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Uniform K-stability and asymptotics of energy functionals in Kähler geometry

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Abstract. Consider a polarized complex manifold (X, L) and a ray of positive metrics on L defined by a positive metric on a test configuration for (X, L). For many common functionals in Kähler geometry, we prove that the slope at infinity along the ray is given by evaluating the non-Archimedean version of the functional (as defined in our earlier paper [BHJ17]) at the non-Archimedean metric on L defined by the test configuration. Using this asymptotic result, we show that coercivity of the Mabuchi functional implies uniform K-stability, as defined in [Der15, BHJ17]. As a partial converse, we show that uniform K-stability implies coercivity of the Mabuchi functional when restricted to Bergman metrics.

Keywords. K-stability, Kähler geometry, canonical metrics, non-Archimedean geometry

Contents

Introduction

Let (X, L) be a polarized complex manifold, i.e. a smooth projective complex variety X endowed with an ample line bundle L. A central problem in Kähler geometry is to

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give necessary and sufficient conditions for the existence of canonical Kähler metrics in the corresponding Kähler class $c_1(L)$, for example, constant scalar curvature Kähler metrics (cscK metrics for short). To fix ideas, suppose the reduced automorphism group $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete. In this case, the celebrated Yau–Tian–Donaldson conjecture asserts that $c_1(L)$ admits a cscK metric iff (X, L) is K-stable. That K-stability follows from the existence of a cscK metric was proved by Stoppa [Stop09], building upon work of Donaldson [Don05], but the reverse direction is considered wide open in general.

This situation has led people to introduce stronger stability conditions that would hopefully imply the existence of a cscK metric. Building upon ideas of Donaldson [Don05], Székelyhidi [Szé06] proposed to use a version of K-stability in which, for any test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ is bounded below by a positive constant times a suitable *norm* of $(\mathcal{X}, \mathcal{L})$. (See also [Szé15] for a related notion.)

Following this lead we defined, in the prequel [BHJ17] to this paper, (X, L) to be *uniformly K-stable* if there exists $\delta > 0$ such that

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any normal and ample test configuration $(\mathcal{X}, \mathcal{L})$. Here $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ is a non-Archimedean analogue of Aubin's J-functional. It is equivalent to the L^1-norm of $(\mathcal{X}, \mathcal{L})$ as well as the minimum norm considered by Dervan [Der15]. The norm is zero iff the normalization of $(\mathcal{X}, \mathcal{L})$ is trivial, so uniform K-stability implies K-stability.

In [BHJ17] we advocated the point of view that a test configuration defines a *non-Archimedean metric* on L, that is, a metric on the Berkovich analytification of (X, L) with respect to the trivial norm on the ground field $\mathbb{C}$. Further, we defined non-Archimedean analogues of many classical functionals in Kähler geometry. One example is the functional J^{NA} above. Another is M^{NA}, a non-Archimedean analogue of the Mabuchi K-energy functional M. It agrees with the Donaldson–Futaki invariant, up to an explicit error term, and uniform K-stability is equivalent to

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any ample test configuration $(\mathcal{X}, \mathcal{L})$. In [BHJ17] we proved that canonically polarized manifolds and polarized Calabi–Yau manifolds are always uniformly K-stable.

A first goal of this paper is to exhibit precise relations between the non-Archimedean functionals and their classical counterparts. From now on we do not *a priori* assume that the reduced automorphism group of (X, L) is discrete. We prove

Theorem A. *Let $(\mathcal{X}, \mathcal{L})$ be an ample test configuration for a polarized complex manifold (X, L). Consider any smooth strictly positive S^1-invariant metric Φ on $\mathcal{L}$ defined near the central fiber, and let $(\phi^s)_s$ be the corresponding ray of smooth positive metrics on L. Denoting by M and J the Mabuchi K-energy functional and Aubin J-functional, respectively, we then have*

$$\lim_{s\to+\infty} \frac{M(\phi^s)}{s} = M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \quad \textit{and} \quad \lim_{s\to+\infty} \frac{J(\phi^s)}{s} = J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

The corresponding equalities also hold for several other functionals (see Theorem 3.6). More generally, we prove that these asymptotic properties hold in the logarithmic setting, for subklt pairs (X, B) and with weaker positivity assumptions (see Theorem 4.2).

At least when the total space $\mathcal{X}$ is smooth, the assertion in Theorem A regarding the Mabuchi functional is closely related to several statements appearing in the literature [PRS08, Corollary 2], [PT09, Corollary 1], [Li12, Remark 12, p. 38], [Tia17, Lemma 2.1], following the seminal work [Tia97]. A special case appears already in [DT92, p. 328]. However, to the best of our knowledge, neither the general and precise statement given here nor its proof is available in the literature.

As in [PRS08], the proof of Theorem A uses Deligne pairings, but the analysis here is more delicate since the test configuration $\mathcal{X}$ is not smooth. Using resolution of singularities, we can make $\mathcal{X}$ smooth, but then we lose the strict positivity of Φ. It turns out that the situation can be analyzed by estimating integrals of the form $\int_{\mathcal{X}_\tau} e^{2\Psi|_{\mathcal{X}_\tau}}$ as $\tau \to 0$, where $\mathcal{X} \to \mathbb{C}$ is an snc test configuration for X, and Ψ is a smooth metric on the (logarithmic) relative canonical bundle of $\mathcal{X}$ near the central fiber (see Lemma 3.11).

Donaldson [Don99] (see also [Mab87, Sem92]) has advocated the point of view that the space $\mathcal{H}$ of positive metrics on L is an infinite-dimensional symmetric space. One can view the space $\mathcal{H}^{\mathrm{NA}}$ of positive non-Archimedean metrics on L as (a subset of) the associated (conical) Tits building. Theorem A gives justification to this paradigm.

The asymptotic formulas in Theorem A allow us to study coercivity properties of the Mabuchi functional. As an immediate consequence of Theorem A, we have

Corollary B. *If the Mabuchi functional is coercive in the sense that*

$$M \geq \delta J - C$$

on $\mathcal{H}$ for some positive constants δ and C, then (X, L) is uniformly K-stable, that is,

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any normal ample test configuration $(\mathcal{X}, \mathcal{L})$.

Coercivity of the Mabuchi functional is known to hold if X is a Kähler–Einstein manifold without vector fields. This was first established in the Fano case by [PS+08]; an elegant proof can be found in [DR17]. As a special case of a very recent result of Berman, Darvas and Lu [BDL16], coercivity of the Mabuchi functional also holds for general polarized varieties admitting a metric of constant scalar curvature and having discrete reduced automorphism group. Thus, if (X, L) admits a constant scalar curvature metric and $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete, then (X, L) is uniformly K-stable. The converse statement is not currently known in general, but see below for the Fano case.

Next, we study coercivity of the Mabuchi functional when restricted to the space of Bergman metrics. For any $m \geq 1$ such that mL is very ample, let $\mathcal{H}_m$ be the space of Fubini–Study type metrics on L, induced by the embedding of $X \hookrightarrow \mathbb{P}^{N_m}$ via mL.

Theorem C. *Fix m such that (X, mL) is linearly normal, and $\delta > 0$. Then the following conditions are equivalent:*

(i) *there exists $C > 0$ such that $M \geq \delta J - C$ on $\mathcal{H}_m$.*
(ii) $\mathrm{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) \geq \delta J^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ *for all one-parameter subgroups λ of* $\mathrm{GL}(N_m, \mathbb{C})$*;*
(iii) $M^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) \geq \delta J^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ *for all one-parameter subgroups λ of* $\mathrm{GL}(N_m, \mathbb{C})$.

Here $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is the test configuration for (X, L) defined by λ.

Note that a different condition equivalent to (i)–(iii) appears in [Pau13, Theorem 1.1].

The equivalence of (ii) and (iii) stems from the close relation between the Donaldson–Futaki invariant and the non-Archimedean Mabuchi functional. In view of Theorem A, the equivalence between (i) and (iii) can be viewed as a generalization of the Hilbert–Mumford criterion. The proof uses in a crucial way a deep result of Paul [Pau12], which states that the restrictions to $\mathcal{H}_m$ of the Mabuchi functional and the J-functional have log norm singularities (see §5).

Since every ample test configuration arises as a one-parameter subgroup λ of $\mathrm{GL}(N_m, \mathbb{C})$ for some m, Theorem C implies

Corollary D. *A polarized manifold (X, L) is uniformly K-stable iff there exist $\delta > 0$ and a sequence $C_m > 0$ such that $M \geq \delta J - C_m$ on $\mathcal{H}_m$ for all sufficiently divisible m.*

Following Paul and Tian [PT06, PT09], we say that (X, mL) is *CM-stable* when there exist $C, \delta > 0$ such that $M \geq \delta J - C$ on $\mathcal{H}_m$.

Corollary E. *If (X, L) is uniformly K-stable, then (X, mL) is CM-stable for any sufficiently divisible positive integer m. Hence the reduced automorphism group is finite.*

Here the last statement follows from a result by Paul [Pau13, Corollary 1.1].

Let us now comment on the relation of uniform K-stability to the existence of Kähler–Einstein metrics on Fano manifolds. In [CDS15], Chen, Donaldson and Sun proved that a Fano manifold X admits a Kähler–Einstein metric iff it is K-polystable; see also [Tia15]. Since then, several new proofs have appeared. Datar and Székelyhidi [DSz15] proved an equivariant version of the conjecture, using Aubin's original continuity method. Chen, Sun and Wang [CSW18] gave a proof using the Kähler–Ricci flow.

In [BBJ15], Berman and the first and last authors of the current paper used a variational method to prove a slightly different statement: in the absence of vector fields, the existence of a Kähler–Einstein metric is equivalent to uniform K-stability. In fact, the direct implication uses Corollary B above.

In §6 we outline a different proof of the fact that a uniformly K-stable Fano manifold admits a Kähler–Einstein metric. Our method, which largely follows ideas of Tian, relies on Székelyhidi's partial C^0-estimates [Szé16] along the Aubin continuity path, together with Corollary D.

As noted above, uniform K-stability implies that the reduced automorphism group of (X, L) is discrete. In the presence of vector fields, there should presumably be a natural notion of uniform K-polystability. We hope to address this in future work.

There have been several important developments since a first draft of the current paper was circulated. First, Z. Sjöström Dyrefelt [SD18] and independently R. Dervan and

J. Ross [DR17] proved a transcendental version of Theorem A. Second, as mentioned above, it was proved in [BBJ15] that in the case of a Fano manifold without holomorphic vector fields, uniform K-stability is equivalent to coercivity of the Mabuchi functional, and hence to the existence of a Kähler–Einstein metric. Finally, the results in this paper were used in [BDL16] to prove that an arbitrary polarized pair (X, L) admitting a cscK metric must be K-polystable.

The organization of the paper is as follows. In the first section, we review several classical energy functionals in Kähler geometry and their interpretation as metrics on suitable Deligne pairings. Then, in §2, we recall some non-Archimedean notions from [BHJ17]. Specifically, a non-Archimedean metric is an equivalence class of test configurations, and the non-Archimedean analogues of the energy functionals in §1 are defined using intersection numbers. In §3 we prove Theorem A relating the classical and non-Archimedean functionals via subgeodesic rays. These results are generalized to the logarithmic setting in §4. Section 5 is devoted to the relation between uniform K-stability and CM-stability. In particular, we prove Theorem C and Corollaries D and E. Finally, in §6, we show how to use Székelyhidi's partial C^0-estimates along the Aubin continuity path together with CM-stability to prove that a uniformly K-stable Fano manifold admits a Kähler–Einstein metric.

1. Deligne pairings and energy functionals

In this section we recall the definition and main properties of the Deligne pairing, as well as its relation to classical functionals in Kähler geometry.

1.1. Metrics on line bundles

We use additive notation for line bundles and metrics. If, for $i = 1, 2$, ϕ_i is a metric on a line bundle L_i on X and $a_i \in \mathbb{Z}$, then $a_1\phi_1 + a_2\phi_2$ is a metric on $a_1L_1 + a_2L_2$. This allows us to define metrics on $\mathbb{Q}$-line bundles. A metric on the trivial line bundle will be identified with a function on X.

If σ is a (holomorphic) section of a line bundle L on a complex analytic space X, then $\log|\sigma|$ stands for the corresponding (possibly singular) metric on L. For any metric ϕ on L, $\log|\sigma| - \phi$ is therefore a function, and

$$|\sigma|_\phi := |\sigma|e^{-\phi} = \exp(\log|\sigma| - \phi)$$

is the length of σ in the metric ϕ.

We normalize the operator d^c so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$, and set (somewhat abusively)

$$dd^c\phi := -dd^c \log|\sigma|_\phi$$

for any local trivializing section σ of L. The globally defined $(1, 1)$-form (or current) $dd^c\phi$ is the curvature of ϕ, normalized so that it represents the (integral) first Chern class of L.

If X is a complex manifold of dimension n and η is a holomorphic n-form on X, then

$$|\eta|^2 := \frac{i^{n^2}}{2^n}\eta \wedge \bar{\eta}$$

defines a natural (smooth, positive) volume form on X. More generally, there is a bijection between smooth metrics on the canonical bundle K_X and (smooth, positive) volume forms on X, which associates to a smooth metric ϕ on K_X the volume form $e^{2\phi}$ locally defined by

$$e^{2\phi} := |\eta|^2/|\eta|^2_\phi$$

for any local section η of K_X.

If ω is a positive $(1,1)$-form on X and $n = \dim X$, then ω^n is a volume form, so $-\frac{1}{2}\log\omega^n$ is a metric on $-K_X$ in our notation. The *Ricci form* of ω is defined as the curvature

$$\operatorname{Ric}\omega := -dd^c\tfrac{1}{2}\log\omega^n$$

of this metric; it is thus a smooth $(1,1)$-form in the cohomology class $c_1(X)$ of $-K_X$.

If ϕ is a smooth positive metric on a line bundle L on X, we denote by $S_\phi \in C^\infty(X)$ the *scalar curvature* of the Kähler form $dd^c\phi$; it satisfies

$$S_\phi(dd^c\phi)^n = n\operatorname{Ric}(dd^c\phi)\wedge(dd^c\phi)^{n-1}. \tag{1.1}$$

1.2. Deligne pairings

While the construction below works in greater generality [Elk89, Zha96, MG00], we will restrict ourselves to the following setting. Let $\pi: Y \to T$ be a flat, projective morphism between smooth complex algebraic varieties, of relative dimension $n \geq 0$. Given line bundles $L_0, \dots, L_n$ on Y, consider the intersection product

$$L_0 \cdot \ldots \cdot L_n \cdot [Y] \in \mathrm{CH}_{\dim Y-(n+1)}(Y) = \mathrm{CH}_{\dim T-1}(Y).$$

Its push-forward belongs to $\mathrm{CH}_{\dim T-1}(T) = \operatorname{Pic}(T)$ since T is smooth, and hence defines an *isomorphism class* of line bundle on T. The *Deligne pairing* of $L_0, \dots, L_n$ selects in a canonical way a specific representative of this isomorphism class, denoted by

$$\langle L_0, \dots, L_n\rangle_{Y/T}.$$

The pairing is functorial, multilinear, and commutes with base change. It further satisfies the following key inductive property: if Z_0 is a non-singular divisor in Y, flat over T and defined by a section $\sigma_0 \in H^0(Y, L_0)$, then we have a canonical identification

$$\langle L_0, \dots, L_n\rangle_{Y/T} = \langle L_1|_{Z_0}, \dots, L_n|_{Z_0}\rangle_{Z_0/T}. \tag{1.2}$$

For $n = 0$, $\langle L_0\rangle_{Y/T}$ coincides with the norm of L_0 with respect to the finite flat morphism $Y \to T$. These properties uniquely characterize the Deligne pairing. Indeed, if we write each L_i as a difference of very ample line bundles, multilinearity reduces the situation to the case where the L_i are very ample. We may thus find non-singular divisors $Z_i \in |L_i|$ with $\bigcap_{i\in I} Z_i$ non-singular and flat over T for each set I of indices, and we get

$$\langle L_0, \dots, L_n\rangle_{Y/T} = \langle L_n|_{Z_0\cap\cdots\cap Z_{n-1}}\rangle_{Z_0\cap\cdots\cap Z_{n-1}/T}.$$

1.3. Metrics on Deligne pairings

We use [Elk90, Zha96, Mor99] as references. Given a smooth metric ϕ_j on each L_j, the Deligne pairing $\langle L_0, \dots, L_n\rangle_{Y/T}$ can be endowed with a continuous metric

$$\langle \phi_0, \dots, \phi_n\rangle_{Y/T},$$

smooth over the smooth locus of π, the construction being functorial, multilinear, and commuting with base change. It is basically constructed by requiring that

$$\langle \phi_0, \dots, \phi_n\rangle_{Y/T} = \langle \phi_1|_{Z_0}, \dots, \phi_n|_{Z_0}\rangle_{Z_0/T} - \int_{Y/T} \log|\sigma_0|_{\phi_0} dd^c\phi_1 \wedge \dots \wedge dd^c\phi_n \tag{1.3}$$

in the notation of (1.2), with $\int_{Y/T}$ denoting fiber integration, i.e. the push-forward by π as a current. By induction, the continuity of the metric $\langle \phi_0, \dots, \phi_n\rangle$ reduces to that of $\int_{Y/T} \log|\sigma_0|_{\phi_0} dd^c\phi_1 \wedge \dots \wedge dd^c\phi_n$, and thus follows from [Stol66, Theorem 4.9].

Remark 1.1. As explained in [Elk90, I.1], arguing by induction, the key point in checking that (1.3) is well-defined is the following symmetry property: if $\sigma_1 \in H^0(Y, L_1)$ is a section with divisor Z_1 such that both Z_1 and $Z_0 \cap Z_1$ are non-singular and flat over T, then

$$\int_{Y/T} \log|\sigma_0|_{\phi_0} dd^c\phi_1 \wedge \alpha + \int_{Z_0/T} \log|\sigma_1|_{\phi_1}\alpha = \int_{Y/T} \log|\sigma_1|_{\phi_1} dd^c\phi_0 \wedge \alpha + \int_{Z_1/T} \log|\sigma_0|_{\phi_0}\alpha$$

with $\alpha = dd^c\phi_2 \wedge \dots \wedge dd^c\phi_n$. By the Lelong–Poincaré formula, the above equality reduces to

$$\pi_*(\log|\sigma_0|_{\phi_0} dd^c \log|\sigma_1|_{\phi_1} \wedge \alpha) = \pi_*(\log|\sigma_1|_{\phi_1} dd^c \log|\sigma_0|_{\phi_0} \wedge \alpha),$$

which holds by Stokes' formula applied to a monotone regularization of the quasi-psh functions $\log|\sigma_i|_{\phi_i}$.

Metrics on Deligne pairings satisfy the following two crucial properties, which are direct consequences of (1.3).

(i) The curvature current of $\langle \phi_0, \dots, \phi_n\rangle_{Y/T}$ satisfies

$$dd^c\langle \phi_0, \dots, \phi_n\rangle_{Y/T} = \int_{Y/T} dd^c\phi_0 \wedge \dots \wedge dd^c\phi_n, \tag{1.4}$$

where again $\int_{Y/T}$ denotes fiber integration.

(ii) Given another smooth metric ϕ_0' on L_0, we have the change of metric formula

$$\langle \phi_0', \phi_1, \dots, \phi_n\rangle_{Y/T} - \langle \phi_0, \phi_1, \dots, \phi_n\rangle_{Y/T} = \int_{Y/T} (\phi_0' - \phi_0) dd^c\phi_1 \wedge \dots \wedge dd^c\phi_n. \tag{1.5}$$

1.4. Energy functionals

Let (X, L) be a polarized manifold, i.e. a smooth projective complex variety X with an ample line bundle L. Set

$$V := (L^n) \quad \text{and} \quad \bar{S} := -nV^{-1}(K_X \cdot L^{n-1}),$$

where $n = \dim X$. Denote by $\mathcal{H}$ the set of smooth positive metrics ϕ on L. For $\phi \in \mathcal{H}$, set $\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n$. Then $\mathrm{MA}(\phi)$ is a probability measure equivalent to Lebesgue measure, and $\int_X S_\phi \, \mathrm{MA}(\phi) = \bar{S}$ by (1.1).

We recall the following functionals in Kähler geometry. Fix a reference metric $\phi_{\mathrm{ref}} \in \mathcal{H}$. Our notation largely follows [BB+13, BB+11].

(i) The *Monge–Ampère energy functional* is given by

$$E(\phi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X (\phi - \phi_{\mathrm{ref}})(dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-j}. \tag{1.6}$$

(ii) The *J-functional* is a translation invariant version of E, defined as

$$J(\phi) := \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - E(\phi). \tag{1.7}$$

The closely related *I-functional* is defined by

$$I(\phi) := \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi). \tag{1.8}$$

(iii) For any closed $(1, 1)$-form θ, the *θ-twisted Monge–Ampère energy* is given by

$$E_\theta(\phi) = \frac{1}{n} \sum_{j=0}^{n-1} V^{-1} \int_X (\phi - \phi_{\mathrm{ref}})(dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-1-j} \wedge \theta. \tag{1.9}$$

Taking $\theta := -n \,\mathrm{Ric}(dd^c\phi_{\mathrm{ref}})$, we obtain the *Ricci energy* $R := -E_{n \,\mathrm{Ric}(dd^c\phi_{\mathrm{ref}})}$.

(iv) The *entropy* of $\phi \in \mathcal{H}$ is defined as

$$H(\phi) := \frac{1}{2} \int_X \log\left[\frac{\mathrm{MA}(\phi)}{\mathrm{MA}(\phi_{\mathrm{ref}})}\right] \mathrm{MA}(\phi), \tag{1.10}$$

that is, (half) the relative entropy of the probability measure $\mathrm{MA}(\phi)$ with respect to $\mathrm{MA}(\phi_{\mathrm{ref}})$. We have $H(\phi) \geq 0$, with equality iff $\phi - \phi_{\mathrm{ref}}$ is constant.

(v) The *Mabuchi functional* (or K-energy) can now be defined via the Chen–Tian formula [Che00] (see also [BB17, Proposition 3.1]) as

$$M(\phi) = H(\phi) + R(\phi) + \bar{S}E(\phi). \tag{1.11}$$

These functionals vanish at $\phi_{\rm ref}$ and satisfy the variational formulas

$$\begin{aligned}
\delta E(\phi) &= \mathrm{MA}(\phi) = V^{-1}(dd^c\phi)^n,\\
\delta E_\theta(\phi) &= V^{-1}(dd^c\phi)^{n-1}\wedge\theta,\\
\delta R(\phi) &= -nV^{-1}(dd^c\phi)^{n-1}\wedge \mathrm{Ric}(dd^c\phi_{\rm ref}),\\
\delta H(\phi) &= nV^{-1}(dd^c\phi)^{n-1}\wedge (\mathrm{Ric}(dd^c\phi_{\rm ref})-\mathrm{Ric}(dd^c\phi)),\\
\delta M(\phi) &= (\bar S - S_\phi)\,\mathrm{MA}(\phi).
\end{aligned}$$

In particular, ϕ is a critical point of M iff $dd^c\phi$ is a cscK metric.

The functionals I, J and $I-J$ are comparable in the sense that

$$\frac{1}{n}J \le I - J \le nJ \tag{1.12}$$

on $\mathcal{H}$. For $\phi\in\mathcal{H}$ we have $J(\phi)\ge 0$, with equality iff $\phi-\phi_{\rm ref}$ is constant. These properties are thus also shared by I and $I-J$.

The functionals H, I, J, M are translation invariant in the sense that $H(\phi+c) = H(\phi)$ for $c\in\mathbb{R}$. For E and R we instead have $E(\phi+c)=E(\phi)+c$ and $R(\phi+c) = R(\phi)-\bar S c$, respectively.

1.5. Energy functionals as Deligne pairings

The functionals above can be expressed using Deligne pairings, an observation going back at least to [PS04]. Note that any metric $\phi\in\mathcal{H}$ induces a smooth metric $\frac12\log\mathrm{MA}(\phi)$ on K_X. The following identities are now easy consequences of the change of metric formula (1.5).

Lemma 1.2. *For any $\phi\in\mathcal{H}$ we have*

$$\begin{aligned}
(n+1)VE(\phi) &= \langle\phi^{n+1}\rangle_X - \langle\phi_{\rm ref}^{n+1}\rangle_X,\\
VJ(\phi) &= \langle\phi,\phi_{\rm ref}^n\rangle_X - \langle\phi_{\rm ref}^{n+1}\rangle_X - \frac{1}{n+1}[\langle\phi^{n+1}\rangle_X - \langle\phi_{\rm ref}^{n+1}\rangle_X],\\
VI(\phi) &= \langle\phi-\phi_{\rm ref},\phi_{\rm ref}^n\rangle_X - \langle\phi-\phi_{\rm ref},\phi^n\rangle_X,\\
VR(\phi) &= \left\langle\tfrac12\log\mathrm{MA}(\phi_{\rm ref}),\phi^n\right\rangle_X - \left\langle\tfrac12\log\mathrm{MA}(\phi_{\rm ref}),\phi_{\rm ref}^n\right\rangle_X,\\
VH(\phi) &= \left\langle\tfrac12\log\mathrm{MA}(\phi),\phi^n\right\rangle_X - \left\langle\tfrac12\log\mathrm{MA}(\phi_{\rm ref}),\phi^n\right\rangle_X,\\
VM(\phi) &= \left\langle\tfrac12\log\mathrm{MA}(\phi),\phi^n\right\rangle_X - \left\langle\tfrac12\log\mathrm{MA}(\phi_{\rm ref}),\phi_{\rm ref}^n\right\rangle_X\\
&\quad + \frac{\bar S}{n+1}[\langle\phi^{n+1}\rangle_X - \langle\phi_{\rm ref}^{n+1}\rangle_X],
\end{aligned}$$

where $\langle\ \rangle_X$ denotes the Deligne pairing with respect to the constant map $X\to\{\mathrm{pt}\}$.

Remark 1.3. The formulas above make it evident that instead of fixing a reference metric $\phi_{\rm ref}\in\mathcal{H}$, we could view E, $H+R$ and M as metrics on suitable multiples of the complex lines $\langle L^{n+1}\rangle_X$, $\langle K_X, L^n\rangle_X$, and $(n+1)\langle K_X, L^n\rangle_X + \bar S\langle L^{n+1}\rangle_X$, respectively.

Remark 1.4. In the definition of R, we could replace $-\mathrm{Ric}(dd^c\phi_{\mathrm{ref}})$ by $dd^c\psi_{\mathrm{ref}}$ for any smooth metric ψ_{ref} on K_X. Similarly, in the definition of H, we could replace the reference measure $\mathrm{MA}(\phi_{\mathrm{ref}})$ by $e^{2\psi_{\mathrm{ref}}}$. Doing so, and keeping the Chen–Tian formula, would only change the Mabuchi functional M by an additive constant.

1.6. The Ding functional

Now suppose X is a Fano manifold, that is, $L := -K_X$ is ample. Any metric ϕ on L then induces a positive volume form $e^{-2\phi}$ on X. The *Ding functional* [Din88] on $\mathcal{H}$ is defined by

$$D(\phi) = L(\phi) - E(\phi), \quad \text{where} \quad L(\phi) = -\frac{1}{2}\log\int_X e^{-2\phi}.$$

This functional has proven an extremely useful tool for the study of the existence of Kähler–Einstein metrics, which are realized as the critical points of D (see e.g. [Berm16, BBJ15]).

2. Test configurations as non-Archimedean metrics

In this section we recall some notions and results from [BHJ17]. Let X be a smooth projective complex variety and L a line bundle on X.

2.1. Test configurations

As in [BHJ17] we adopt the following flexible terminology for test configurations.

Definition 2.1. A *test configuration* $\mathcal{X}$ for X consists of the following data:

(i) a flat, projective morphism of schemes $\pi : \mathcal{X} \to \mathbb{C}$;
(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;
(iii) an isomorphism $\mathcal{X}_1 \simeq X$.

We denote by τ the coordinate on $\mathbb{C}$, and by $\mathcal{X}_\tau$ the fiber over τ.

These conditions imply that $\mathcal{X}$ is reduced and irreducible [BHJ17, Proposition 2.6]. If $\mathcal{X}, \mathcal{X}'$ are test configurations for X, then there is a unique $\mathbb{C}^*$-equivariant birational map $\mathcal{X}' \dashrightarrow \mathcal{X}$ compatible with the isomorphism in (iii). We say that $\mathcal{X}'$ *dominates* $\mathcal{X}$ if this birational map is a morphism; when it is an isomorphism we somewhat abusively identify $\mathcal{X}$ and $\mathcal{X}'$. Any test configuration $\mathcal{X}$ is dominated by its *normalization* $\widetilde{\mathcal{X}}$.

An *snc* test configuration for X is a smooth test configuration $\mathcal{X}$ whose central fiber $\mathcal{X}_0$ has simple normal crossing support (but is not necessarily reduced).

When $\mathcal{X}$ is a test configuration, we define the *logarithmic canonical bundle* as

$$K^{\log}_{\mathcal{X}} := K_{\mathcal{X}} + \mathcal{X}_{0,\mathrm{red}}.$$

Setting $K^{\log}_{\mathbb{C}} := K_{\mathbb{C}} + [0]$, we define the *relative logarithmic canonical bundle* as

$$K^{\log}_{\mathcal{X}/\mathbb{C}} := K^{\log}_{\mathcal{X}} - \pi^* K^{\log}_{\mathbb{C}} = K_{\mathcal{X}/\mathbb{C}} + \mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0;$$

this is well behaved under base change $\tau \mapsto \tau^d$ (see [BHJ17, §4.4]). Despite the terminology, $K_{\mathcal{X}}$, $K_{\mathcal{X}/\mathbb{C}}$, $K_{\mathcal{X}}^{\log}$ and $K_{\mathcal{X}/\mathbb{C}}^{\log}$ are only Weil divisor classes in general; they are line bundles when $\mathcal{X}$ is smooth.

Definition 2.2. A *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of a test configuration $\mathcal{X}$ for X, together with the following additional data:

(iv) a $\mathbb{C}^*$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;
(v) an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$.

A *pull-back* of a test configuration $(\mathcal{X}, \mathcal{L})$ is a test configuration $(\mathcal{X}', \mathcal{L}')$ where $\mathcal{X}'$ dominates $\mathcal{X}$ and $\mathcal{L}'$ is the pull-back of $\mathcal{L}$. In particular, the *normalization* $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ is the pull-back of $(\mathcal{X}, \mathcal{L})$ with $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the normalization morphism.

A test configuration $(\mathcal{X}, \mathcal{L})$ is *trivial* if $\mathcal{X} = X \times \mathbb{C}$ with $\mathbb{C}^*$ acting trivially on X. This implies that $(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = (X, L) \times \mathbb{C}$ for some constant $c \in \mathbb{Q}$. A test configuration for (X, L) is *almost trivial* if its normalization is trivial.

We say that $(\mathcal{X}, \mathcal{L})$ is ample (resp. semiample, resp. nef) when $\mathcal{L}$ is relatively ample (resp. relatively semiample, resp. nef). The pull-back of a semiample (resp. nef) test configuration is semiample (resp. nef).

If L is ample, then for every semiample test configuration $(\mathcal{X}, \mathcal{L})$ there exists a unique ample test configuration $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ that is dominated by $(\mathcal{X}, \mathcal{L})$ and satisfies $\mu_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathrm{amp}}}$, where $\mu\colon \mathcal{X} \to \mathcal{X}_{\mathrm{amp}}$ is the canonical morphism (see [BHJ17, Proposition 2.17]).

Note that while $\mathcal{X}$ can often be chosen smooth $\mathcal{X}_{\mathrm{amp}}$ will not be smooth, in general. It is, however, normal whenever $\mathcal{X}$ is.

2.2. *One-parameter subgroups*

Suppose L is ample. Ample test configurations are then essentially equivalent to one-parameter degenerations of X. See [BHJ17, §2.3] for details on what follows.

Fix $m \geq 1$ such that mL is very ample, and consider the corresponding closed embedding $X \hookrightarrow \mathbb{P}^{N_m - 1}$ with $N_m := h^0(X, mL)$. Then every one-parameter subgroup (1-PS for short) $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$ induces an ample test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ for (X, L). By definition, $\mathcal{X}_\lambda$ is the Zariski closure in $\mathbb{P}(V) \times \mathbb{C}$ of the image of the closed embedding $X \times \mathbb{C}^* \hookrightarrow \mathbb{P}(V) \times \mathbb{C}^*$ mapping (x, τ) to $(\lambda(\tau)x, \tau)$. Note that $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is trivial iff λ is a multiple of the identity. We emphasize that $\mathcal{X}_\lambda$ is not normal in general.

In fact, every ample test configuration may be obtained as above. Using one-parameter subgroups, we can produce test configurations that are almost trivial but not trivial, as observed in [LX14, Remark 5]. See [BHJ17, Proposition 2.12] for an elementary proof of the following result.

Proposition 2.3. *For every m divisible enough, there exists a 1-PS $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$ such that the test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is non-trivial but almost trivial.*

2.3. *Valuations and log discrepancies*

By a *valuation on* X we mean a real-valued valuation v on the function field $\mathbb{C}(X)$ (trivial on the ground field $\mathbb{C}$). The *trivial valuation* v_{triv} is defined by $v_{\mathrm{triv}}(f) = 0$ for

$f \in \mathbb{C}(X)^*$. A valuation v is *rational divisorial* if it is of the form $v = c\,\mathrm{ord}_F$, where $c \in \mathbb{Q}_{>0}$ and F is a prime divisor on a projective normal variety Y admitting a birational morphism onto X. We denote by $X_{\mathbb{Q}}^{\mathrm{div}}$ the set of valuations on X that are either rational divisorial or trivial, and equip it with the weakest topology such that $v \mapsto v(f)$ is continuous for every $f \in \mathbb{C}(X)^*$.

The *log discrepancy* $A_X(v)$ of a valuation in $X_{\mathbb{Q}}^{\mathrm{div}}$ is defined as follows. First, we set $A_X(v_{\mathrm{triv}}) = 0$. For a rational divisorial valuation $v = c\,\mathrm{ord}_F$ as above, we set $A_X(v) = c(1 + \mathrm{ord}_F(K_{Y/X}))$, where $K_{Y/X}$ is the relative canonical (Weil) divisor.

Now consider a normal test configuration $\mathcal{X}$ of X. Since $\mathbb{C}(\mathcal{X}) \simeq \mathbb{C}(X)(\tau)$, any valuation w on $\mathcal{X}$ restricts to a valuation $r(w)$ on X. Let E be an irreducible component of the central fiber $\mathcal{X}_0 = \sum b_E E$. Then ord_E is a $\mathbb{C}^*$-invariant rational divisorial valuation on $\mathbb{C}(\mathcal{X})$ and satisfies $\mathrm{ord}_E(t) = b_E$. If we set $v_E := r(b_E^{-1}\,\mathrm{ord}_E)$, then v_E is a valuation in $X_{\mathbb{Q}}^{\mathrm{div}}$. Conversely, every valuation $v \in X_{\mathbb{Q}}^{\mathrm{div}}$ has a unique $\mathbb{C}^*$-invariant preimage w under r normalized by $w(\tau) = 1$, and w is associated to an irreducible component of the central fiber of some test configuration for X (cf. [BHJ17, Theorem 4.6]).

Note that ord_E is a rational divisorial valuation on $X \times \mathbb{C}$. By [BHJ17, Proposition 4.11], the log discrepancies of ord_E and v_E are related as follows: $A_{X\times\mathbb{C}}(\mathrm{ord}_E) = b_E(1 + A_X(v_E))$.

2.4. *Compactifications*

For some purposes it is convenient to compactify test configurations. The following notion provides a canonical way of doing so.

Definition 2.4. The *compactification* $\bar{\mathcal{X}}$ of a test configuration $\mathcal{X}$ for X is defined by gluing together $\mathcal{X}$ and $X \times (\mathbb{P}^1 \setminus \{0\})$ along their respective open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{C} \setminus \{0\})$, using the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{C} \setminus \{0\})$.

The compactification $\bar{\mathcal{X}}$ comes with a $\mathbb{C}^*$-equivariant flat morphism $\bar{\mathcal{X}} \to \mathbb{P}^1$, still denoted by π. By construction, $\pi^{-1}(\mathbb{P}^1 \setminus \{0\})$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times (\mathbb{P}^1 \setminus \{0\})$ over $\mathbb{P}^1 \setminus \{0\}$.

Similarly, a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) admits a compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is a $\mathbb{C}^*$-linearized $\mathbb{Q}$-line bundle on $\bar{\mathcal{X}}$. Note that $\bar{\mathcal{L}}$ is relatively (semi)ample iff $\mathcal{L}$ is.

The relative canonical bundle and relative logarithmic canonical bundle are now defined by

$$K_{\bar{\mathcal{X}}/\mathbb{P}^1} := K_{\bar{\mathcal{X}}} - \pi^* K_{\mathbb{P}^1},$$
$$K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} := K^{\log}_{\bar{\mathcal{X}}} - \pi^* K^{\log}_{\mathbb{P}^1} = K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0.$$

2.5. *Non-Archimedean metrics*

Following [BHJ17, §6] (see also [BJ18]) we introduce:

Definition 2.5. Two test configurations $(\mathcal{X}_1, \mathcal{L}_1)$, $(\mathcal{X}_2, \mathcal{L}_2)$ for (X, L) are *equivalent* if there exists a test configuration $(\mathcal{X}_3, \mathcal{L}_3)$ that is a pull-back of both $(\mathcal{X}_1, \mathcal{L}_1)$ and $(\mathcal{X}_2, \mathcal{L}_2)$. An equivalence class is called a *non-Archimedean metric* on L, and is denoted by ϕ. We denote by ϕ_{triv} the equivalence class of the trivial test configuration $(X, L) \times \mathbb{C}$.

A non-Archimedean metric ϕ is called *semipositive* if some (or, equivalently, any) representative $(\mathcal{X}, \mathcal{L})$ of ϕ is nef. Note that this implies that L is nef.

When L is ample, we say that a non-Archimedean metric ϕ on L is *positive* if some (or, equivalently, any) representative $(\mathcal{X}, \mathcal{L})$ of ϕ is semiample. We denote by $\mathcal{H}^{\mathrm{NA}}$ the set of all non-Archimedean positive metrics on L. By [BHJ17, Lemma 6.3], every $\phi \in \mathcal{H}^{\mathrm{NA}}$ is represented by a unique normal, ample test configuration.

The set of non-Archimedean metrics on a line bundle L admits two natural operations:

(i) a *translation action* of $\mathbb{Q}$, denoted by $\phi \mapsto \phi + c$ and induced by the map $(\mathcal{X}, \mathcal{L}) \mapsto (\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)$;

(ii) a *scaling action* of the semigroup $\mathbb{N}^*$ of positive integers, denoted by $\phi \mapsto \phi_d$ and induced by the base change of $(\mathcal{X}, \mathcal{L})$ by $\tau \mapsto \tau^d$.

When L is ample (resp. nef) these operations preserve the set of positive (resp. semipositive) metrics. The trivial metric ϕ_{triv} is fixed by the scaling action.

As in §1.1, we use additive notation for non-Archimedean metrics. A non-Archimedean metric on $\mathcal{O}_X$ induces a bounded (and continuous) function on $X^{\mathrm{div}}_{\mathbb{Q}}$.

Remark 2.6. As explained in [BHJ17, §6.8], a non-Archimedean metric ϕ on L, as defined above, can be viewed as a metric on the Berkovich analytification [Berk90] of L with respect to the trivial absolute value on the ground field $\mathbb{C}$. See also [BJ18] for a more systematic analysis, itself building upon [BFJ16, BFJ15a].

2.6. *Intersection numbers and Monge–Ampère measures*

Following [BHJ17, §6.6] we define the intersection number $(\phi_0 \cdot \ldots \cdot \phi_n)$ of non-Archimedean metrics $\phi_0, \ldots, \phi_n$ on line bundles $L_0, \ldots, L_n$ on X as follows. Pick representatives $(\mathcal{X}, \mathcal{L}_i)$ of ϕ_i, $0 \le i \le n$, with the same test configuration $\mathcal{X}$ for X and set

$$(\phi_0 \cdot \ldots \cdot \phi_n) := (\bar{\mathcal{L}}_0 \cdot \ldots \cdot \bar{\mathcal{L}}_n),$$

where $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ is the compactification of $(\mathcal{X}, \mathcal{L}_i)$. It follows from the projection formula that this does not depend of the choice of the $\mathcal{L}_i$. Note that $(\phi_{\mathrm{triv}}^{n+1}) = 0$. When $L_0 = \mathcal{O}_X$, so that $\mathcal{L}_0 = \mathcal{O}_X(D)$ for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D = \sum r_E E$ supported on $\mathcal{X}_0$, we can compute the intersection number as $(\phi_0 \cdot \ldots \cdot \phi_n) = \sum_E r_E (\mathcal{L}_1|_E \cdot \ldots \cdot \mathcal{L}_n|_E)$.

To a non-Archimedean metric ϕ on a big and nef line bundle L on X we associate, as in [BHJ17, §6.7], a signed finite atomic *Monge–Ampère measure* on $X^{\mathrm{div}}_{\mathbb{Q}}$. Pick a representative $(\mathcal{X}, \mathcal{L}_i)$ of ϕ, and set

$$\mathrm{MA}^{\mathrm{NA}}(\phi) = V^{-1} \sum_E b_E (\mathcal{L}|_E^n) \delta_{v_E},$$

where E ranges over irreducible components of $\mathcal{X}_0 = \sum_E b_E E$, $v_E = r(b_E^{-1} \operatorname{ord}_E) \in X^{\mathrm{div}}_{\mathbb{Q}}$, and $V = (L^n)$. When the ϕ_i are semipositive, the mixed Monge–Ampère measure is a probability measure.

2.7. Functionals on non-Archimedean metrics

Following [BHJ17, §7] we define non-Archimedean analogues of the functionals considered in §1.4. Fix a line bundle L.

Definition 2.7. Let W be a set of non-Archimedean metrics on L that is closed under translation and scaling. A functional $F\colon W \to \mathbb{R}$ is

(i) *homogeneous* if $F(\phi_d) = dF(\phi)$ for all $\phi \in W$ and $d \in \mathbb{N}^*$;
(ii) *translation invariant* if $F(\phi + c) = F(\phi)$ for all $\phi \in W$ and $c \in \mathbb{Q}$.

When L is ample, a functional F on $\mathcal{H}^{\mathrm{NA}}$ may be viewed as a function $F(\mathcal{X}, \mathcal{L})$ on the set of all semiample test configurations $(\mathcal{X}, \mathcal{L})$ that is invariant under pull-back, i.e. $F(\mathcal{X}', \mathcal{L}') = F(\mathcal{X}, \mathcal{L})$ whenever $(\mathcal{X}', \mathcal{L}')$ is a pull-back of $(\mathcal{X}, \mathcal{L})$ (and, in particular, invariant under normalization). Homogeneity amounts to $F(\mathcal{X}_d, \mathcal{L}_d) = d\,F(\mathcal{X}, \mathcal{L})$ for all $d \in \mathbb{N}^*$, and translation invariance to $F(\mathcal{X}, \mathcal{L}) = F(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)$ for all $c \in \mathbb{Q}$.

For each non-Archimedean metric ϕ on L, choose a normal representative $(\mathcal{X}, \mathcal{L})$ that dominates $X \times \mathbb{C}$ via $\rho\colon \mathcal{X} \to X \times \mathbb{C}$. Then $\mathcal{L} = \rho^*(L \times \mathbb{C}) + D$ for a uniquely determined $\mathbb{Q}$-Cartier divisor D supported on $\mathcal{X}_0$. Write $\mathcal{X}_0 = \sum_E b_E E$ and let $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ be the compactification of $(\mathcal{X}, \mathcal{L})$.

In this notation, we may describe our non-Archimedean functionals as follows. Assume L is big and nef. Let ϕ_{triv} and ψ_{triv} be the trivial metrics on L and K_X, respectively.

(i) The *non-Archimedean Monge–Ampère energy* of ϕ is

$$E^{\mathrm{NA}}(\phi) := \frac{(\phi^{n+1})}{(n+1)V} = \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)V}.$$

(ii) The *non-Archimedean I-functional* and *J-functional* are given by

$$\begin{aligned} I^{\mathrm{NA}}(\phi) &:= V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) - V^{-1}((\phi - \phi_{\mathrm{triv}}) \cdot \phi^n) \\ &= V^{-1}(\bar{\mathcal{L}} \cdot (\rho^*(L \times \mathbb{P}^1))^n) - V^{-1}(D \cdot \bar{\mathcal{L}}^n) \end{aligned}$$

and

$$\begin{aligned} J^{\mathrm{NA}}(\phi) &:= V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) - E^{\mathrm{NA}}(\phi) \\ &= \frac{1}{V}(\bar{\mathcal{L}} \cdot (\rho^*(L \times \mathbb{P}^1))^n) - \frac{1}{(n+1)V}(\bar{\mathcal{L}}^{n+1}). \end{aligned}$$

(iii) The *non-Archimedean Ricci energy* is

$$R^{\mathrm{NA}}(\phi) := V^{-1}(\psi_{\mathrm{triv}} \cdot \phi^n) = V^{-1}(\rho^* K^{\log}_{X \times \mathbb{P}^1/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n).$$

(iv) The *non-Archimedean entropy* is

$$\begin{aligned} H^{\mathrm{NA}}(\phi) &:= \int_{X^{\mathrm{div}}_{\mathbb{Q}}} A_X(v)\, \mathrm{MA}^{\mathrm{NA}}(\phi) \\ &= V^{-1}(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) - V^{-1}(\rho^* K^{\log}_{X \times \mathbb{P}^1/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n). \end{aligned}$$

(v) The *non-Archimedean Mabuchi functional* (or K-energy) is

$$\begin{aligned} M^{\mathrm{NA}}(\phi) &:= H^{\mathrm{NA}}(\phi) + R^{\mathrm{NA}}(\phi) + \bar{S} E^{\mathrm{NA}}(\phi) \\ &= V^{-1}(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + \frac{\bar{S}}{(n+1)V}(\bar{\mathcal{L}}^{n+1}). \end{aligned}$$

Note the resemblance to the formulas in §1.5. All of these functionals are homogeneous. They are also translation invariant, except for E^{NA} and R^{NA}, which satisfy

$$E^{\mathrm{NA}}(\phi + c) = E^{\mathrm{NA}}(\phi) + c \quad \text{and} \quad R^{\mathrm{NA}}(\phi + c) = R^{\mathrm{NA}}(\phi) - \bar{S}c \tag{2.1}$$

for all $\phi \in \mathcal{H}^{\mathrm{NA}}$ and $c \in \mathbb{Q}$.

The functionals I^{NA}, J^{NA} and $I^{\mathrm{NA}} - J^{\mathrm{NA}}$ are comparable on semipositive metrics in the same way as in (1.12). By [BHJ17, Lemma 7.7, Theorem 5.16], when ϕ is positive, the first term in the definition of J^{NA} satisfies

$$V^{-1}(\phi \cdot \phi^n_{\mathrm{triv}}) = (\phi - \phi_{\mathrm{triv}})(v_{\mathrm{triv}}) = \max_{X^{\mathrm{div}}_{\mathbb{Q}}}(\phi - \phi_{\mathrm{triv}}) = \max_E b_E^{-1} \operatorname{ord}_E(D).$$

Further, $J^{\mathrm{NA}}(\phi) \geq 0$, with equality iff $\phi = \phi_{\mathrm{triv}} + c$ for some $c \in \mathbb{Q}$, and J^{NA} is comparable to both a natural L^1-norm and the minimum norm in the sense of Dervan [Der15] (see [BHJ17, Theorem 7.9, Remark 7.12]). For a normal ample test configuration $(\mathcal{X}, \mathcal{L})$ representing $\phi \in \mathcal{H}^{\mathrm{NA}}$ we also denote the J-norm by $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.

2.8. *The Donaldson–Futaki invariant*

As explained in [BHJ17], the non-Archimedean Mabuchi functional is closely related to the Donaldson–Futaki invariant. We have

Proposition 2.8. *Assume L is ample. Let $\phi \in \mathcal{H}^{\mathrm{NA}}$ be the class of an ample test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), and denote by $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ its normalization, which is thus the unique normal, ample representative of ϕ. Then*

$$M^{\mathrm{NA}}(\phi) = \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) - V^{-1}((\widetilde{\mathcal{X}}_0 - \widetilde{\mathcal{X}}_{0,\mathrm{red}}) \cdot \tilde{\mathcal{L}}^n), \tag{2.2}$$

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) + 2V^{-1} \sum_E m_E (E \cdot \mathcal{L}^n), \tag{2.3}$$

where E ranges over the irreducible components of $\mathcal{X}_0$ contained in the singular locus of $\mathcal{X}$ and $m_E \in \mathbb{N}^*$ is the length of $(\nu_* \mathcal{O}_{\widetilde{\mathcal{X}}})/\mathcal{O}_{\mathcal{X}}$ at the generic point of E, with $\nu \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the normalization.

In particular, $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq M^{\mathrm{NA}}(\phi)$*, and equality holds iff $\mathcal{X}$ is regular in codimension one and $\mathcal{X}_0$ is generically reduced.*

Indeed, formulas (2.2) and (2.3) follow from the discussion in [BHJ17, §7.3] and from [BHJ17, Proposition 3.15], respectively. Note that intersection-theoretic formulas for the Donaldson–Futaki invariant appeared already in [Wan12] and [Oda13].

For a general non-Archimedean metric ϕ on L we can define

$$\begin{aligned}\mathrm{DF}(\phi) &= M^{\mathrm{NA}}(\phi) + V^{-1}((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \bar{\mathcal{L}}^n)\\ &= V^{-1}(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + \frac{\bar{S}}{(n+1)V}(\bar{\mathcal{L}}^{n+1})\end{aligned}$$

for any normal representative $(\mathcal{X}, \mathcal{L})$ of ϕ. Clearly $M^{\mathrm{NA}}(\phi) \le \mathrm{DF}(\phi)$ when ϕ is semipositive.

2.9. *The non-Archimedean Ding functional [BHJ17, §7.7]*

Suppose X is weakly Fano, that is, $L := -K_X$ is big and nef. In this case, we define the *non-Archimedean Ding functional* on the space of non-Archimedean metrics on L by

$$D^{\mathrm{NA}}(\phi) = L^{\mathrm{NA}}(\phi) - E^{\mathrm{NA}}(\phi),$$

where L^{NA} is defined by

$$L^{\mathrm{NA}}(\phi) = \inf_v \big(A_X(v) + (\phi - \phi_{\mathrm{triv}})(v)\big),$$

the infimum taken over all valuations v on X that are rational divisorial or trivial. Recall from §2.5 that $\phi - \phi_{\mathrm{triv}}$ is a non-Archimedean metric on $\mathcal{O}_X$ and induces a bounded function on $X^{\mathrm{div}}_{\mathbb{Q}}$. Note that $L^{\mathrm{NA}}(\phi+c) = L^{\mathrm{NA}}(\phi)+c$; hence D^{NA} is translation invariant.

We always have $D^{\mathrm{NA}} \le J^{\mathrm{NA}}$ (see [BHJ17, Proposition 7.28]). When ϕ is semipositive, we have $D^{\mathrm{NA}}(\phi) \le M^{\mathrm{NA}}(\phi)$ (see [BHJ17, Proposition 7.32]).

2.10. *Uniform K-stability*

As in [BHJ17, §8] we make the following definition.

Definition 2.9. A polarized complex manifold (X, L) is *uniformly K-stable* if there exists a constant $\delta > 0$ such that the following equivalent conditions hold:

(i) $M^{\mathrm{NA}}(\phi) \ge \delta J^{\mathrm{NA}}(\phi)$ for every $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$;
(ii) $\mathrm{DF}(\phi) \ge \delta J^{\mathrm{NA}}(\phi)$ for every $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$;
(iii) $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \ge \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for any normal ample test configuration $(\mathcal{X}, \mathcal{L})$.

Here the equivalence between (ii) and (iii) is definitional, and (i)⇒(ii) follows immediately from $\mathrm{DF} \le M^{\mathrm{NA}}$. The implication (ii)⇒(i) follows from the homogeneity of M^{NA} together with the fact that $\mathrm{DF}(\phi_d) = M^{\mathrm{NA}}(\phi_d)$ for d sufficiently divisible. See [BHJ17, Proposition 8.2] for details.

The fact that $J^{\mathrm{NA}}(\phi) = 0$ iff $\phi = \phi_{\mathrm{triv}} + c$ implies that uniform K-stability is stronger than K-stability as introduced by [Tia97, Don02]. Our notion of uniform K-stability is equivalent to uniform K-stability defined either with respect to the L^1-norm or the minimum norm in the sense of [Der15] (see [BHJ17, Remark 8.3]).

In the Fano case, uniform K-stability is further equivalent to *uniform Ding stability*:

Theorem 2.10. *Assume* $L := -K_X$ *is ample and fix a number* δ *with* $0 \le \delta \le 1$. *Then the following conditions are equivalent:*

(i) $M^{\mathrm{NA}} \ge \delta J^{\mathrm{NA}}$ *on* $\mathcal{H}^{\mathrm{NA}}$;
(ii) $D^{\mathrm{NA}} \ge \delta J^{\mathrm{NA}}$ *on* $\mathcal{H}^{\mathrm{NA}}$.

This is proved in [BBJ15] using the Minimal Model Program as in [LX14]. See [Fuj16] for a more general result, and also [Fuj18].

3. Non-Archimedean limits

In this section we prove Theorem A and Corollary B.

3.1. Rays of metrics and non-Archimedean limits

For any line bundle L on X, there is a bijection between smooth rays $(\phi^s)_{s>0}$ of metrics on L and S^1-invariant smooth metrics Φ on the pull-back of L to $X\times\Delta^*$, with $\Delta^* = \Delta_1^* \subset \mathbb{C}$ the punctured unit disc. The restriction of Φ to $\mathcal{X}_\tau$ for $\tau \in \Delta^*$ is given by pull-back of $\phi^{\log|\tau|^{-1}}$ under the map $\mathcal{X}_\tau \to X$ given by the $\mathbb{C}^*$-action. Similarly, smooth rays $(\phi^s)_{s>s_0}$ correspond to S^1-invariant smooth metrics on the pull-back of L to $X \times \Delta_{r_0}^*$, with $r_0 = e^{-s_0}$.

A *subgeodesic ray* is a ray (ϕ^s) whose corresponding metric Φ is semipositive. Such rays can of course only exist when L is nef.

Definition 3.1. We say that a smooth ray (ϕ^s) admits a non-Archimedean metric ϕ^{NA} as *non-Archimedean limit* if there exists a test configuration $(\mathcal{X}, \mathcal{L})$ representing ϕ^{NA} such that the metric Φ on $L \times \Delta^*$ corresponding to $(\phi^s)_s$ extends to a smooth metric on $\mathcal{L}$ over Δ.

In other words, a non-Archimedean limit exists iff Φ has *analytic singularities* along $X \times \{0\}$, i.e. splits into a smooth part and a divisorial part after pulling back to a blow-up.

Lemma 3.2. *Given a ray* $(\phi^s)_s$ *in* $\mathcal{H}$, *the non-Archimedean limit* $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$ *is unique, if it exists.*

Proof. Let ψ_1 and ψ_2 be non-Archimedean limits of $(\phi^s)_s$ and let Φ be the smooth metric on $L \times \Delta^*$ defined by the ray (ϕ^s). For $i = 1, 2$, pick a representative $(\mathcal{X}_i, \mathcal{L}_i)$ of ψ_i such that Φ extends as a smooth metric on $\mathcal{L}_i$ over Δ. After replacing $(\mathcal{X}_i, \mathcal{L}_i)$ by suitable pull-backs, we may assume that $\mathcal{X}_1 = \mathcal{X}_2 =: \mathcal{X}$ and $\mathcal{X}$ is normal. Then $\mathcal{L}_2 = \mathcal{L}_1 + D$ for a $\mathbb{Q}$-divisor D supported on $\mathcal{X}_0$. Now a smooth metric on $\mathcal{L}_1$ induces a singular metric on $\mathcal{L}_1 + D$ that is smooth iff $D = 0$. Hence $\mathcal{L}_1 = \mathcal{L}_2$, so that $\psi_1 = \psi_2$. □

Remark 3.3. Following [Berk09, §2] (see also [Jon16, BJ17]) one can construct a compact Hausdorff space X^{An} fibering over the interval $[0, 1]$ such that the fiber X_ρ^{An} over any point $\rho \in (0, 1]$ is homeomorphic to the complex manifold X, and the fiber X_0^{An} over 0 is homeomorphic to the Berkovich analytification of X with respect to the trivial norm on $\mathbb{C}$. Similarly, the line bundle L induces a line bundle L^{An} over X^{An}. If a ray $(\phi^s)_{s>0}$ admits a non-Archimedean limit ϕ^{NA}, then it induces a continuous metric on L^{An} whose restriction to L_ρ^{An} is given by $\phi^{\log \rho^{-1}}$ and whose restriction to X_0^{an} is given by ϕ^{NA}. In this way, ϕ^{NA} is indeed the limit of ϕ^s as $s \to +\infty$.

3.2. *Non-Archimedean limits of functionals*

For the rest of §3, assume that L is ample.

Definition 3.4. A functional $F\colon \mathcal{H} \to \mathbb{R}$ admits a functional $F^{\mathrm{NA}}\colon \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$ as a *non-Archimedean limit* if, for every smooth subgeodesic ray (ϕ^s) in $\mathcal{H}$ admitting a non-Archimedean limit $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$, we have

$$\lim_{s\to+\infty} \frac{F(\phi^s)}{s} = F^{\mathrm{NA}}(\phi^{\mathrm{NA}}). \tag{3.1}$$

Proposition 3.5. *If $F\colon \mathcal{H} \to \mathbb{R}$ admits a non-Archimedean limit $F^{\mathrm{NA}}\colon \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$, then F^{NA} is homogeneous.*

Proof. Consider a semiample test configuration $(\mathcal{X}, \mathcal{L})$ representing a non-Archimedean metric $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$, and let $(\phi^s)_s$ be a smooth subgeodesic ray admitting ϕ^{NA} as a non-Archimedean limit. For $d \geq 1$, let $(\mathcal{X}_d, \mathcal{L}_d)$ be the normalized base change induced by $\tau \to \tau^d$. The associated non-Archimedean metric ϕ_d^{NA} is then the non-Archimedean limit of the subgeodesic ray (ϕ^{ds}), so $\lim_{s\to+\infty} s^{-1} F(\phi_{ds}) = F^{\mathrm{NA}}(\phi_d^{\mathrm{NA}})$. On the other hand, we clearly have $\lim_{s\to+\infty}(ds)^{-1} F(\phi^{ds}) = \lim_{s\to+\infty} s^{-1} F(\phi^s) = F^{\mathrm{NA}}(\phi^{\mathrm{NA}})$. The result follows. □

3.3. *Asymptotics of the functionals*

The following result immediately implies Theorem A and Corollary B.

Theorem 3.6. *The functionals E, H, I, J, M and R on $\mathcal{H}$ admit non-Archimedean limits on $\mathcal{H}^{\mathrm{NA}}$ given, respectively, by E^{NA}, H^{NA}, I^{NA}, J^{NA}, M^{NA} and R^{NA}.*

In addition, we have the following result due to Berman [Berm16, Proposition 3.8]. See also [BBJ15, Theorem 3.1] for a more general result.

Theorem 3.7. *If $L := -K_X$ is ample, then the Ding functional D on $\mathcal{H}$ admits D^{NA} on $\mathcal{H}^{\mathrm{NA}}$ as non-Archimedean limit.*

Remark 3.8. In §4 we will extend the previous two results to the logarithmic setting and with relaxed positivity assumptions.

The main tool in the proof of Theorem 3.6 is the following result (compare [PRS08, Lemma 6]).

Lemma 3.9. *For $i = 0, \dots, n$, let L_i be a line bundle on X with a smooth reference metric $\phi_{i,\mathrm{ref}}$. Let also $(\mathcal{X}, \mathcal{L}_i)$ be a smooth test configuration for (X, L_i), Φ_i an S^1-invariant smooth metric on $\mathcal{L}_i$ near $\mathcal{X}_0$, and denote by (ϕ_i^s) the corresponding ray of smooth metrics on L_i. Then*

$$\langle \phi_0^s, \dots, \phi_n^s \rangle_X - \langle \phi_{0,\mathrm{ref}}, \dots, \phi_{n,\mathrm{ref}} \rangle_X = s(\bar{\mathcal{L}}_0 \cdot \dots \cdot \bar{\mathcal{L}}_n) + O(1)$$

as $s \to +\infty$. Here $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ is the compactification of $(\mathcal{X}, \mathcal{L}_i)$ for $0 \le i \le n$ and $\langle \cdot, \dots, \cdot \rangle_X$ denotes the Deligne pairing with respect to the constant morphism $X \to \{\mathrm{pt}\}$.

Proof. The Deligne pairing $F := \langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle_{\mathcal{X}/\mathbb{C}}$ is a line bundle on $\mathbb{C}$, endowed with a $\mathbb{C}^*$-action and a canonical identification of its fiber at $\tau = 1$ with the complex line $\langle L_0, \dots, L_n \rangle_X$. It extends to a line bundle $\langle \bar{\mathcal{L}}_0, \dots, \bar{\mathcal{L}}_n \rangle_{\bar{\mathcal{X}}/\mathbb{P}^1}$ on $\mathbb{P}^1$ that is $\mathbb{C}^*$-equivariantly trivial on $\mathbb{P}^1 \setminus \{0\}$. Denoting by $w \in \mathbb{Z}$ the weight of the $\mathbb{C}^*$-action on the fiber at 0, we have

$$w = \deg \langle \bar{\mathcal{L}}_0, \dots, \bar{\mathcal{L}}_n \rangle_{\bar{\mathcal{X}}/\mathbb{P}^1} = (\bar{\mathcal{L}}_0, \dots, \bar{\mathcal{L}}_n).$$

Pick a non-zero vector $v \in F_1 = \langle L_0, \dots, L_n \rangle_X$. The $\mathbb{C}^*$-action produces a section $\tau \mapsto \tau \cdot v$ of F on $\mathbb{C}^*$, and $\sigma := \tau^{-w}(\tau \cdot v)$ is a nowhere vanishing section of F on $\mathbb{C}$ (see [BHJ17, Corollary 1.4]).

Since the metrics Φ_i are smooth and S^1-invariant, $\Psi := \langle \Phi_0, \dots, \Phi_n \rangle_{\mathcal{X}/\mathbb{C}}$ is a continuous S^1-invariant metric on F near $0 \in \mathbb{C}$. Hence the function $\log |\sigma|_\Psi$ is bounded near $0 \in \mathbb{C}$.

The S^1-invariant metric Ψ defines a ray (ψ^s) of metrics on the line F_1 through $|v|_{\psi^s} = |\tau \cdot v|_{\Psi_\tau}$, for $s = \log |\tau|^{-1}$, where Ψ_τ is the restriction of Ψ to F_τ. Thus

$$\log |v|_{\psi^s} = \log |\tau \cdot v|_{\Psi_\tau} = w \log |\tau| + \log |\sigma|_{\Psi_\tau} = -sw + O(1).$$

By functoriality, the metric ψ^s on F_1 is nothing but the Deligne pairing $\langle \phi_0^s, \dots, \phi_n^s \rangle$. If we set $\psi_{\mathrm{ref}} = \langle \phi_{0,\mathrm{ref}}, \dots, \phi_{n,\mathrm{ref}} \rangle_X$, it therefore follows that

$$\langle \phi_0^s, \dots, \phi_n^s \rangle_X - \langle \phi_{0,\mathrm{ref}}, \dots, \phi_{n,\mathrm{ref}} \rangle_X = \log |v|_{\psi_{\mathrm{ref}}} - \log |v|_{\psi^s} = sw + O(1),$$

which completes the proof. □

Proof of Theorem 3.6. Let $(\phi^s)_s$ be a smooth subgeodesic ray in $\mathcal{H}$ admitting a non-Archimedean limit $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$. Pick a test configuration $(\mathcal{X}, \mathcal{L})$ representing ϕ^{NA} such that $\mathcal{X}$ is smooth and $\mathcal{X}_0$ has snc support. Thus $\mathcal{L}$ is relatively semiample and $(\phi^s)_s$ corresponds to a smooth S^1-invariant semipositive metric Φ on $\mathcal{L}$ over Δ. By Lemma 1.2, we have

$$(n+1)V(E(\phi^s) - E(\phi_{\mathrm{ref}})) = \langle \phi^s, \dots, \phi^s \rangle_X - \langle \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X.$$

From Lemma 3.9, it follows that

$$\lim_{s \to +\infty} \frac{E(\phi^s)}{s} = \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)V} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$

which proves the result for the Monge–Ampère energy E. The case of the functionals I, J and R is similarly a direct consequence of Lemmas 1.2 and 3.9. In view of the Chen–Tian formulas for M and M^{NA}, it remains to consider the case of the entropy functional H. In fact, it turns out to be easier to treat the functional $H + R$.

By Lemma 1.2 we have

$$V(H(\phi^s) + R(\phi^s)) = \left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \right\rangle_X - \langle \psi_{\mathrm{ref}}, \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X,$$

where $\psi_{\mathrm{ref}} = \frac{1}{2}\log \mathrm{MA}(\phi_{\mathrm{ref}})$, so we must show that

$$\left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \right\rangle_X - \langle \psi_{\mathrm{ref}}, \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X = s(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + o(s). \tag{3.2}$$

The collection of metrics $\frac{1}{2}\log \mathrm{MA}(\Phi|_{\mathcal{X}_\tau})$ with $\tau \neq 0$ defines a smooth metric Ψ on $K^{\log}_{\mathcal{X}/\mathbb{C}}$ over Δ^*, but the difficulty here (as opposed to the situation in [PRS08]) is that Ψ will not a priori extend to a smooth (or even locally bounded) metric on $K^{\log}_{\mathcal{X}/\mathbb{C}}$ over Δ. Indeed, since we have assumed that $\mathcal{X}$ is smooth, there is no reason for Φ to be strictly positive.

Instead, pick a smooth, S^1-invariant reference metric Ψ_{ref} on $K^{\log}_{\mathcal{X}/\mathbb{C}}$ over Δ, and denote by $(\psi^s_{\mathrm{ref}})_{s>0}$ the corresponding ray of smooth metrics on K_X. By Lemma 3.9 we have

$$\langle \psi^s_{\mathrm{ref}}, \phi^s, \dots, \phi^s \rangle_X - \langle \psi_{\mathrm{ref}}, \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X = s(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + O(1).$$

Since

$$\left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \right\rangle_X - \langle \psi^s_{\mathrm{ref}}, \phi^s, \dots, \phi^s \rangle_X = \frac{1}{2}\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n,$$

Theorem 3.6 is a consequence of the following result. □

Lemma 3.10. *We have* $\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n = O(\log s)$ *as* $s \to +\infty$.

Let us first prove an estimate of independent interest. See [BJ17] for more precise results.

Lemma 3.11. *Let $\mathcal{X}$ be an snc test configuration for X and Ψ a smooth metric on $K^{\log}_{\mathcal{X}/\mathbb{C}}$ near $\mathcal{X}_0$. Denote by $e^{2\Psi_\tau}$ the induced volume form on $\mathcal{X}_\tau$ for $\tau \neq 0$. Then*

$$\int_{\mathcal{X}_\tau} e^{2\Psi_\tau} \sim (\log|\tau|^{-1})^d \quad \text{as } \tau \to 0, \tag{3.3}$$

with d denoting the dimension of the dual complex of $\mathcal{X}_0$, so that $d+1$ is the largest number of local components of $\mathcal{X}_0$.

Here $A \sim B$ means that A/B is bounded from above and below by positive constants.

Proof. Since $\mathcal{X}_0$ is an snc divisor, every point of $\mathcal{X}_0$ admits local coordinates $(z_0, \dots, z_n)$ that are defined in a neighborhood of $B := \{|z_i| \le 1\}$ and such that $z_0^{b_0} \dots z_p^{b_p} = \varepsilon\tau$ with $0 \le p \le n$ and $\varepsilon > 0$. Here $b_i \in \mathbb{Z}_{>0}$ is the multiplicity of $\mathcal{X}_0$ along $\{z_i = 0\}$. The integer d in the statement of the theorem is then the largest such integer p. By compactness of $\mathcal{X}_0$, it will be enough to show that

$$\int_{B\cap\mathcal{X}_\tau} e^{2\Psi_\tau} \sim (\log|\tau|^{-1})^p.$$

The holomorphic n-form

$$\eta := \frac{1}{p+1}\sum_{j=0}^{p}\frac{(-1)^j}{b_j}\frac{dz_0}{z_0}\wedge\dots\wedge\widehat{\frac{dz_j}{z_j}}\wedge\dots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\dots\wedge dz_n$$

satisfies

$$\eta\wedge\frac{d\tau}{\tau} = \frac{dz_0}{z_0}\wedge\dots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\dots\wedge dz_n.$$

Thus η defines a local frame of $K^{\log}_{\mathcal{X}/\mathbb{C}}$ on B, so the holomorphic n-form $\eta_\tau := \eta|_{\mathcal{X}_\tau}$ satisfies

$$C^{-1}|\eta_\tau|^2 \le e^{2\Psi_\tau} \le C|\eta_\tau|^2$$

for a constant $C > 0$ independent of τ. Hence it suffices to prove $\int_{B\cap\mathcal{X}_\tau}|\eta_\tau|^2 \sim (\log|\tau|^{-1})^p$.

To this end, we parametrize $B\cap\mathcal{X}_\tau$ in (logarithmic) polar coordinates as follows. Consider the p-dimensional simplex

$$\sigma = \Big\{w\in\mathbb{R}^{p+1}_{\ge 0}\ \Big|\ \sum_{j=0}^{p} b_j w_j = 1\Big\},$$

the p-dimensional (possibly disconnected) commutative compact Lie group

$$T = \Big\{\theta\in(\mathbb{R}/\mathbb{Z})^{p+1}\ \Big|\ \sum_{j=0}^{p} b_j\theta_j = 0\Big\},$$

and the polydisc $\mathbb{D}^{n-p}\subset\mathbb{C}^{n-p}$. We may cover $\mathbb{C}^*$ by two simply connected open sets, on each of which we fix a branch of the complex logarithm. We then define a diffeomorphism χ_τ from $\sigma\times T\times\mathbb{D}^{n-p}$ to $B\cap\mathcal{X}_\tau$ by setting

$$z_j = e^{w_j\log(\varepsilon\tau)+2\pi i\theta_j}\quad\text{for } 0\le j\le p.$$

A simple computation shows that

$$\chi_\tau^*(|\eta_\tau|^2) = \text{const}\,(\log|\varepsilon\tau|^{-1})^p dV,$$

where dV denotes the natural volume form on $\sigma\times T\times\mathbb{D}^{n-p}$. It follows that, for $|\tau|\ll 1$,

$$\int_{B\cap\mathcal{X}_\tau}|\eta_\tau|^2 \sim \int_{\sigma\times T\times\mathbb{D}^{n-p}}\chi_\tau^*(|\eta_\tau|^2)\sim(\log|\tau|^{-1})^p,$$

which completes the proof. □

Proof of Lemma 3.10. On the one hand, we have

$$V^{-1}\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n$$
$$= \int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}/\int_X e^{2\psi^s_{\mathrm{ref}}}}\right]\mathrm{MA}(\phi^s) - \log\int_X e^{2\psi^s_{\mathrm{ref}}} \geq -\log\int_X e^{2\psi^s_{\mathrm{ref}}},$$

since the first term on the second line is the relative entropy of the probability measure $\mathrm{MA}(\phi^s)$ with respect to the probability measure $e^{2\psi^s_{\mathrm{ref}}}/\int_X e^{2\psi^s_{\mathrm{ref}}}$. By Lemma 3.11 we have $\int_X e^{2\psi^s_{\mathrm{ref}}} = O(s^d)$, where $0 \leq d \leq n$. This gives the lower bound in Lemma 3.10.

To get the upper bound, it suffices to prove that the function $g_\tau := \frac{(dd^c\Phi|_{\mathcal{X}_\tau})^n}{e^{2\Psi_\tau}}$ on $\mathcal{X}_\tau$ is uniformly bounded from above. Indeed, if $\tau = e^{-s}$, we then see that

$$\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n = \int_{\mathcal{X}_\tau}(\log V^{-1} + \log g_\tau)(dd^c\Phi|_{\mathcal{X}_\tau})^n$$

is uniformly bounded from above, since $(dd^c\Phi|_{\mathcal{X}_\tau})^n$ has fixed mass V for all τ.

To bound g_τ from above, we use local coordinates $(z_j)_{j=0}^n$ as in the proof of Lemma 3.11. With the notation in that proof, it suffices to prove that the function $(\Omega|_{\mathcal{X}_\tau})^n/e^{2\Psi_\tau}$ on $\mathcal{X}_\tau$ is uniformly bounded from above, where $\Omega := \frac{i}{2}\sum_{j=0}^n dz_j \wedge d\bar{z}_j$. Indeed, we have $dd^c\Phi \leq C\Omega$ for some constant $C > 0$. It then further suffices to prove the bound

$$i^n dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge \widehat{dz_j \wedge d\bar{z}_j} \wedge \cdots \wedge dz_n \wedge dz_n\big|_{\mathcal{X}_\tau} \leq Ce^{2\Psi_\tau} \tag{3.4}$$

for $0 \leq j \leq p$ and a uniform constant $C > 0$.

To prove (3.4) we use the logarithmic polar coordinates in the proof of Lemma 3.10. Namely, if $\chi_\tau : \sigma \times T \times \mathbb{D}^{n-p} \to B \cap X_\tau$ is the diffeomorphism in that proof, we have

$$\chi_\tau^*(e^{2\Psi_\tau}) \sim (\log|\tau|^{-1})^p dV,$$
$$\chi_\tau^*(i^n dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge \widehat{dz_j \wedge d\bar{z}_j} \wedge \cdots \wedge dz_n \wedge dz_n) \sim (\log|\tau|^{-1})^p \prod_{0\leq l\leq p,\, l\neq j}|z_l|^2 dV.$$

Thus (3.4) holds, which completes the proof. □

4. The logarithmic setting

In this section we extend, for completeness, Theorem 3.6—and hence Theorem A and Corollary B—to the logarithmic setting. We will also relax the positivity assumptions used. Our conventions and notation largely follow [BB+11].

4.1. Preliminaries

If X is a normal projective variety of dimension n, and $\phi_1, \ldots, \phi_n$ are smooth metrics on $\mathbb{Q}$-line bundles $L_1, \ldots, L_n$ on X, then we define $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ as the push-forward of the measure $dd^c\phi_1|_{X_{\mathrm{reg}}} \wedge \cdots \wedge dd^c\phi_n|_{X_{\mathrm{reg}}}$ from X_{reg} to X. This is a signed Radon measure of total mass $(L_1 \cdot \ldots \cdot L_n)$, positive if the ϕ_i are semipositive.

A *boundary* on X is a Weil $\mathbb{Q}$-divisor B on X such that the Weil $\mathbb{Q}$-divisor class

$$K_{(X,B)} := K_X + B$$

is $\mathbb{Q}$-Cartier. Note that B is not necessarily effective. We call (X, B) a *pair*.

The *log discrepancy* of a rational divisorial valuation $v = c\,\mathrm{ord}_F$ with respect to (X, B) is defined as in §2.3, using $A_{(X,B)}(v) = c(1 + \mathrm{ord}_F(K_{Y/(X,B)}))$ and $A_{(X,B)}(v_{\mathrm{triv}}) = 0$. The pair (X, B) is *subklt* if $A_{(X,B)}(v) > 0$ for all rational divisorial valuations v. (It is klt when B is further effective.)

A pair (X, B) is *log smooth* if X is smooth and B has simple normal crossing support. A *log resolution* of (X, B) is a projective birational morphism $f: X' \to X$, with X' smooth, such that $\mathrm{Exc}(f) + f_*^{-1}(B)$ has simple normal crossing support. In this case, there is a unique snc divisor B' on X' such that $f_*B' = B$ and $K_{(X',B')} = f^*K_{(X,B)}$. In particular the pair (X', B') is log smooth. The pair (X, B) is subklt iff (X', B') is subklt, and the latter is equivalent to B' having coefficients < 1.

A smooth metric ψ on $K_{(X,B)}$ canonically defines a smooth positive measure μ_ψ on $X_{\mathrm{reg}} \setminus B$ as follows. Let ϕ_B be the canonical singular metric on $\mathcal{O}_{X_{\mathrm{reg}}}(B)$, with curvature current given by $[B]$. Then $\psi - \phi_B$ is a smooth metric on $K_{X_{\mathrm{reg}}\setminus B}$, and hence induces a smooth positive measure

$$\mu_\psi := e^{2(\psi - \phi_B)}$$

on $X_{\mathrm{reg}} \setminus B$. The fact that (X, B) is subklt means precisely that the total mass of μ_ψ is finite. Thus we can view μ_ψ as a finite positive measure on X that is smooth on $X_{\mathrm{reg}} \setminus B$ and gives no mass to B or X_{sing}.

4.2. *Archimedean functionals*

Let X be a normal complex projective variety of dimension n. Fix a big and nef $\mathbb{Q}$-line bundle L on X and set $V := (L^n) > 0$. For a smooth metric ϕ on L, set $\mathrm{MA}(\phi) = V^{-1}(dd^c\phi)^n$.

Fix a smooth positive reference metric ϕ_{ref} on L The energy functionals E, I and J are defined on smooth metrics on L exactly as in (1.6), (1.8) and (1.7), respectively; they are normalized by $E(\phi_{\mathrm{ref}}) = I(\phi_{\mathrm{ref}}) = J(\phi_{\mathrm{ref}}) = 0$. The functionals I and J are translation invariant, whereas $E(\phi + c) = E(\phi) + c$. All three functionals are pull-back invariant in the following sense. Let $q: X' \to X$ be a birational morphism, with X' normal and projective, and set $L' := q^*L$. For any smooth metric ϕ on L, we have $E(\phi') = E(\phi)$, $I(\phi') = I(\phi)$ and $J(\phi') = J(\phi)$, where $\phi' = q^*\phi$ and where the functionals are computed with respect to the reference metric $\phi'_{\mathrm{ref}} := q^*\phi_{\mathrm{ref}}$.

Now consider a boundary B on X. Set $\bar{S}_B := -nV^{-1}(K_{(X,B)} \cdot L^{n-1})$ and fix a smooth reference metric ψ_{ref} on $K_{(X,B)}$. When X is smooth and $B = 0$, we could pick $\psi_{\mathrm{ref}} = \frac{1}{2}\log \mathrm{MA}(\phi_{\mathrm{ref}})$, but in general there seems to be no canonical way to get ψ_{ref} from ϕ_{ref}.

The analogue of the Ricci energy R is defined on smooth metrics ϕ on L by

$$R_B(\phi) := \sum_{j=0}^{n-1} \frac{1}{V} \int_{X_{\mathrm{reg}}} (\phi - \phi_{\mathrm{ref}}) dd^c\psi_{\mathrm{ref}} \wedge (dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-1-j}.$$

It satisfies $R_B(\phi + c) = R_B(\phi) - \bar{S}_B c$ and is pull-back invariant in the following sense. Suppose $q: X' \to X$ is a birational morphism, with X' projective normal, and define B' by $q_* B' = B$ and $q^* K_{(X,B)} = K_{(X',B')}$. Set $\phi'_{\mathrm{ref}} = q^* \phi_{\mathrm{ref}}$ and $\psi'_{\mathrm{ref}} := q^* \psi_{\mathrm{ref}}$. Then $R_B(\phi) = R_{B'}(\phi')$, where $\phi' = q^* \phi$.

Now assume (X, B) is subklt and let $\mu_{\mathrm{ref}} = \mu_{\psi_{\mathrm{ref}}}$ be the finite positive measure defined in §4.1. It is smooth and positive on $X_{\mathrm{ref}} \setminus B$, and may be assumed to have mass 1, after adding a constant to ψ_{ref}. For a smooth semipositive metric ϕ on L, set

$$H_B(\phi) := \frac{1}{2} \int_{X_{\mathrm{reg}}} \log \frac{\mathrm{MA}(\phi)}{\mu_{\mathrm{ref}}} \mathrm{MA}(\phi) = \frac{1}{2} \int_{X_{\mathrm{reg}}} \log \frac{\mathrm{MA}(\phi)}{e^{2(\psi_{\mathrm{ref}} - \phi_B)}} \mathrm{MA}(\phi).$$

We may have $H_B(\phi_{\mathrm{ref}}) \neq 0$. However, H_B is bounded from below and translation invariant. It is also pull-back invariant in the sense above, with reference measure $\mu'_{\mathrm{ref}} = \mu_{\psi'_{\mathrm{ref}}}$ on X'.

Lemma 4.1. *If ϕ is a smooth semipositive metric on L, then $H_B(\phi) < \infty$.*

Proof. By pull-back invariance we may assume that (X, B) is log smooth. In this case $\mathrm{MA}(\phi)$ and μ_{ref} are smooth measures on X that are strictly positive on X_{reg}. Consider any point $\xi \in B$ and pick local coordinates $(z_1, \dots, z_n)$ at ξ such that the irreducible components of B are given by $\{z_i = 0\}$, $0 \leq i \leq p$. Fix a volume form dV near ξ. Then $\mu_{\mathrm{ref}} = g \prod_{i=0}^{p} |z_i|^{2a_i} dV$, and $\mathrm{MA}(\phi) = h dV$, with $a_i > -1$, $g > 0$ and $h \geq 0$ smooth. If $f = h \log\big(\frac{h}{g} \prod_{i=0}^{p} |z_i|^{-2a_i}\big)$, then f is locally integrable with respect to dV. This completes the proof. □

As in §1.4, we define the Mabuchi functional on semipositive smooth metrics by

$$M_B := H_B + R_B + \bar{S}_B E.$$

Then M_B is translation invariant and pull-back invariant in the sense above. At least formally, the critical points of M_B satisfy

$$n(\mathrm{Ric}(dd^c\phi) - [B]) \wedge (dd^c\phi)^{n-1} = \bar{S}_B (dd^c\phi)^n$$

and should be conical cscK metrics (see [Li18]).

Finally consider the (weak) *log Fano case*, in which $L := -K_{(X,B)}$ is big and nef. The Ding functional is then defined on smooth metrics as $D_B = L_B - E$, with

$$L_B(\phi) := -\frac{1}{2} \log \int_{X_{\mathrm{reg}}} e^{-2(\phi + \phi_B)}.$$

If we use $\psi_{\mathrm{ref}} = -\phi_{\mathrm{ref}}$, then the formula for the Mabuchi functional simplifies to

$$M_B(\phi) = H_B(\phi) - \left(E(\phi) - \int_{X_{\mathrm{reg}}} (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi) \right).$$

We have $D_B \leq M_B$ on smooth semipositive metrics.

4.3. *Non-Archimedean functionals*

The extensions of the non-Archimedean functionals in §2.7 to the logarithmic setting were studied in [BHJ17, §7]. Let us briefly review them.

Consider a normal complex projective variety X and a big and nef $\mathbb{Q}$-line bundle L on X. Let ϕ be a non-Archimedean metric on L, represented by a normal test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), that we assume dominates $(X \times \mathbb{C}, L \times \mathbb{C})$ via $\rho \colon \mathcal{X} \to X \times \mathbb{C}$. The formulas in §2.7 for $E^{\mathrm{NA}}(\phi)$, $I^{\mathrm{NA}}(\phi)$ and $J^{\mathrm{NA}}(\phi)$ are still valid.

Given a boundary B on X we set

$$R_B^{\mathrm{NA}}(\phi) := V^{-1}(\psi_{\mathrm{triv}} \cdot \phi^n) = V^{-1}(\rho^* K^{\log}_{(X\times\mathbb{P}^1, B\times\mathbb{P}^1)/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n).$$

Now assume (X, B) is subklt and let $\mathcal{B}$ (resp. $\bar{\mathcal{B}}$) be the (componentwise) Zariski closure of $B \times \mathbb{C}^*$ in $\mathcal{X}$ (resp. $\bar{\mathcal{X}}$). Then

$$\begin{aligned} H_B^{\mathrm{NA}}(\phi) &:= \int_{X_{\mathbb{Q}}^{\mathrm{div}}} A_{(X,B)}(v)\, \mathrm{MA}^{\mathrm{NA}}(\phi) \\ &= V^{-1}(K^{\log}_{(\bar{\mathcal{X}}, \bar{\mathcal{B}})/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) - V^{-1}(\rho^* K^{\log}_{(X\times\mathbb{P}^1, B\times\mathbb{P}^1)/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n). \end{aligned}$$

and

$$\begin{aligned} M_B^{\mathrm{NA}}(\phi) &:= H_B^{\mathrm{NA}}(\phi) + R_B^{\mathrm{NA}}(\phi) + \bar{S}_B E^{\mathrm{NA}}(\phi) \\ &= \frac{1}{V}(K^{\log}_{(\bar{\mathcal{X}}, \bar{\mathcal{B}})/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + \frac{\bar{S}_B}{(n+1)V}(\bar{\mathcal{L}}^{n+1}). \end{aligned}$$

While the definitions of $H_B^{\mathrm{NA}}(\phi)$ and $M_B^{\mathrm{NA}}(\phi)$ make sense for arbitrary non-Archimedean metrics ϕ, we will usually assume that ϕ is semipositive.

All the functionals above have the same invariance properties as their Archimedean cousins. They are also homogeneous in the sense of Definition 2.7.

Finally, when (X, B) is weakly log Fano, so that (X, B) is subklt and $L := -K_{(X,B)}$ is big and nef, the non-Archimedean Ding functional is defined by

$$D_B^{\mathrm{NA}}(\phi) = L_B^{\mathrm{NA}}(\phi) - E^{\mathrm{NA}}(\phi),$$

where

$$L_B^{\mathrm{NA}}(\phi) = \inf_v \big(A_{(X,B)}(v) + (\phi - \phi_{\mathrm{triv}})(v)\big),$$

the infimum taken over all valuations $v \in X_{\mathbb{Q}}^{\mathrm{div}}$.

The Ding functional D_B^{NA} is translation invariant and pull-back invariant. The formula for the Mabuchi functional simplifies in the log Fano case to

$$M_B^{\mathrm{NA}}(\phi) = H_B^{\mathrm{NA}}(\phi) - \left(E^{\mathrm{NA}}(\phi) - \int_{X_{\mathbb{Q}}^{\mathrm{div}}} (\phi - \phi_{\mathrm{ref}})\, \mathrm{MA}^{\mathrm{NA}}(\phi)\right).$$

We have $D_B^{\mathrm{NA}} \le \min\{M_B^{\mathrm{NA}}, J^{\mathrm{NA}}\}$ on semipositive metrics (see [BHJ17, Propositions 7.28 and 7.32].

4.4. Asymptotics

The following result generalizes Theorem 3.6 and shows that if F is one of the functionals E, I, J, H_B, R_B or M_B on $\mathcal{H}$, then F admits a non-Archimedean limit on $\mathcal{H}^{\mathrm{NA}}$ given by F^{NA}. For future reference, we state the result in detail.

Theorem 4.2. *Let X be a normal projective variety, L a big and nef $\mathbb{Q}$-line bundle on X, and $(\mathcal{X}, \mathcal{L})$ a test configuration for (X, L) inducing a non-Archimedean metric ϕ^{NA} on L. Further, let Φ be a smooth, S^1-invariant metric on $\mathcal{L}$ near $\mathcal{X}_0$, inducing a smooth ray $(\phi^s)_{s>s_0}$ of metrics on L. Fix a smooth reference metric ϕ_{ref} on L. Then*

$$\lim_{s\to+\infty} \frac{F(\phi^s)}{s} = F^{\mathrm{NA}}(\phi^{\mathrm{NA}}), \tag{4.1}$$

where F is any of the functionals E, I, J.

Further, if B is a boundary on X and ψ_{ref} is a smooth reference metric on $K_{(X,B)}$, then (4.1) also holds for $F = R_B$. Finally, if (X, B) is subklt and Φ is semipositive, then (4.1) holds for $F = H_B$ and $F = M_B$.

In addition, Berman proved that in the log Fano case, the Ding functional D_B admits D_B^{NA} as non-Archimedean limit. Indeed, the following result follows from [Berm16, Proposition 3.8 and §4.3].

Theorem 4.3. *Let (X, B) be a subklt pair with $L := -K_{(X,B)}$ big and nef, $(\mathcal{X}, \mathcal{L})$ a test configuration for (X, L) inducing a non-Archimedean metric ϕ^{NA} on L, and Φ a semipositive smooth, S^1-invariant metric on $\mathcal{L}$ near $\mathcal{X}_0$, inducing a smooth ray $(\phi^s)_{s>s_0}$ of semipositive metrics on L. Then $\lim_{s\to+\infty} \frac{1}{s} D_B(\phi^s) = D_B^{\mathrm{NA}}(\phi^{\mathrm{NA}})$.*

In fact, it is enough to assume Φ is semipositive and locally bounded in Theorem 4.3.

Remark 4.4. Theorems 4.2 and 4.3 remain true even when Φ is not S^1-invariant, in the following sense. For $\tau \in \Delta^*$, let ϕ_τ be the metric on L defined as the pull-back of $\Phi|_{\mathcal{X}_\tau}$ under the $\mathbb{C}^*$-action. Then $\lim_{\tau\to 0} (\log|\tau|^{-1})^{-1} F(\phi_\tau) = F^{\mathrm{NA}}(\phi^{\mathrm{NA}})$.

4.5. Proof of Theorem 4.2

By pull-back invariance, we may assume that X is smooth. After further pull-back, we may also assume that $\mathcal{X}$ is smooth and dominates $X \times \mathbb{C}$. In this case, the asymptotic formulas for E, I and J follow immediately from Lemma 3.9.

When considering the remaining functionals, we may similarly, by pull-back invariance, assume that the pair (X, B) is log smooth. The asymptotic formula for R_B now follows from Lemma 3.9 since we can express $R_B(\phi)$ in terms of Deligne pairings:

$$R_B(\phi) = \langle \psi_{\mathrm{ref}}, \phi^n \rangle_X - \langle \psi_{\mathrm{ref}}, \phi_{\mathrm{ref}}^n \rangle_X,$$

whereas the non-Archimedean counterpart is given by the intersection number

$$R_B^{\mathrm{NA}}(\phi) = V^{-1} (\rho^* K^{\log}_{(X\times\mathbb{P}^1, B\times\mathbb{P}^1)/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n)_{\bar{\mathcal{X}}}.$$

Finally we consider the functionals H_B and M_B. Thus assume (X, B) is log smooth and subklt. We may further assume that the divisor $\mathcal{X}_0 + \mathcal{B}$ has simple normal crossing support, where $\mathcal{B}$ is the (componentwise) Zariski closure of the pull-back of $B \times \mathbb{C}^*$ in $\mathcal{X}$.

As in §3.3 it suffices to prove the asymptotic formula for the functional $H_B + R_B$. To this end, we express H_B in terms of Deligne pairings. Write $B = \sum_i c_i B_i$, where B_i, $i \in I$, are the irreducible components of B and $c_i \in \mathbb{Q}$. Fix a smooth metric ψ_i on $\mathcal{O}_X(B_i)$ for $i \in I$. Then $\psi_B := \sum_i c_i \psi_i$ is a smooth metric on $\mathcal{O}_X(B)$, and it follows from (1.3) that

$$\begin{aligned} V H_B(\phi) &= \frac{1}{2}\int_X \log \frac{\mathrm{MA}(\phi)}{e^{2(\psi_{\mathrm{ref}}-\psi_B)}}(dd^c\phi)^n + \sum_{i\in I} c_i \int_X \log|\sigma_i|_{\psi_i}(dd^c\phi)^n \\ &= \left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi), \phi^n\right\rangle_X - \langle \psi_{\mathrm{ref}}, \phi^n\rangle_X + \langle \psi_B, \phi^n\rangle_X + \sum_{i\in I} c_i(\langle \phi^n\rangle_{B_i} - \langle \psi_i, \phi^n\rangle_X) \\ &= \left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi), \phi^n\right\rangle_X - \langle \psi_{\mathrm{ref}}, \phi^n\rangle_X + \sum_{i\in I} c_i \langle \phi^n\rangle_{B_i} \end{aligned}$$

for any smooth semipositive metric ϕ on L. This implies

$$\begin{aligned} V(H_B(\phi) + R_B(\phi)) &= \left\langle \tfrac{1}{2}\log \mathrm{MA}(\phi), \phi^n\right\rangle_X - \langle \psi_{\mathrm{ref}}, \phi_{\mathrm{ref}}^n\rangle_X + \sum_{i\in I} c_i \langle \phi^n\rangle_{B_i} \\ &= V(H(\phi) + R(\phi)) + n\sum_{i\in I} c_i (L^{n-1}\cdot B_i) E(\phi|_{B_i}) + O(1). \end{aligned}$$

On the non-Archimedean side, we have

$$\begin{aligned} V(H_B^{\mathrm{NA}}(\phi^{\mathrm{NA}}) + R_B^{\mathrm{NA}}(\phi^{\mathrm{NA}})) &= (K^{\log}_{(\bar{\mathcal{X}},\bar{\mathcal{B}})/\mathbb{P}^1}\cdot \bar{\mathcal{L}}^n)_{\bar{\mathcal{X}}} = (K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1}\cdot \bar{\mathcal{L}}^n)_{\bar{\mathcal{X}}} + (\bar{\mathcal{B}}\cdot \bar{\mathcal{L}}^n)_{\bar{\mathcal{X}}} \\ &= V(H^{\mathrm{NA}}(\phi^{\mathrm{NA}}) + R^{\mathrm{NA}}(\phi^{\mathrm{NA}})) + \sum_{i\in I} c_i (\bar{\mathcal{L}}|^n_{\bar{\mathcal{B}}_i})_{\bar{\mathcal{B}}_i} \\ &= V(H^{\mathrm{NA}}(\phi^{\mathrm{NA}}) + R^{\mathrm{NA}}(\phi^{\mathrm{NA}})) + n\sum_{i\in I} c_i (L^{n-1}\cdot B_i) E^{\mathrm{NA}}(\phi_i^{\mathrm{NA}}), \end{aligned}$$

where ϕ_i^{NA} is the non-Archimedean metric on $L|_{B_i}$ represented by $\mathcal{L}|_{\mathcal{B}_i}$.

It now follows from Theorem 3.6 that[1]

$$\lim_{s\to+\infty} \frac{1}{s}(H(\phi^s) + R(\phi^s)) = H^{\mathrm{NA}}(\phi^{\mathrm{NA}}) + R(\phi^{\mathrm{NA}}).$$

Applying Theorem 3.6 on B_i and $\mathcal{B}_i$, we also get $\lim_{s\to+\infty} \frac{1}{s}E(\phi_i^s) = E^{\mathrm{NA}}(\phi_i^{\mathrm{NA}})$. Thus

$$\lim_{s\to+\infty} \frac{1}{s}(H_B(\phi^s) + R_B(\phi^s)) = H_B^{\mathrm{NA}}(\phi^{\mathrm{NA}}) + R_B(\phi^{\mathrm{NA}}),$$

which completes the proof of Theorem 4.2.

[1] While Theorem 3.6 is stated in the case when L and $\mathcal{L}$ are ample and Φ is positive, the proof extends to the weaker positivity assumptions used here.

4.6. Coercivity and uniform K-stability

Let us finally extend Corollary B to the logarithmic setting. Consider a pair (X, B) and a big and nef line bundle L on X. The Donaldson–Futaki invariant of a normal test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is given by

$$\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) := \frac{1}{V}(K_{(\bar{\mathcal{X}}.\bar{\mathcal{B}})/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) + \bar{S}_B \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)V} = M_B^{\mathrm{NA}}(\phi) + \frac{1}{V}((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \mathcal{L}^n),$$

where ϕ is the non-Archimedean metric on L represented by ϕ. Now assume L is ample. We then define $((X, B); L)$ to be *uniformly K-stable* if the following two equivalent conditions hold:

(i) there exists $\delta > 0$ such that $M_B^{\mathrm{NA}}(\phi) \geq \delta J^{\mathrm{NA}}(\phi)$ for every $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$;
(ii) there exists $\delta > 0$ such that $\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for any normal ample test configuration $(\mathcal{X}, \mathcal{L})$.

The equivalence between the two conditions is proved in [BHJ17, Proposition 8.2].

Corollary 4.5. *Let (X, B) be a subklt pair and L an ample line bundle on X. Suppose that the Mabuchi functional is coercive in the sense that there exist positive constants δ and C such that $M_B(\phi) \geq \delta J(\phi) - C$ for every positive smooth metric ϕ on L. Then $((X, B); L)$ is uniformly K-stable; more precisely* $\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) \geq M_B(\phi) \geq \delta J^{\mathrm{NA}}(\phi)$ *for every positive non-Archimedean metric on L, where $(\mathcal{X}, \mathcal{L})$ is the unique normal ample representative of ϕ.*

5. Uniform K-stability and CM-stability

From now on, X is smooth. In this section we explore the relationship between uniform K-stability and (asymptotic) CM-stability. In particular we prove Theorem C, Corollary D and Corollary E.

5.1. Functions with log norm singularities

In this section, G denotes a reductive complex algebraic group.

Definition 5.1. We say that a function $f : G \to \mathbb{R}$ has *log norm singularities* if there exist finitely many rational numbers a_i, finite-dimensional complex vector spaces V_i endowed with an algebraic G-action and non-zero vectors $v_i \in V_i$ such that

$$f(g) = \sum_i a_i \log \|g \cdot v_i\| + O(1)$$

for some choice of norms on the V_i's.

Remark 5.2. By the equivalence of norms on a finite-dimensional vector space, the description of f is independent of the choice of norms on the V_i. In particular, given a maximal compact subgroup K of G, the norms may be assumed to be K-invariant, so that f descends to a function on the Riemannian symmetric space G/K.

Remark 5.3. Taking appropriate tensor products, it is easy to see that every function f on G with log norm singularities may be written as

$$f(g) = a(\log \|g \cdot v\| - \log \|g \cdot w\|) + O(1), \tag{5.1}$$

where $a \in \mathbb{Q}_{>0}$ and v, w are vectors in a normed vector space V endowed with a G-action.

An algebraic group homomorphism $\lambda : \mathbb{C}^* \to G$ is called a *one-parameter subgroup* (1-PS for short). The following generalization of the Kempf–Ness/Hilbert–Mumford criterion is closely related to [Pau13]. Our argument, which is based on Mumford's original proof of the Hilbert–Mumford criterion [MFF, §2.1], fixes in particular the proof of [Pau13, Theorem 4.2], as well as an incorrect argument provided in a previous version of the present paper.

Theorem 5.4. *Let f be a function on G with log norm singularities.*

(i) *For each* 1*-PS* $\lambda\colon \mathbb{C}^* \to G$*, there exists* $f^{\mathrm{NA}}(\lambda) \in \mathbb{Q}$ *such that*

$$(f \circ \lambda)(\tau) = f^{\mathrm{NA}}(\lambda) \log |\tau|^{-1} + O(1) \quad \textit{for } |\tau| \le 1.$$

(ii) *The function f is bounded below on G iff $f^{\mathrm{NA}}(\lambda) \ge 0$ for all* 1*-PS* λ*.*

The chosen notation stems from the fact that f^{NA} induces a function on the (conical) Tits building of G, i.e. the non-Archimedean analogue of G/K (compare [MFF, §2.2]).

By Remark 5.3 we may and do assume that f is of the form

$$f(g) := \log \|g \cdot v\| - \log \|g \cdot w\|,$$

where v, w are nonzero vectors in a finite-dimensional normed vector space V equipped with a linear G-action. In that case, the following variant of the Kempf–Ness criterion, observed in [Pau13, Proposition 4], translates Theorem 5.4 into an algebro-geometric statement.

Lemma 5.5. *The function $f(g) = \log \|g \cdot v\| - \log \|g \cdot w\|$ is bounded below on G if and only if the Zariski closure of the orbit of $[v, w] \in \mathbb{P}(V \oplus W)$ does not intersect the subspace $\mathbb{P}(\{0\} \oplus W)$.*

Proof. As with any algebraic group action, the orbit $G \cdot [v, w]$ is a complex algebraic subvariety of $\mathbb{P}(V \oplus W)$, i.e. a locally closed subset in the Zariski topology. Its Zariski closure therefore coincides with its closure in the Euclidean topology, and the argument is then elementary. Indeed, assume $f(g_i) \to -\infty$ for some sequence $g_i \in G$, i.e. $\|g_i \cdot v\| = o(\|g_i \cdot w\|)$. After passing to a subsequence, $\tilde{w}_i := (g_i \cdot w)/\|g_i \cdot w\|$ converges (in the Euclidean topology) to a non-zero vector in W, while $\tilde{v}_i := (g_i \cdot v)/\|g_i \cdot w\|$ tends to 0; hence $g_i \cdot [v, w] \in G \cdot [v, w]$ converges to $[0, \tilde{w}] \in \mathbb{P}(\{0\} \oplus W)$. Conversely, if $g_i \cdot [v, w] \to [0, \tilde{w}]$ for some sequence $g_i \in G$ and non-zero $\tilde{w} \in W$, then $c_i(g_i \cdot v) \to 0$ in V and $c_i(g_i \cdot w) \to \tilde{w}$ in W with $c_i \in \mathbb{C}^*$, and hence $f(g_i) = \log \|c_i(g_i \cdot v)\| - \log \|c_i(g_i \cdot w)\| \to -\infty$. □

The key ingredient in the proof of Theorem 5.4 is the following algebro-geometric result, which will be obtained as a consequence of the Iwahori decomposition theorem, very much as in [MFF].

Theorem 5.6. *Let G be a complex reductive group with a linear action on a finite-dimensional complex vector space U. If the (Zariski) closure of the G-orbit of a point $x \in \mathbb{P}(U)$ meets a G-invariant Zariski closed subset $Z \subset \mathbb{P}(U)$, then some $z \in Z \cap \overline{G \cdot x}$ can be reached by a 1-PS λ of G, i.e. $\lim_{\tau \to 0} \lambda(\tau) \cdot x = z$.*

Remark 5.7. As explained in [Don12, §5], it is however not true in general that *any* $z \in Z \cap \overline{G \cdot x}$ can be reached by a 1-PS λ, unless the stabilizer of z in G is reductive.

Introduce the formal power series ring $R = \mathbb{C}[\![t]\!]$ and its fraction field $K := \mathbb{C}(\!(t)\!)$, and let X be a complex algebraic variety. Viewed as a $\mathbb{C}$-scheme, X is separated, and thus the set $X(R)$ of R-points, i.e. morphisms $\gamma : \operatorname{Spec} R \to X$ over $\operatorname{Spec} \mathbb{C}$, injects into $X(K)$. Further, each $\gamma \in X(R)$ admits a reduction $\tilde{\gamma} \in X(\mathbb{C})$. If X is proper (i.e. $X(\mathbb{C})$ is compact), the valuative criterion yields $X(R) = X(K)$, which means that any 'meromorphic arc' $\gamma : \operatorname{Spec} K \to X$ uniquely extends across the closed point of $\operatorname{Spec} R$, whose image is $\tilde{\gamma}$. In case $X = \mathbb{P}(U)$ for a complex vector space U, this becomes very concrete: for each $\gamma \in X(K) = \mathbb{P}(U_K)$, there exists $u \in U_R$, unique up to multiplication by a unit of R, such that $\gamma = [u]$ and $\tilde{u} \neq 0$ in U, and we then have $\tilde{\gamma} = [\tilde{u}]$.

The following valuative criterion was used without precise reference in Mumford's proof of the Hilbert–Mumford criterion [MFF, p. 54]. We provide here some details (see [Ant, §4] for a closely related discussion).

Lemma 5.8. *Let $\phi : Y \to X$ be a morphism between complex algebraic varieties, and let $x \in X(\mathbb{C})$ be a closed point. Then x belongs to the (Zariski) closure of the image $\phi(Y)$ if and only if there exists $\gamma \in Y(K)$ with $\phi(\gamma) \in X(R)$ and $\widetilde{\phi(\gamma)} = x$.*

Proof. The condition is clearly sufficient. Assume conversely that x is in the Zariski closure of $\phi(Y)$. Replacing X with the closure of $\phi(Y)$, we may assume that ϕ is dominant. By Chevalley's theorem, $\phi(Y)$ is constructible, i.e. a finite union of locally closed subsets; being dense in X, it thus contains a non-empty open subset $U \subset X$. Using for instance Noether normalization, it is easy to construct a closed point $p \in C$ on a smooth algebraic curve and a morphism $f : C \to X$ with $f(p) = x$ and $f^{-1}(U)$ non-empty [Kem, Lemma 7.2.1]. It follows that the restriction of the induced morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to X$ to the generic point lifts to Y, and passing to the formal completion of C at p yields the result. □

Proof of Theorem 5.6. The action of G on $X := \mathbb{P}(U)$, being algebraic, induces an action of the group $G(K)$ on the set $X(K)$. Since K is an extension of $\mathbb{C}$, the closed point $x \in X$ can be viewed as an element of $X(K)$, and our goal is to find a point $\lambda \in G(K)$ corresponding to a one-parameter subgroup of G such that the reduction of $\lambda \cdot x \in X(K)$ belongs to Z.

Given any 1-PS $\lambda \in G(K)$ and $\xi \in X(K)$, we first claim that the reduction of $\lambda \cdot \xi \in X(K)$ only depends on $\tilde{\xi} \in X(\mathbb{C})$. Indeed, denote by $U = \bigoplus_{m \in \mathbb{Z}} U_m$ the weight decomposition with respect to λ. As mentioned above, there exists $u \in U_R$, unique up to a unit in R, such that $\xi = [u]$ and $\tilde{u} \neq 0$. The reduction of

$$\lambda \cdot \xi = \Big[\sum_m t^m u_m\Big]$$

is equal to $[\widetilde{u_p}]$ with $p := \min\{m \mid \widetilde{u_m} \neq 0\}$, and hence only depends on $\tilde{\xi} = [\tilde{u}]$.

Now let $\phi : G \to X$ be the orbit morphism $\phi(g) = g \cdot x$. By assumption, $\phi(G)$ contains a closed point $z \in Z$ in its Zariski closure, and Lemma 5.8 thus implies the existence of $\gamma \in G(K)$ such that the reduction of $\phi(\gamma) = \gamma \cdot x \in X(K)$ is equal to z.

By Iwahori's theorem (cf. [MFF, p. 52]), we can find a decomposition $\gamma = \alpha\lambda\beta$ in $G(K)$ with $\alpha, \beta \in G(R)$ and $\lambda \in G(K)$ induced by a 1-PS. By G-invariance of Z, the reduction of $(\lambda\beta) \cdot x$ belongs to Z. After replacing λ with $\tilde{\beta}\lambda\tilde{\beta}^{-1}$ and β with $\tilde{\beta}^{-1}\beta$, we may assume that $\tilde{\beta} = e \in G(\mathbb{C})$. As a result, $\tilde{x} = \widetilde{\beta \cdot x}$, and the above claim implies that $\widetilde{\lambda \cdot x} = \widetilde{(\lambda\beta) \cdot x}$ belongs to Z. □

Proof of Theorem 5.4. (i) Let $\lambda \colon \mathbb{C}^* \to G$ be a one-parameter subgroup, and denote by $V = \bigoplus_{m \in \mathbb{Z}} V_m$ the corresponding weight decompositon. For $\tau \in \mathbb{C}^*$, we then have

$$\lambda(\tau) \cdot v = \sum_m \tau^m v_m,$$

and hence

$$\log \|\lambda(\tau) \cdot v\| = \max_{v_m \neq 0}(m \log|\tau| + \log\|v_m\|) + O(1) = -\Big(\min_{v_m \neq 0} m\Big) \log|\tau|^{-1} + O(1)$$

for $|\tau| \leq 1$. This proves (i) with with $f^{\mathrm{NA}}(\lambda) = \min\{m \mid w_m \neq 0\} - \min\{m \mid v_m \neq 0\}$.

(ii) By (i), $f^{\mathrm{NA}}(\lambda) \geq 0$ for all 1-PS λ if and only if $f \circ \lambda$ is bounded below on $\mathbb{C}^*$ for all λ. By Lemma 5.5, $f \circ \lambda$ is bounded below on $\mathbb{C}^*$ iff $\lim_{\tau \to 0} \lambda(\tau) \cdot [v, w]$ does not belong to the G-invariant Zariski closed subset $Z := \mathbb{P}(\{0\} \oplus W)$, while f is bounded below on G iff $Z \cap \overline{G \cdot [v, w]} = \emptyset$. The equivalence now follows from Theorem 5.6. □

5.2. *Proof of Theorem C and Corollaries D and E*

Replacing L with mL, we may assume for notational simplicity that $m = 1$. Set $N := h^0(L)$ and $G := \mathrm{SL}(N, \mathbb{C})$, so that each $\sigma \in G$ defines a Fubini–Study type metric ϕ_σ on L. Note that $M - \delta J$ is bounded below on $\mathcal{H}_1 \simeq \mathrm{GL}(N, \mathbb{C})/\mathrm{U}(N)$ iff $M(\phi_\sigma) - \delta J(\phi_\sigma)$ is bounded below for $\sigma \in G$, by translation invariance of M and J.

The key ingredient is the following result of S. Paul [Pau12] (see also [Kap13]):

Theorem 5.9. *The functionals E, J and M all have log norm singularities on G.*

Granted this result we can deduce Theorem C. The equivalence of (ii) and (iii) follows from the same argument as for [BHJ17, Proposition 8.2], so it suffices to show that (i) and (iii) are equivalent. By Theorem 5.9, the function $f(\sigma) := M(\phi_\sigma) - \delta J(\phi_\sigma)$ on G has log norm singularities. By Theorem 5.4, it is thus bounded below iff

$$\lim_{s \to +\infty} \frac{(f \circ \lambda)(e^{-s})}{s} \geq 0$$

for each one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$. We obtain the desired result since by Theorem B, this limit is equal to $M^{\mathrm{NA}}(\phi_\lambda) - \delta J^{\mathrm{NA}}(\phi_\lambda)$, where $\phi_\lambda \in \mathcal{H}^{\mathrm{NA}}$ is the non-Archimedean metric on L defined by λ.

Corollary D follows since every ample test configuration of (X, L) is induced by a 1-PS (see §2.2). The first assertion of Corollary E follows immediately, and the fact that the reduced automorphism group of (X, L) is finite is a consequence of [Pau13, Corollary 1.1].

Proof of Theorem 5.9. Recall from [Pau12] that to the linearly normal embedding $X \hookrightarrow \mathbb{P}(H^0(X, L)^*) \simeq \mathbb{P}^{N-1}$ are associated the X-resultant R, i.e. the Chow coordinate of X, and the X-hyperdiscriminant Δ, which cuts out the dual variety of

$$X \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{Nn-1},$$

the second arrow being the Segre embedding.

In our notation, we then have $\deg R = V(n+1)$ and $\deg \Delta = V(n(n+1)-\bar{S})$ [Pau12, Proposition 5.7], and [Pau12, Theorem A] becomes

$$M(\phi_\sigma) = V^{-1} \log \|\sigma \cdot \Delta\| - V^{-1} \frac{\deg \Delta}{\deg R} \log \|\sigma \cdot R\| + O(1), \tag{5.2}$$

which proves the assertion for $M(\phi_\sigma)$.

We next consider

$$J(\phi_\sigma) = \int_X (\phi_\sigma - \phi_{\mathrm{ref}})\, \mathrm{MA}(\phi_{\mathrm{ref}}) - E(\phi_\sigma).$$

On the one hand, by [Pau04, Theorem 1] (or [Zha96, Theorems 1.6 and 3.6]) we have

$$E(\phi_\sigma) = \frac{1}{\deg R} \log \|\sigma \cdot R\| + O(1). \tag{5.3}$$

On the other hand, choosing any norm on the space of complex $N \times N$-matrices (in which G of course embeds), it is observed in [Tia17, proof of Lemma 3.2] that

$$\int_X (\phi_\sigma - \phi_{\mathrm{ref}})\, \mathrm{MA}(\phi_{\mathrm{ref}}) = \log \|\sigma\| + O(1).$$

The assertion for $J(\phi_\sigma)$ follows. □

5.3. *Discussion of [Tia17]*

The statement of [Tia17, Lemma 3.1] sounds overoptimistic from the GIT point of view, as it would mean that CM-stability can be tested by only considering one-parameter subgroups of a fixed maximal torus T.

At least, the proof is incorrect, the problem being the estimate (3.1), which claims that $\phi_{\tau k} - \phi_\tau$ is uniformly bounded with respect to $\tau \in T$ and $k \in K$. As the next example shows, this is not even true for a fixed $k \in K$.

Example 5.10. Assume (s_1, s_2) is a basis of $H^0(X, L)$, let $k \in \mathrm{U}(2)$ be the unitary transformation exchanging s_1 and s_2, $\tau = (t, t^{-1})$, and pick a point x with $s_1(x) = 0$. Then

$$\phi_{\tau k}(x) - \phi_\tau(x) = 4 \log |\tau|$$

is unbounded.

In any case, the methods here do not seem to be able to deduce CM-stability from K-stability, because of the following fact (cf. [Li12, p. 39]).

Proposition 5.11. *For each polarized manifold (X, L) and each m large and divisible enough, there exists a non-trivial 1-PS λ in $\mathrm{GL}(N_m, \mathbb{C})$ such that J and M remain bounded on the corresponding Fubini–Study ray $\phi^s := \phi_{\lambda(e^{-s})}$.*

Proof. As originally observed in [LX14] (cf. Proposition 2.3), (X, L) admits a non-trivial ample test configuration $(\mathcal{X}, \mathcal{L})$ that is almost trivial, i.e. with trivial normalization. As recalled in §2.2, for each m large and divisible enough, $(\mathcal{X}, \mathcal{L})$ can be realized as the test configuration induced by a 1-PS $\lambda : \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$, which is non-trivial since $(\mathcal{X}, \mathcal{L})$ is. Since the normalization of $(\mathcal{X}, \mathcal{L})$ is trivial, the associated non-Archimedean metric is of the form $\phi_{\mathrm{triv}} + c$ for some $c \in \mathbb{Q}$, and hence $M^{\mathrm{NA}}(\phi_\lambda) = J^{\mathrm{NA}}(\phi_\lambda) = 0$. Since M and J have log norm singularities on $\mathrm{GL}(N_m, \mathbb{C})$ by Theorem 5.9, M and J are indeed bounded on ϕ^s by Theorem 5.4. □

6. Remarks on the Yau–Tian–Donaldson conjecture

As explained in the introduction, we will here give a simple argument, following ideas of Tian, for the existence of a Kähler–Einstein metric on a Fano manifold X, assuming $(X, -K_X)$ is uniformly K-stable and the partial C^0-estimates due to Székelyhidi.

6.1. *Partial C^0-estimates and the continuity method*

For the moment, consider an arbitrary polarized manifold (X, L). For each m such that mL is very ample, we have a 'Bergman kernel approximation' map $P_m : \mathcal{H} \to \mathcal{H}_m$, defined by setting $P_m(\phi)$ to be the Fubini–Study metric induced by the L^2-scalar product on $H^0(X, mL)$ defined by $m\phi$.

Definition 6.1. A subset $A \subset \mathcal{H}$ satisfies *partial C^0-estimates at level m* if there exists $C > 0$ such that $|P_m(\phi) - \phi| \le C$ for all $\phi \in A$.

Now assume X is Fano, and set $L := -K_X$. Given a Kähler form $\alpha \in c_1(X)$, consider Aubin's continuity method

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1 - t)\alpha. \tag{6.1}$$

It is well-known that there exists a unique maximal solution $(\omega_t)_{t \in [0,T)}$, where $0 < T \le 1$. The following important result, due to Székelyhidi [Szé16], confirms a conjecture of Tian.

Theorem 6.2. *The set* $A := \{\omega_t \mid t \in [0, T)\}$ *satisfies partial* C^0*-estimates at level* m *for arbitrarily large positive integers* m.

Given this result, we shall prove

Theorem 6.3. *Any uniformly K-stable Fano manifold admits a Kähler–Einstein metric.*

By working (much) harder, Datar and Székelyhidi [DSz15] have in fact been able to deduce from Theorem 6.2 a much better result dealing with K-polystability and allowing a compact group action.

6.2. CM-stability and partial C^0*-estimates*

We first present in some detail well-known ideas due to Tian [Tia12, §4.3]. In this section, (X, L) is an arbitrary polarized manifold.

Proposition 6.4. *Assume that* (X, mL) *is CM-stable, and that* $A \subset \mathcal{H}$ *satisfies partial* C^0*-estimates at level* m*. Then there exist* $\delta, C > 0$ *such that* $M \geq \delta J - C$ *on* A.

The proof, which is similar to the arguments in [Szé16, §5], is based on two lemmas.

Lemma 6.5. *For any two metrics* $\phi, \psi \in \mathcal{H}$*, we have*

(i) $|J(\phi) - J(\psi)| \leq 2 \sup(\phi - \psi)$;
(ii) $M(\phi) \geq M(\psi) - C \sup|\phi - \psi|$ *for some* $C > 0$ *only depending on a one-sided bound (either upper or lower) for the Ricci curvature of the Kähler metric* $dd^c\psi$.

Proof. Recall that

$$E(\phi) - E(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}.$$

As a consequence, $|E(\phi) - E(\psi)| \leq \sup|\phi - \psi|$, and (i) follows immediately.

For (ii), we basically argue as in [Tia17, proof of Lemma 3.1]. By the Chen–Tian formula (1.11), we have

$$M(\phi) - M(\psi) = H_\psi(\phi) + \bar{S}(E(\phi) - E(\psi)) + E_{\mathrm{Ric}(dd^c\psi)}(\psi) - E_{\mathrm{Ric}(dd^c\psi)}(\phi).$$

Here the entropy term $H_\psi(\phi)$ is non-negative, and we have

$$E_{\mathrm{Ric}(dd^c\psi)}(\phi) - E_{\mathrm{Ric}(dd^c\psi)}(\psi) = \sum_{j=0}^{n-1} V^{-1} \int_X (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j-1} \wedge \mathrm{Ric}(dd^c\psi).$$

Assume $\mathrm{Ric}(dd^c\psi) \leq C dd^c\psi$ for some constant $C > 0$. We may then write

$$(dd^c\phi)^j \wedge (dd^c\psi)^{n-j-1} \wedge \mathrm{Ric}(dd^c\psi) = C(dd^c\phi)^j \wedge (dd^c\psi)^{n-j} - (dd^c\phi)^j \wedge (dd^c\psi)^{n-j-1} \wedge (C' dd^c\psi - \mathrm{Ric}(dd^c\psi)),$$

a difference of two positive measures of mass CV and $CV + (L^{n-1} \cdot K_X)$, respectively, and the desired estimate follows.

The case where $\mathrm{Ric}(dd^c\psi) \geq -C' dd^c\psi$ is treated similarly (and will anyway not be used in what follows). □

We next recall a well-known upper bound for the Ricci curvature of restrictions of Fubini–Study metrics.

Lemma 6.6. *We have* $\mathrm{Ric}(dd^c\phi) \leq N_m dd^c\phi$ *for all* $\phi \in \mathcal{H}_m$.

Proof. Choose a basis of $H^0(X, mL)$, and let ω be the corresponding Fubini–Study metric on $\mathbb{P} := \mathbb{P}H^0(X, mL)^*$. Its curvature tensor

$$\Theta(T_{\mathbb{P}}, \omega) \in C^\infty(\mathbb{P}, \Lambda^{1,1}T^*_{\mathbb{P}} \otimes \mathrm{End}(T_{\mathbb{P}}))$$

is Griffiths positive and satisfies

$$\mathrm{Tr}_{T_{\mathbb{P}}} \Theta(T_{\mathbb{P}}, \omega) = \mathrm{Ric}(\omega) = N_m\omega.$$

For each complex submanifold $Y \subset \mathbb{P}$, the curvature of its tangent bundle T_Y with respect to $\omega|_Y$ satisfies $\Theta(T_Y, \omega|_Y) \leq \Theta(T_{\mathbb{P}}, \omega)|_{T_Y}$ as $(1, 1)$-forms on Y with values in the endomorphisms of T_Y, as a consequence of a well-known curvature monotonicity property going back to Griffiths. We thus have

$$\mathrm{Ric}(\omega|_Y) = \mathrm{Tr}_{T_Y} \Theta(T_Y, \omega|_Y) \leq \mathrm{Tr}_{T_Y} \Theta(T_{\mathbb{P}}, \omega)|_{T_Y}.$$

Using now $\Theta(T_{\mathbb{P}}, \omega) \geq 0$, we have on the other hand

$$\mathrm{Tr}_{T_Y} \Theta(T_{\mathbb{P}}, \omega)|_{T_Y} \leq \mathrm{Tr}_{T_{\mathbb{P}}} \Theta(T_{\mathbb{P}}, \omega)|_Y = N_m\omega|_Y,$$

and hence $\mathrm{Ric}(\omega|_Y) \leq N_m\omega|_Y$. Applying this to the images of $X \subset \mathbb{P}$ under projective transformations yields the desired result. □

Proof of Proposition 6.4. Since (X, mL) is CM-stable, there exist $\delta, C > 0$ such that

$$M(P_m(\phi)) \geq \delta J(P_m(\phi)) - C \quad \text{for all } \phi \in \mathcal{H}. \tag{6.2}$$

By assumption on A, we also have $|P_m(\phi) - \phi| \leq C$ for all $\phi \in A$, and by Lemma 6.6, the Ricci curvature of $dd^c P_m(\phi)$ is uniformly bounded above. Hence Lemma 6.5 shows, as desired, that there exists $C' > 0$ with $M(\phi) \geq \delta J(\phi) - C'$ for all $\phi \in A$. □

6.3. *Proof of Theorem 6.3*

Assume now that X is a Fano manifold and set $L := -K_X$. Consider the continuity method (6.1). Pick metrics ψ and ϕ_t on $-K_X$ such that $\alpha = dd^c\psi$ and $\omega_t = dd^c\phi_t$, respectively. After adding a constant to ϕ_t, (6.1) may be written

$$(dd^c\phi_t)^n = e^{-2(t\phi_t + (1-t)\psi)}. \tag{6.3}$$

We recall the proof of the following well-known monotonicity property.

Lemma 6.7. *The function $t \mapsto M(\phi_t)$ is non-increasing.*

Proof. We have

$$\begin{aligned} -\frac{d}{dt}M(\phi_t) &= nV^{-1}\int_X \dot\phi_t\big(\mathrm{Ric}(\omega_t)\wedge\omega_t^{n-1}-\omega_t^n\big) \\ &= nV^{-1}(1-t)\int_X \dot\phi_t dd^c(\psi-\phi_t)\wedge(dd^c\phi_t)^{n-1} \\ &= nV^{-1}(1-t)\int_X (\psi-\phi_t)dd^c\dot\phi_t\wedge(dd^c\phi_t)^{n-1}. \end{aligned}$$

Since d^c is normalized so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$, we have

$$n\frac{dd^c\dot\phi_t\wedge\omega_t^{n-1}}{\omega_t^n} = \mathrm{tr}_{\omega_t} dd^c\dot\phi_t = -\frac{1}{2\pi}\Delta_t''\dot\phi_t$$

with Δ_t'' denoting the $\bar\partial$-Laplacian with respect to ω_t. On the other hand, differentiating (6.3) yields

$$n dd^c\dot\phi_t\wedge\omega_t^{n-1} = 2(\psi-\phi_t-t\dot\phi_t)\omega_t^n,$$

and hence

$$\psi-\phi_t = \left(t-\frac{1}{\pi}\Delta_t''\right)\dot\phi_t.$$

We get

$$\begin{aligned} -\frac{d}{dt}M(\phi_t) &= \frac{1-t}{2\pi}\int_X\left(\left(\frac{1}{\pi}\Delta_t''-t\right)\dot\phi_t\right)(\Delta_t''\dot\phi_t)\,\mathrm{MA}(\phi_t) \\ &= \frac{1-t}{2\pi}\int_X\left\langle\left(\frac{1}{\pi}\Delta_t''-t\right)\bar\partial\dot\phi_t, \bar\partial\dot\phi_t\right\rangle_{\omega_t}\mathrm{MA}(\phi_t). \end{aligned}$$

Since $\mathrm{Ric}(\omega_t)\ge t\omega_t$, the $\bar\partial$-Laplacian Δ_t'' satisfies $\frac{1}{\pi}\Delta_t''\ge t$ on $(0,1)$-forms, and the last integral is thus non-negative. Indeed, this follows from the Bochner–Kodaira–Nakano identity applied to

$$C^\infty(X,\Lambda^{0,1}T_X^*)\simeq C^\infty(X,\Lambda^{n,1}T_X^*\otimes K_X^*)$$

with the fiber metric $\psi_t = -\frac{1}{2}\log\omega_t^n$ on $K_X^* = -K_X$, with curvature $dd^c\psi_t = \mathrm{Ric}(\omega_t)$. □

We may now complete the proof of Theorem 6.3. By Corollary E, $(X, -mK_X)$ is CM-stable for all m divisible enough. Theorem 6.2 and Proposition 6.4 therefore yield $\delta, C>0$ such that $M(\phi_t)\ge\delta J(\phi_t)-C$ along Aubin's continuity path (6.1). Since $M(\phi_t)$ is bounded above by Lemma 6.7, it follows that $J(\phi_t)$ remains bounded. By [Tia00, Lemma 6.19], the oscillation of ϕ_t is bounded, and well-known arguments allow us to conclude the proof (see [Tia00, §6.2]).

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