi_i$ with $\pi^\star\psi_i \in \mathrm{PSH}(\tilde{\omega})$, and it follows that $\tilde{\tau}^\star$ is $\tilde{\omega}$-psh, and coincides with $\tilde{\tau} = \sup_i \pi^\star\psi_i = \lim_i \sup \pi^\star\psi_i$ on $\tilde{X}^{\mathrm{div}}$. By [BJ22a, Theorem 7.38], we get $(\pi^\star\omega, \pi^\star\psi_i)^{n+1} \to (\pi^\star\omega, \tilde{\tau}^\star)^{n+1}$. On the other hand, Lemma 3.1 yields

$$(\pi^\star\omega, \pi^\star\psi_i)^{n+1} = (\omega,\psi_i)^{n+1} = (n+1)\,\mathrm{vol}(\omega)\,\mathrm{E}_\omega(\psi_i) \le (n+1)\,\mathrm{vol}(\omega)\,\mathrm{E}_\omega(\tau),$$

and we infer

$$(\pi^\star\omega, \tilde{\tau}^\star)^{n+1} \le (n+1)\,\mathrm{vol}(\omega)\,\mathrm{E}_\omega(\tau). \tag{3.1}$$

By [BJ22a, Theorem 5.6] we also have $\tau^\star = \tau$ on X^{div}. Each $\psi \in \mathrm{PSH}(\omega)$ such that $\psi \le \tau^\star$ on X^{an} therefore satisfies $\psi \le \tau$ on X^{div} (see [BJ22a, Theorem 5.6]); hence $\pi^\star\psi \le \tilde{\tau} \le \tilde{\tau}^\star$ on $\tilde{X}^{\mathrm{div}}$, which implies $\pi^\star\psi \le \tilde{\tau}^\star$ on $\tilde{X}^{\mathrm{an}}$ (see [BJ22a, Theorem 4.22]). Assuming ψ bounded, we get

$$(\omega, \psi)^{n+1} = (\pi^\star\omega, \pi^\star\psi)^{n+1} \le (\pi^\star\omega, \tilde{\tau}^\star)^{n+1},$$

where the equality follows from Lemma 3.1, and the inequality from the monotonicity of the energy pairing, see [BJ22a, Theorem 7.1]. Taking the supremum over ψ now yields

$$(n+1)\,\mathrm{vol}(\omega)\,\mathrm{E}_\omega(\tau^\star) \le (\pi^\star\omega, \tilde{\tau}^\star)^{n+1}.$$

Combined with (3.1), this implies $\mathrm{E}_\omega(\tau^\star) \le \mathrm{E}_\omega(\tau)$, and the result follows. □

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