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Differentiability of Relative Volumes Over an Arbitrary Non-Archimedean Field

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Given an ample line bundle L on a geometrically reduced projective scheme defined over an arbitrary non-Archimedean field, we establish a differentiability property for the relative volume of two continuous metrics on the Berkovich analytification of L, extending previously known results in the discretely valued case. As applications, we provide fundamental solutions to certain non-Archimedean Monge–Ampère equations and generalize an equidistribution result for Fekete points. Our main technical input comes from determinant of cohomology and Deligne pairings.

Introduction

In [5], a variational approach to the resolution of complex Monge–Ampère equations was introduced, inspired by the classical work of Aleksandrov on real Monge–Ampère equations and the Minkowski problem. A key ingredient in this approach is a differentiability property for relative volumes, previously established in [3].

This variational approach was adapted in [7] to non-Archimedean Monge–Ampère equations in the context of Berkovich geometry. While most of the results in that paper assumed the non-Archimedean ground field K to be discretely valued and of residue characteristic 0, the proof of the differentiability property required a stronger algebraicity assumption that was later removed in [11]. Building on these results, a

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version for trivially valued fields was obtained in [10], with a view towards the study of K-stability [9].

The main result of the present paper establishes the differentiability property over an arbitrary non-Archimedean field. While only one ingredient in the variational approach, it can already be used to construct fundamental solutions to Monge–Ampère equations and to generalize the results of [6] on equidistribution of Fekete points. Our strategy follows overall that of [11], itself inspired by techniques of Abbes–Bouche [1] and Yuan [33] in the context of Arakelov geometry. As in [6], the extra technical input enabling us to deal with possibly non-Noetherian valuation rings is provided by the Deligne pairings machinery.

Working over non-discretely valued fields arises naturally in several contexts. First, Berkovich analytifications over trivially valued fields form a natural setting to study K-stability, as advocated in [9]. Next, any non-Archimedean field that is non-trivially valued and algebraically closed (such as $\mathbb{C}_p$) is densely valued. Another instance is in Arakelov theory, where computing the relative height of a projective variety X defined over the function field F of an adelically polarized projective variety B over $\mathbb{Q}$ leads naturally to a bunch of non-Archimedean absolute values on F satisfying a product formula. Here, the absolute values over a prime p are induced by Zariski dense points of the Berkovich analytification of $B \otimes \mathbb{Q}_p$ and are usually not discrete. For details about this generalization of Moriwaki's heights, we refer to [20, §3].

Differentiability of relative volumes

In what follows, K denotes an arbitrary (complete) non-Archimedean field, X is a geometrically reduced projective K-scheme, and L is an ample line bundle on X. Set $n := \dim X$, and denote by X^{an} the associated Berkovich analytic space.

The data of a continuous metric ϕ on (the analytification of) L induces for each $m \in \mathbb{N}$ a supnorm $\|\cdot\|_{m\phi}$ on the space of sections $H^0(mL) = H^0(X, mL)$. Here and throughout the paper, we use additive notation for line bundles and metrics, see §1.2. Given a second continuous metric ψ on L, one defines the *relative volume* of the associated supnorms as

$$\mathrm{vol}(\|\cdot\|_{m\phi}, \|\cdot\|_{m\psi}) := \log\left(\frac{\det\|\cdot\|_{m\psi}}{\det\|\cdot\|_{m\phi}}\right),$$

where $\det\|\cdot\|_{m\phi}, \det\|\cdot\|_{m\psi}$ denote the induced norms on the determinant line $\det H^0(mL)$. This notion of relative volume, introduced in [6, 14], can be described in terms of (virtual) lengths in the discretely valued case as in [11]. As a consequence of Chen and Maclean's work [14], it is proved in [6, Theorem 9.8] that the *relative volume* of ϕ, ψ

$$\mathrm{vol}(L, \phi, \psi) := \lim_{m\to\infty} \frac{n!}{m^{n+1}} \mathrm{vol}(\|\cdot\|_{m\phi}, \|\cdot\|_{m\psi})$$

exists in $\mathbb{R}$.

When K is non-trivially valued, a continuous metric on L is called *psh* (a shorthand for *plurisubharmonic*) if it can be written as a uniform limit of metrics on L induced by nef models of L. This definition, which goes back to the work of Shou-Wu Zhang [34], is not adapted to the trivially valued case, where the trivial metric on L is the only model metric. An alternative description of psh metrics relying on Fubini–Study metrics can, however, be adopted [6, 10], the upshot being that a continuous metric ϕ on L is psh if and only if it becomes psh after base change to some (equivalently, any) non-Archimedean extension of K. Both approaches give the same psh metrics on L in the non-trivially valued case and the latter works also in the trivially valued case.

For a continuous psh metric ϕ on L, a positive Radon measure $(dd^c\phi)^n$ on X^{an} was constructed by Chambert-Loir [13] for K discretely valued; the general case can be obtained from [19] by base change to an algebraically closed non-trivially valued extension of K, or directly from the local approach in [12]. The main result of the present paper is as follows.

Theorem A. Let K be an arbitrary non-Archimedean field, X a projective, geometrically reduced K-scheme, and L an ample line bundle on X. For any continuous psh metric ϕ on L and any continuous function f on X^{an}, we then have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(L, \phi + tf, \phi) = \int_{X^{\mathrm{an}}} f\,(dd^c\phi)^n. \tag{0.1}$$

Such a differentiability property was already predicted by Kontsevich and Tschinkel in their pioneering investigations of non-Archimedean pluripotential theory [25]. A version of Theorem A when L is merely nef will be established in a subsequent paper.

In the discretely valued case, Theorem A was proved in [11] and the present proof follows the same overall strategy. As a first step, we reduce to the case where K is algebraically closed and non-trivially valued, and $\phi = \phi_{\mathscr{L}}, f = \pm\phi_D$ are respectively induced by an ample model $\mathscr{L}$ of L and a vertical effective Cartier divisor D, both living on some model $\mathscr{X}$ of X. A filtration argument that goes back to Yuan's work [33] yields an estimate for

$$\mathrm{vol}(\|\cdot\|_{m(\phi+f)}, \|\cdot\|_{m\phi})$$

in terms of the *content* $h^0(D, m\mathcal{A}|_D)$ of the torsion K°-module $H^0(D, m\mathcal{A}|_D)$, where $\mathcal{A}$ is a certain ample line bundle on $\mathscr{X}$, and the content is a version of the length adapted to the non-Noetherian valuation ring K°. The key ingredient is then the asymptotic Riemann–Roch formula

$$h^0(D, m\mathcal{A}|_D) \sim \frac{m^n}{n!} \int_{X^{\mathrm{an}}} \phi_D \, (dd^c \phi_{\mathcal{A}})^n,$$

which we obtain as a consequence of the results on determinant of cohomology and metrics on Deligne pairings established in [6].

Applications to non-Archimedean pluripotential theory

The *relative Monge–Ampère energy* of two continuous psh metrics ϕ, ψ on L is defined as

$$\mathrm{E}(\phi, \psi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j},$$

where $\phi - \psi$ is a continuous function on X^{an}, in our additive notation for metrics. Given any other continuous psh metric ϕ', we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{E}\left((1-t)\phi + t\phi', \psi\right) = \int_{X^{\mathrm{an}}} (\phi' - \phi)\,(dd^c\phi)^n,$$

which means that $\phi \mapsto \mathrm{E}(\phi, \psi)$ is the unique antiderivative of the Monge–Ampère operator $\phi \mapsto (dd^c\phi)^n$ that vanishes at ψ, and implies the cocycle property

$$\mathrm{E}(\phi_1, \phi_2) = \mathrm{E}(\phi_1, \phi_3) + \mathrm{E}(\phi_3, \phi_2)$$

for any three continuous psh metrics ϕ_1, ϕ_2, ϕ_3 on L.

Next, the *psh envelope* $\mathrm{P}(\phi)$ of a continuous metric ϕ on L is defined as the pointwise supremum of the family of (continuous) psh metrics ψ on L such that $\psi \le \phi$. We say that *continuity of envelopes* holds for (X, L) if $\mathrm{P}(\phi)$ is continuous, hence also psh, for all continuous metrics ϕ. As observed in [6, Lemma 7.30], continuity of envelopes is equivalent to the fact that the usc upper envelope of any bounded above family of psh metrics on L remains psh, a classical property in (complex) pluripotential theory which leads to the natural conjecture that continuity of envelopes holds as soon as X is normal.

At present, continuity of envelopes has been established when X is smooth, and one of the following holds:

- X is a curve, as a consequence of Thuillier's work [31] (see [21]);
- K discretely or trivially valued, of residue characteristic 0 [8, 10], building on multiplier ideals and the Nadel vanishing theorem;
- K is discretely valued of characteristic p, (X, L) is defined over a function field of transcendence degree d, and resolution of singularities is assumed in dimension $d+n$ [21], replacing multiplier ideals with test ideals.

Generalizing [11], which dealt with the discretely valued case, the main result of [6, Theorem A] states that any two continuous metrics ϕ, ψ on L with continuous envelope satisfy

$$\mathrm{vol}(L, \phi, \psi) = \mathrm{E}(\mathrm{P}(\phi), \mathrm{P}(\psi)). \tag{0.2}$$

In the present non-Archimedean context, the relative Monge–Ampère energy can be interpreted as a local height, and (0.2) as a local Hilbert–Samuel formula. Combined with Theorem A, it enables us to prove the following analogue of [3, Theorem B].

Theorem B. Assume that continuity of envelopes holds for (X, L), and let ϕ be a continuous metric on L.

(i) The Monge–Ampère measure $(dd^c\mathrm{P}(\phi))^n$ is supported on the contact locus $\{\mathrm{P}(\phi) = \phi\}$. In other words, the *orthogonality property*

$$\int_{X^{\mathrm{an}}} (\phi - \mathrm{P}(\phi))(dd^c\mathrm{P}(\phi))^n = 0$$

is satisfied.

(ii) For any continuous function f and continuous psh metric ψ, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{E}(\mathrm{P}(\phi + tf), \psi) = \int_{X^{\mathrm{an}}} f\,(dd^c\mathrm{P}(\phi))^n.$$

It is in fact essentially formal to show that (i) and (ii) are equivalent, and are also equivalent to the special case of (ii) where ϕ is psh, which corresponds precisely to Theorem A, thanks to (0.2).

Using Theorem B and the variational argument of [5, 7], we are able to produce 'fundamental solutions' to Monge–Ampère equations, as follows.

Corollary C. Assume continuity of envelopes for (X, L). Let $x \in X^{\mathrm{an}}$ be a nonpluripolar point, ϕ a continuous metric on L, and assume that x is L-regular, in the sense that

$$\phi_x := \sup\{\psi \text{ psh metric on } L \mid \psi(x) \le \phi(x)\}$$

is continuous (and hence psh). Then

$$V^{-1}(dd^c\phi_x)^n = \delta_x,$$

with $V := (L^n)$ and δ_x the Dirac mass at x.

Here again, L-regularity is expected to be automatic for nonpluripolar points on a normal variety. It is established in [9, Theorem 5.13] when X is smooth and K is trivially or discretely valued, of residue characteristic 0.

As a final consequence of Theorem A, we generalize the equidistribution of Fekete points in Berkovich spaces, which was established in [6] following the variational strategy going back to [4] in the complex analytic case, under assumptions guaranteeing the differentiability property (ii) of Theorem B. For any basis $\mathbf{s} = (s_1, \dots, s_N)$ of $H^0(X, L)$, the Vandermonde (or Slater) determinant $\det(s_i(x_j))_{1\le i,j\le N}$ can be seen as a global section $\det(\mathbf{s}) \in H^0(X^N, L^{\boxtimes N})$. Given a continuous metric ϕ on L, a *Fekete configuration* for ϕ is a point $P \in (X^N)^{\mathrm{an}}$ achieving the supremum of $|\det(\mathbf{s})|_{\phi^{\boxtimes N}}$, a condition that does not depend on the choice of the basis $\mathbf{s}$. By Theorem A, the differentiability property (0.1) holds for any continuous psh metric ϕ of L and hence we get the following result as a direct application of [6, Theorem 10.10].

Corollary D. Let K be any non-Archimedean field, and let L be an ample line bundle on a projective, geometrically reduced K-scheme X. Set $n := \dim X$, $N_m := h^0(X, mL)$ and $V := (L^n)$. Pick a continuous psh metric ϕ on L, and choose for each $m \gg 1$ a Fekete configuration $P_m \in (X^{N_m})^{\mathrm{an}}$ for $m\phi$. Then P_m equidistributes to the probability measure $V^{-1}(dd^c\phi)^n$, i.e.

$$\lim_{m\to\infty} \int_{X^{\mathrm{an}}} f\, \delta_{P_m} = \int_{X^{\mathrm{an}}} f\, V^{-1}(dd^c\phi)^n.$$

for each continuous function f on X^{an} where δ_{P_m} is the discrete probability measure on X^{an} obtained by averaging over the components of the image of P_m in $(X^{\mathrm{an}})^{N_m}$.

Organization of paper

Section 1 collects preliminary material on norms, metrics, and their relative volumes. We recall also properties of the energy and the Monge–Ampère measures. Section 2 reviews some facts on the determinant of cohomology, and proves the key Riemann–Roch type formula. In Section 3, we prove first Theorem A. Assuming continuity of envelopes, we then deduce Corollary B and Corollary C.

Notation and Conventions

Throughout the paper, we work over a *non-Archimedean field* K, that is, a field complete with respect to a non-Archimedean absolute value $|\cdot|$, which might be the trivial absolute value. The corresponding valuation is denoted by $v_K := -\log|\cdot|$. The valuation ring, maximal ideal and residue field are respectively denoted by

$$K^\circ := \{a \in K \mid |a| \leq 1\}, \quad K^{\circ\circ} := \{a \in K \mid |a| < 1\}, \quad \tilde{K} := K^\circ/K^{\circ\circ}.$$

We assume that the reader is familiar with the basics of non-Archimedean geometry given in [2]. If X is a scheme of finite type over K, we denote by X^{an} its Berkovich analytification. The space of continuous, real valued functions on X^{an} is denoted by $C^0(X^{\mathrm{an}})$.

We use additive notation for line bundles and metrics. If L, M are line bundles on X endowed with metrics ϕ and ψ, then $L + M$ denotes the tensor product of the line bundles and $\phi + \psi$ the induced metric, respectively. The norm on L associated to ϕ is denoted by $|\cdot|_\phi$ and $\|\cdot\|_\phi$ is the associated supnorm on $H^0(X, L)$, which is a norm if X is reduced. See §1.2 for more details.

For line bundles $L_1, \ldots, L_n$ on an n-dimensional projective scheme X over a field, we use $(L_1 \cdot \ldots \cdot L_n)$ for the intersection number of the 1st Chern classes of $L_1, \ldots, L_n$. Usually, we will have $L = L_1 = \cdots = L_n$ and we then simply write (L^n) for this intersection number, which agrees with the degree of X with respect to L.

1 Preliminaries

We collect here some background results on the norms, lattices, models, Monge–Ampère measures, energy, and volumes. In what follows, X denotes an n-dimensional, geometrically reduced projective K-scheme. Recall that geometrically reduced simply amounts to X reduced whenever K is perfect.

1.1 Norms, lattices, and content

Let V be a finite dimensional K-vector space, and set $r = \dim V$. By a *norm* on V, we always mean an ultrametric norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ compatible with the given absolute value of K. It induces a *determinant norm* $\det\|\cdot\|$ on the determinant line $\det V = \Lambda^r V$, given by

$$\det\|\tau\| := \inf_{\tau = v_1\wedge\cdots\wedge v_r} \|v_1\|\cdots\|v_r\|$$

for any $\tau \in \det V$. Given two norms $\|\cdot\|, \|\cdot\|'$, the *relative volume* of $\|\cdot\|$ with respect to $\|\cdot\|'$ is defined as

$$\mathrm{vol}(\|\cdot\|, \|\cdot\|') := \log\left(\frac{\det\|\tau\|'}{\det\|\tau\|}\right)$$

for any nonzero $\tau \in \det V$. For more details on the determinant norm and relative volumes, we refer to [6, §2.1–2.3].

A *lattice* in V is a finitely generated K°-submodule $\mathcal{V} \subset V$ that spans V over K. The *lattice norm* $\|\cdot\|_{\mathcal{V}}$ associated to a lattice $\mathcal{V}$ is given for $v \in V$ by

$$\|v\|_{\mathcal{V}} := \inf_{a\in K,\, v\in a\mathcal{V}} |a|.$$

Relative volumes of lattice norms admit the following algebraic interpretation. By [29, Proposition 2.10 (i)] (see also [6, Lemma 2.17]), every finitely presented, torsion K°-module M satisfies

$$M \cong K^\circ/(a_1) \oplus \ldots \oplus K^\circ/(a_r)$$

for some nonzero $a_1, \ldots, a_r \in K^{\circ\circ}$, where r and the sequence $v_K(a_i)$ are further uniquely determined by M, up to reordering. The *content* (this quantity was called length in [29], and corresponds to $-\log$ of the content as defined in [30]) of M is defined as

$$\mathrm{c}(M) = \sum_{i=1}^{r} v_K(a_i) \in \mathbb{R}_{\geq 0}.$$

When K is discretely valued with uniformizer $\pi \in K^{\circ\circ}$, then $\mathrm{c}(M)$ is the usual length of M, multiplied by $v_K(\pi)$ [6, Example 2.19].

Now every finitely presented torsion K°-module M arises as a quotient $M = \mathcal{V}/\mathcal{V}'$ for lattices $\mathcal{V}' \subset \mathcal{V}$ in a finite dimensional K-vector space, and

$$\mathrm{c}(M) = \mathrm{vol}(\|\cdot\|_{\mathcal{V}}, \|\cdot\|_{\mathcal{V}'}). \tag{1.1}$$

1.2 Metrics

As in [6, §5], we use additive notation for metrics on a line bundle L over X. Then a *metric* ϕ on L is a family of functions $\phi_x : L \otimes_X \mathscr{H}(x) \to \mathbb{R} \cup \{\infty\}$ such that $|\ |_{\phi_x} := e^{-\phi_x}$ is a norm on the 1-dimensional $\mathscr{H}(x)$-vector space $L \otimes_X \mathscr{H}(x)$ for every $x \in X^{\mathrm{an}}$. Here, $\mathscr{H}(x)$ is the completed residue field of x endowed with its canonical absolute value [2, Remark 1.2.2]. We usually skip the x and write simply $|\ |_\phi$ for the norms. Note that $L \otimes_X \mathscr{H}(x)$ is the non-Archimedean analogue of the fiber of a holomorphic line bundle.

Given two metrics ϕ, ψ on line bundles L, M over X, we denote by $\phi \pm \psi$ the induced metric on $L \pm M = L \otimes M^{\pm 1}$. The corresponding norms thus satisfy $|\cdot|_{\phi\pm\psi} = |\cdot|_\phi \otimes |\cdot|_\psi^{\pm 1}$.

A metric ϕ on L is called *continuous* if the function $x \mapsto |t(x)|_\phi$, induced by any local section t of L, is continuous with respect to the Berkovich topology. For $s \in H^0(X, L)$, the associated *supremum norm* is denoted by

$$\|s\|_\phi := \sup_{x \in X^{\mathrm{an}}} |s(x)|_\phi.$$

1.3 Models

In this paper, a *model* $\mathscr{X}$ of X is a flat projective K°-scheme, together with an identification of the generic fiber $\mathscr{X}_\eta$ of $\mathscr{X} \to \mathrm{Spec}(K^\circ)$ with X. There is a canonical *reduction map* $\mathrm{red}_{\mathscr{X}} : X^{\mathrm{an}} \to \mathscr{X}_s$ to the special fiber $\mathscr{X}_s$ of $\mathscr{X}$ (see [22, Remark 2.3] and [23, §2] for details).

We say that a model $\mathscr{X}$ of X is *dominated* by another model $\mathscr{X}'$ if the identity on X extends to a (unique) morphism $\mathscr{X}' \to \mathscr{X}$ over K°. This induces a partial order on the set of models of X modulo isomorphism, which turns it into a directed system.

If K is algebraically closed and nontrivially valued, then it follows from the reduced fiber theorem (see for instance [6, Theorem 4.20]) that models $\mathscr{X}$ with reduced special fiber $\mathscr{X}_s$ are cofinal among all models. On the other hand, in the trivially valued case, X is its only model, up to isomorphism.

Now let L be a line bundle on X. A *model* $(\mathscr{X},\mathscr{L})$ of (X,L) consists of a model $\mathscr{X}$ of X and a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with an identification $\mathscr{L}|_{\mathscr{X}_\eta} \simeq L$ compatible with the identification $\mathscr{X}_\eta \simeq X$. We then say that $\mathscr{L}$ is a *model of* L determined on $\mathscr{X}$. Every model of the trivial line bundle $L = \mathcal{O}_X$ determined on a model $\mathscr{X}$ is of the form $\mathscr{L} = \mathcal{O}_{\mathscr{X}}(D)$, where D is a Cartier divisor which is *vertical*, that is, supported in the special fiber.

Lemma 1.1. Assume K is algebraically closed and non-trivially valued, and let (L_i) be a finite collection of ample line bundles on X. Then models $\mathscr{X}$ of X that have reduced special fiber and such that all L_i extend to an ample $\mathbb{Q}$-line bundles on $\mathscr{X}$ are cofinal in the set of all models.

Proof. By [22, Proposition 4.11, Lemma 4.12], every model $\mathscr{X}$ of X is dominated by a model $\mathscr{X}'$ on which all L_i extend to ample $\mathbb{Q}$-line bundles $\mathcal{L}'_i$. By [6, Theorem 4.20], the integral closure of $\mathscr{X}'$ in its generic fiber $\mathscr{X}_\eta \simeq X$ is a model $\mathscr{X}''$ with reduced special fiber, which dominates $\mathscr{X}'$ via a finite morphism $\mu : \mathscr{X}'' \to \mathscr{X}'$. As a result, $\mu^*\mathscr{L}'_i$ is an ample $\mathbb{Q}$-line bundle extending L_i, and we are done. ■

If $(\mathscr{X},\mathscr{L})$ is a model of (X,L), then $H^0(\mathscr{X},\mathscr{L})$ is a lattice in $H^0(X,L)$. Indeed, it follows from the direct image theorem given in [32, Theorem 3.5] that $H^0(\mathscr{X},\mathscr{L})$ is a finitely generated K°-module, while flat base change implies $H^0(\mathscr{X},\mathscr{L})\otimes_{K^\circ}K \simeq H^0(X,L)$.

Recall that a section t of a line bundle over a scheme Z is *regular* if its zero subscheme is a Cartier divisor, that is, if the corresponding function in any local trivialization of the line bundle is a nonzero divisor. The section t is *relatively regular* with respect to a flat morphism $Z \to S$ if its zero subscheme is a Cartier divisor and is flat over S.

Given a model $(\mathscr{X},\mathscr{L})$ of (X,L), it follows from [16, 11.3.7] that a section $t \in H^0(\mathscr{X},\mathscr{L})$ is relatively regular (with respect to the structure morphism $\mathcal{X} \to \operatorname{Spec} K^\circ$) if and only if its restriction to the special fiber $\mathscr{X}_s$ is regular. By [6, Proposition A.15], if $\mathscr{L}$ is ample then $H^0(\mathscr{X},m\mathscr{L})$ admits relatively regular sections for all $m \gg 1$. For later use, we note:

Lemma 1.2. Let $(\mathscr{X},\mathscr{L})$ be a model of (X,L), and D be an effective vertical Cartier divisor. If $t \in H^0(\mathscr{X},\mathscr{L})$ is a relatively regular section, then $t|_D$ is regular on D.

Proof. The statement is local, and thus reduces to the following. Let $\mathcal{A}$ be a flat, finite type K°-algebra, $f \in \mathcal{A}$ a relatively regular function, and $a \in \mathcal{A}$ a nonzero divisor whose image in $\mathcal{A}\otimes_{K^\circ} K$ is invertible. We have to show that the image of f in $\mathcal{A}/(a)$ is a nonzero divisor. To see this, pick $g, h \in \mathcal{A}$ such that $fg = ah$. We then need to prove that $g \in (a)$. Since f is relatively regular, $\mathcal{A}/(f)$ is flat over K°, and the map $\mathcal{A}/(f) \to \mathcal{A}/(f) \otimes_{K^\circ} K$ is thus injective. The image of a in $\mathcal{A}/(f) \otimes_{K^\circ} K$ being invertible, the image of h in $\mathcal{A}/(f) \hookrightarrow \mathcal{A}/(f) \otimes_{K^\circ} K$ is zero, and hence $h \in (f)$, that is, $h = h'f$ for some $h' \in \mathcal{A}$. Then $fg = ah'f$, and hence $g = ah' \in (a)$ as f is a nonzero divisor. ■

1.4 Model metrics

Let L be a line bundle on X. To every model $(\mathscr{X}, \mathscr{L})$ of (X, L) is associated a continuous metric $\phi_{\mathscr{L}}$ on L, determined as follows: every point of X^{an} belongs to the affinoid domain $\mathrm{red}_{\mathscr{X}}^{-1}(\mathcal{U})$ induced by an affine open subset $\mathcal{U}$ of $\mathscr{X}$ on which $\mathscr{L}$ admits a trivializing section τ, and $\phi_{\mathscr{L}}$ is determined by requiring that $|\tau|_{\phi_{\mathscr{L}}} \equiv 1$ on $\mathrm{red}_{\mathscr{X}}^{-1}(\mathcal{U})$. This construction is invariant under pull-back to a higher model, that is, $\phi_{\mu^*\mathscr{L}} = \phi_{\mathscr{L}}$ for any morphism of models $\mu : \mathscr{X}' \to \mathscr{X}$. We refer to [6, 5.3] and [22, §2] for more details.

A *model metric* on L is defined as a continuous metric of the form $\phi = m^{-1}\phi_{\mathscr{L}}$ where $\mathscr{L}$ is a model of mL for some nonzero $m \in \mathbb{N}$. We say that ϕ is determined by the $\mathbb{Q}$-model $m^{-1}\mathscr{L}$.

A *model function* is a continuous function on X^{an} corresponding to a model metric on the trivial line bundle $\mathcal{O}_X$. It is thus determined by a vertical $\mathbb{Q}$-Cartier divisor D on some model $\mathscr{X}$ of X, and we write ϕ_D for the corresponding model function. Model functions form a $\mathbb{Q}$-vector space of continuous functions, which is stable under max. When K is non-trivially valued, model functions further separate points, and hence are dense in $C^0(X^{\mathrm{an}})$ by the Stone-Weierstrass theorem, see [17, Theorem 7.12].

The next result explains the importance of models with reduced special fiber in our approach.

Lemma 1.3. Let $\mathscr{X}$ be a model of X with reduced special fiber.

(i) If $\mathscr{L}$ is a model of L determined on $\mathscr{X}$, then the supnorm $\|\cdot\|_{\phi_{\mathscr{L}}}$ coincides with the lattice norm $\|\cdot\|_{H^0(\mathscr{X},\mathscr{L})}$.

(ii) If D is a vertical Cartier divisor on $\mathscr{X}$, then D is effective if and only if $\phi_D \geq 0$.

Proof. Property (i) is [6, Lemma 6.3]. For (ii), note that the vertical Cartier divisor D induces a canonical meromorphic section s_D of $\mathscr{L} = \mathcal{O}(D)$ which restricts to a global section of $\mathcal{O}_X = \mathscr{L}|_X$. By definition of a lattice norm, we have $s_D \in H^0(\mathscr{X}, \mathscr{L})$ if and only if $\|s_D\|_{H^0(\mathscr{X},\mathscr{L})} \leq 1$ and hence (ii) follows from (i). ■

1.5 Plurisubharmonic metrics and envelopes

In this subsection, we recall some facts about plurisubharmonic metrics on an ample line bundle L over X. We refer to [6, §7] for a thorough discussion.

Assume first that K is non-trivially valued. Following Shou-Wu Zhang [34], we then say that a continuous metric ϕ on L is *plurisubharmonic* (*psh* for short) if ϕ can be written as a uniform limit of model metrics $\phi_{\mathscr{L}_i}$ associated to nef $\mathbb{Q}$-models $\mathscr{L}_i$ of L. By [6, Theorem 7.8], this definition is compatible with the point of view of [6, 10], which defines continuous psh metrics as uniform limits of Fubini–Study metrics.

When K is trivially valued, a continuous metric ϕ on L is called *psh* if there exists a non-trivially valued non-Archimedean field extension F of K such that the induced continuous metric ϕ_F on the base change $L \otimes_K F$ is psh in the above sense. By [6, Theorem 7.32], this condition is independent of the choice of F, and compatible with the Fubini–Study approach of [6, 10].

Definition 1.4. We say that *continuity of envelopes* holds for (X, L) if, for any continuous metric ϕ on L, the *psh envelope*

$$\mathrm{P}(\phi) := \sup\{\psi \text{ continuous psh metric on } L \mid \psi \le \phi\}$$

is a continuous metric on L as well.

When this holds, $\mathrm{P}(\phi)$ is automatically psh, and is thus characterized as the greatest continuous psh metric dominated by ϕ. In the complex analytic case, continuity of envelopes holds over any normal complex space, and fails in general otherwise. By analogy, we conjecture that continuity of envelopes holds as soon as X is normal. As recalled in the introduction, it is at present known when X is smooth and one of the following is satisfied:

- X is a curve, as a consequence of A. Thuillier's work [31] (see [21]);
- K discretely or trivially valued, of residue characteristic 0 [8, 10], building on multiplier ideals and the Nadel vanishing theorem;
- K is discretely valued of characteristic p, (X, L) is defined over a function field of transcendence degree d, and resolution of singularities is assumed in dimension $d + n$ [21], replacing multiplier ideals with test ideals.

1.6 Monge–Ampère measures and energy

A construction of A. Chambert-Loir associates to any n-tuple $\phi_1, \ldots, \phi_n$ of continuous psh metrics on ample line bundles $L_1, \ldots, L_n$ over X their *mixed Monge–Ampère* *measure*

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n,$$

a positive Radon measure on X^{an} of total mass equal to the intersection number $(L_1 \cdot \ldots \cdot L_n)$. This measure depends multilinearly and continuously on the tuple $(\phi_1, \ldots, \phi_n)$ with respect to uniform convergence (and weak convergence of measures), and the construction is further compatible with ground field extension.

These measures were first constructed in [13] over non-Archimedean fields K with a countable dense subset. Over an arbitrary non-Archimedean ground field, the measures can be obtained by base change to a non-trivially valued algebraically closed non-Archimedean field F, using [18, §2]. One can also directly rely on the local approach in [13], see [6, §8.1] for details.

Example 1.5. For psh model metrics $\phi_1, \ldots, \phi_n$, the measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ has finite support. When K is algebraically closed, the ϕ_i are determined by nef $\mathbb{Q}$-models $\mathscr{L}_1, \ldots, \mathscr{L}_n$ of L determined on a model $\mathscr{X}$ that can be chosen to have reduced special fiber $\mathscr{X}_s$; each irreducible component Y of $\mathscr{X}_s$ then determines a unique point $x_Y \in X^{\mathrm{an}}$ with $\mathrm{red}_{\mathscr{X}}(x_Y)$ the generic point of Y, and we have

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \sum_Y (\mathscr{L}_1|_Y \cdots \mathscr{L}_n|_Y)\delta_{x_Y},$$

where δ_{x_Y} is the Dirac measure at x_Y, see [18, Corollary 2.8] and [12, Théorème 6.9.3].

From now on we fix an ample line bundle L on X, and denote by $V := (L^n)$ its volume. The *relative Monge–Ampère energy* of ϕ, ψ is defined as

$$\mathrm{E}(\phi, \psi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\mathrm{an}}} (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}. \tag{1.2}$$

We emphasize that the present normalization is not uniform across the literature. For each ψ, the functional $\phi \mapsto \mathrm{E}(\phi, \psi)$ is characterized as the unique antiderivative of the Monge–Ampère operator $\phi \mapsto (dd^c\phi)^n$ that vanishes at ψ, in the sense that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{E}((1-t)\phi + t\phi', \psi) = \int_{X^{\mathrm{an}}} (\phi' - \phi)(dd^c\phi)^n \tag{1.3}$$

for any two continuous psh metrics ϕ, ϕ'. As a consequence, the *cocycle property*

$$\mathrm{E}(\phi_1, \phi_2) + \mathrm{E}(\phi_2, \phi_3) + \mathrm{E}(\phi_3, \phi_1) = 0$$

holds for all triples of continuous psh metrics ϕ_1, ϕ_2, ϕ_3 on L.

Another key property of the Monge–Ampère energy is the concavity of $\phi \mapsto \mathrm{E}(\phi, \psi)$. In view of (1.3) and the cocyle property, this amounts to

$$\mathrm{E}(\phi, \psi) \le \int_{X^{\mathrm{an}}} (\phi - \psi)(dd^c\psi)^n \tag{1.4}$$

for all continuous psh metrics ϕ, ψ on L. Moreover,

$$\mathrm{E}(\phi + c) = \mathrm{E}(\phi) + Vc$$

for all $c \in \mathbb{R}$. We refer to [10, §3.8] for details on the above properties.

1.7 Relative volumes of metrics

Recall that the *volume* of a line bundle L on X is defined as

$$\mathrm{vol}(L) := \lim_{m\to\infty} \frac{n!}{m^n} \dim H^0(X, mL) \in \mathbb{R}_{\ge 0}.$$

For geometrically integral projective schemes, the existence of the limit can be shown by using Okounkov bodies, see for instance [27]. The generalization to geometrically reduced projective schemes can be found in [6, Theorem 9.8]. We have $\mathrm{vol}(L) > 0$ if and only if L is big, and $\mathrm{vol}(L) = (L^n)$ whenever L is nef.

The *relative volume* of two continuous metrics ϕ, ψ on L is

$$\mathrm{vol}(L, \phi, \psi) := \lim_{m\to\infty} \frac{n!}{m^{n+1}} \mathrm{vol}(\|\cdot\|_{m\phi}, \|\cdot\|_{m\psi}) \in \mathbb{R}.$$

The existence of this limit was established is [6, Theorem 9.8], building on the work of Chen and Maclean [14].

Proposition 1.6. The following properties hold for all continuous metrics on a given line bundle L:

(i) *cocycle formula:* $\mathrm{vol}(L, \phi_1, \phi_2) + \mathrm{vol}(L, \phi_2, \phi_3) + \mathrm{vol}(L, \phi_3, \phi_1) = 0$;

(ii) *monotonicity:* $\phi \le \phi' \Longrightarrow \mathrm{vol}(L,\phi,\psi) \le \mathrm{vol}(L,\phi',\psi)$;

(iii) *scaling:* $\mathrm{vol}(L,\phi+c,\psi) = \mathrm{vol}(L,\phi,\psi) + \mathrm{vol}(L)c$ for $c \in \mathbb{R}$;

(iv) *Lipschitz continuity:*

$$\left|\mathrm{vol}(L,\phi,\psi) - \mathrm{vol}(L,\phi',\psi')\right| \le \mathrm{vol}(L)\left(\sup_{x\in X^{\mathrm{an}}} |\phi(x)-\phi'(x)| + \sup_{x\in X^{\mathrm{an}}} |\psi(x)-\psi'(x)|\right);$$

(v) *homogeneity*: $\mathrm{vol}(aL,a\phi,a\psi) = a^{n+1}\mathrm{vol}(L,\phi,\psi)$ for all $a \in \mathbb{N}$;

(vi) *base change invariance*: for any non-Archimedean extension F/K, we have

$$\mathrm{vol}(L_F,\phi_F,\psi_F) = \mathrm{vol}(L,\phi,\psi),$$

with ϕ_F, ψ_F denoting the pullbacks of ϕ, ψ to the base change $L_F = L \otimes_K F$.

In particular, if L is not big, that is, $\mathrm{vol}(L) = 0$, then $\mathrm{vol}(L,\phi,\psi) = 0$ for all continuous metrics on L, by (iv). We refer to [6, Propositions 9.11, 9.12] for proofs of the above properties.

The next result, which equates relative volume and relative energy, goes back to [3] in the complex analytic case. In the non-Archimedean context, the result was established in [11] in the discretely valued case, and in [6, Corollary B] in the general case.

Theorem 1.7. If L is an ample line bundle and ϕ, ψ are continuous metrics on L with continuous psh envelopes $\mathrm{P}(\phi), \mathrm{P}(\psi)$, then

$$\mathrm{vol}(L,\phi,\psi) = \mathrm{E}(\mathrm{P}(\phi),\mathrm{P}(\psi)).$$

2 An Asymptotic Riemann–Roch Theorem

This section reviews some facts on the determinant of cohomology and Deligne pairings, following [6, Appendix A], and uses this to prove a Riemann–Roch-type formula for vertical Cartier divisors on models. We still denote by X a geometrically reduced projective K-scheme of dimension n.

2.1 Determinant of cohomology and Deligne pairings

The determinant of cohomology of a line bundle L on X is a line bundle $\lambda_X(L)$ over Spec K, that is, a one-dimensional K-vector space; it can simply be described as

$$\lambda_X(L) := \sum_{i=0}^{n}(-1)^i \det H^i(X, L),$$

where we use additive notation for tensor products of line bundles.

Consider now a model $(\mathscr{X}, \mathscr{L})$ of (X, L), with structure morphism $\pi : \mathscr{X} \to S := \operatorname{Spec} K^\circ$. Kiehl's theorem on (pseudo)coherence of direct images and the flatness of π imply that the complex $R\pi_*\mathscr{L}$ is perfect. Thus, there exists a bounded complex of vector bundles $\mathcal{E}^\bullet$ on S with a quasi-isomorphism $\mathcal{E}^\bullet \to R\pi_*\mathscr{L}$ and the determinant of cohomology of $\mathcal{L}$ is defined as

$$\lambda_{\mathscr{X}}(\mathscr{L}) := \det \mathcal{E}^\bullet = \sum_i (-1)^i \det \mathcal{E}^i,$$

this line bundle on S being unique up to unique isomorphism of $\mathbb{Q}$-line bundles by [24]. This construction commutes with base change, and $\lambda_{\mathscr{X}}(\mathscr{L})$ is thus a $\mathbb{Q}$-model of $\lambda_X(L)$, cf. [6, Appendix A] for more details.

By flatness of π, the $\mathcal{O}_S$-module $\pi_*\mathscr{L}$ is torsion-free, and hence locally free. When $R^i\pi_*\mathscr{L}$ is locally free for all i, [24, p.43] yields

$$\lambda_{\mathscr{X}}(\mathscr{L}) = \sum_{i=0}^{n}(-1)^i \det R^i\pi_*\mathscr{L}. \tag{2.1}$$

Combining this with Serre vanishing (see [6, Corollary A.12] for the relevant statement), we infer:

Lemma 2.1. If $\mathscr{L}$ is ample and $\mathcal{E}$ is any line bundle on $\mathscr{X}$, then $\lambda_{\mathscr{X}}(m\mathscr{L} + \mathcal{E})$ coincides with the determinant of the vector bundle $\pi_*(m\mathscr{L} + \mathcal{E})$ for all $m \gg 1$.

The fundamental property of the determinant of cohomology, which is extracted in [6, Appendix A] from a paper of François Ducrot [15], is that $\lambda_{\mathscr{X}}$ admits a canonical structure of a *polynomial functor of degree* $n+1$. By definition, this means that the $(n+1)$-st iterated difference

$$\langle \mathscr{L}_0, \ldots, \mathscr{L}_n \rangle_{\mathscr{X}} := \sum_{I \subset \{0,\ldots,n\}} (-1)^{n+1-|I|} \lambda_{\mathscr{X}}(\sum_{i \in I} \mathscr{L}_i) \tag{2.2}$$

has a structure of multilinear functor, compatible with its natural symmetry structure and with base change, and called the *Deligne pairing*. As a consequence, we get for each line bundle $\mathscr{L}$ on a model $\mathscr{X}$ and $m \in \mathbb{Z}$ a polynomial expansion of $\mathbb{Q}$-line bundles

$$\lambda_{\mathscr{X}}(m\mathscr{L}) = \frac{m^{n+1}}{(n+1)!} \langle \mathcal{L}^{n+1} \rangle_{\mathscr{X}} + \ldots, \tag{2.3}$$

called the *Knudsen–Mumford expansion*. Here and thereafter, we use the shorthand notation

$$\langle \mathcal{L}^{n+1} \rangle_{\mathscr{X}} := \langle \underbrace{\mathcal{L}, \ldots, \mathcal{L}}_{n+1\text{-times}} \rangle_{\mathscr{X}}.$$

Lemma 2.2. Let $\mathscr{L}_0$ be a be a line bundle on a model $\mathscr{X}$ of X. The polynomial structure of degree $n+1$ on $\lambda_{\mathscr{X}}$ induces a polynomial structure of degree n on

$$\mathscr{L} \mapsto \lambda_{\mathscr{X}}(\mathscr{L} + \mathscr{L}_0) - \lambda_{\mathscr{X}}(\mathscr{L}),$$

whose n-th iterated difference further identifies with

$$(\mathscr{L}_1, \ldots, \mathscr{L}_n) \mapsto \langle \mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n \rangle_{\mathscr{X}}.$$

Proof. By definition, the n-th iterated difference of $\mathscr{L} \mapsto \lambda_{\mathscr{X}}(\mathscr{L} + \mathscr{L}_0) - \lambda_{\mathscr{X}}(\mathscr{L})$ is equal to

$$\sum_{J \subset \{1,\ldots,n\}} (-1)^{n-|J|} \left(\lambda_{\mathscr{X}}(\mathscr{L}_0 + \sum_{j \in J} \mathscr{L}_j) - \lambda_{\mathscr{X}}(\sum_{j \in J} \mathscr{L}_j) \right)$$

$$= \sum_{I \subset \{0,\ldots,n\},\, 0 \in I} (-1)^{n+1-|I|} \lambda_{\mathscr{X}}(\sum_{i \in I} \mathscr{L}_i) + \sum_{I \subset \{0,\ldots,n\},\, 0 \notin I} (-1)^{n+1-|I|} \lambda_{\mathscr{X}}(\sum_{i \in I} \mathscr{L}_i)$$

$$= \langle \mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n \rangle_{\mathscr{X}},$$

by (2.2). This finishes the proof, since the latter is a multilinear functor of $(\mathscr{L}_1, \dots, \mathscr{L}_n)$. ■

We finally recall the following special case of [6, Theorem 8.18]. We use the terminology for model functions and model metrics introduced in §1.4.

Lemma 2.3. If D is a vertical divisor on a model $\mathscr{X}$ of X with associated model function ϕ_D and $\mathscr{L}_1, \dots, \mathscr{L}_n$ are nef line bundles on $\mathscr{X}$, then

$$\phi_{\langle \mathcal{O}_{\mathscr{X}}(D), \mathscr{L}_1, \dots, \mathscr{L}_n \rangle_{\mathscr{X}}} = \int_{X^{\mathrm{an}}} \phi_D \, dd^c \phi_{\mathscr{L}_1} \wedge \cdots \wedge dd^c \phi_{\mathscr{L}_n},$$

where we identify the model function $\phi_{\langle \mathcal{O}_{\mathscr{X}}(D), \mathscr{L}_1, \dots, \mathscr{L}_n \rangle_{\mathscr{X}}}$ on $\mathrm{Spec}(K)$ with its unique value.

2.2 An asymptotic Riemann–Roch theorem

Pick a model of X, a line bundle $\mathscr{L}$ on $\mathscr{X}$, and an effective vertical Cartier divisor D on $\mathscr{X}$. By the coherence in Kiehl's direct image theorem, the K°-module $H^0(D, \mathscr{L}|_D)$ is finitely presented and torsion (see [6, Corollary A.12]). We denote by $h^0(D, \mathscr{L}|_D)$ its content, as defined in §1.1.

Theorem 2.4. Let $\mathscr{X}$ be a model of X, $\mathscr{L}$ an ample line bundle on $\mathscr{X}$, and D an effective vertical Cartier divisor on $\mathscr{X}$. Then

$$h^0(D, m\mathscr{L}|_D) = \frac{m^n}{n!} \int_{X^{\mathrm{an}}} \phi_D (dd^c \phi_{\mathscr{L}})^n + O(m^{n-1}).$$

Proof. By Serre vanishing [6, Theorem A.6]), we have $H^q(\mathscr{X}, m\mathscr{L}) = H^q(\mathscr{X}, m\mathscr{L} - D) = 0$ for all $q \geq 1$ and $m \gg 1$. Restriction to D thus yields an exact sequence

$$0 \to H^0(\mathscr{X}, m\mathscr{L} - D) \to H^0(\mathscr{X}, m\mathscr{L}) \to H^0(D, m\mathscr{L}|_D) \to 0,$$

which implies

$$h^0(D, m\mathscr{L}|_D) = \phi_{\det H^0(\mathscr{X}, m\mathscr{L})} - \phi_{\det H^0(\mathscr{X}, m\mathscr{L} - D)},$$

by (1.1). By Lemma 2.1, we further have

$$\det H^0(\mathscr{X}, m\mathscr{L}) = \lambda_{\mathscr{X}}(m\mathscr{L}), \quad \det H^0(\mathscr{X}, m\mathscr{L} - D) = \lambda_{\mathscr{X}}(m\mathscr{L} - D)$$

and hence

$$h^0(D, m\mathscr{L}|_D) = \phi_{\lambda_{\mathscr{X}}(m\mathscr{L}) - \lambda_{\mathscr{X}}(m\mathscr{L}-D)}.$$

Now Lemma 2.2 provides a polynomial expansion

$$\lambda_{\mathscr{X}}(m\mathscr{L}) - \lambda_{\mathscr{X}}(m\mathscr{L} - D) = \frac{m^n}{n!}\langle \mathcal{O}_{\mathscr{X}}(D), \mathscr{L}^n \rangle_{\mathscr{X}} + \ldots,$$

and hence

$$\begin{aligned} h^0(D, m\mathscr{L}|_D) &= \frac{m^n}{n!}\phi_{\langle \mathcal{O}_{\mathscr{X}}(D), \mathscr{L}^n\rangle_{\mathscr{X}}} + O(m^{n-1}) \\ &= \frac{m^n}{n!}\int_{X^{\mathrm{an}}} \phi_D (dd^c \phi_{\mathscr{L}})^n + O(m^{n-1}), \end{aligned}$$

by Lemma 2.3. ∎

3 Differentiability and Orthogonality

In this section, we prove our main result on differentiability of relative volumes, which generalizes [11, Theorem B] from discretely valued non-Archimedean fields to arbitrary ones. In what follows, X is a projective, geometrically reduced scheme of dimension n over an arbitrary non-Archimedean field K, and L is an *ample* line bundle on X.

3.1 Proof of Theorem A

The following result corresponds to Theorem A in the introduction.

Theorem 3.1. For any continuous psh metric ϕ on L and continuous function f on X^{an}, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(L, \phi + tf, \phi) = \int_{X^{\mathrm{an}}} f\, (dd^c\phi)^n.$$

The key ingredient in the proof is the following general estimate, which can be viewed as a local analogue of the Siu-type inequality proved in [33]. Note that Yuan's argument was inspired by the proof of Siu's inequalities in algebraic geometry as given in [26, Theorem 2.2.15], see also [26, p. 183] for a historical account of Siu's inequality.

Lemma 3.2. Let ϕ be a continuous psh metric on L, ψ_1, ψ_2 be continuous psh metrics on an auxiliary ample line bundle M, and set $f := \psi_1 - \psi_2$ and $C := ((L+M)^n) - (L^n) > 0$. Then

$$C \inf_{x \in X^{\mathrm{an}}} f(x) \le \int_{X^{\mathrm{an}}} f\,(dd^c\phi + dd^c\psi_1)^n - \mathrm{vol}(L, \phi + f, \phi) \le C \sup_{x \in X^{\mathrm{an}}} f(x). \tag{3.1}$$

Proof. In the proof, we assume that the reader is familiar with the properties of Monge–Ampère measures and relative volumes given in §1.6 and in §1.7. First, we give a few reduction steps.

By the invariance of relative volumes under ground field extension, we can pass to a non-Archimedean extension and assume that K is algebraically closed and non-trivially valued (as we did at the beginning of §1.6 for Monge–Ampère measures). Every continuous psh metric on an ample line bundle is then a uniform limit of metrics induced by nef $\mathbb{Q}$-models of L. By continuity of Monge–Ampère measures and relative volumes with respect to uniform convergence, we may thus assume that there exist nef $\mathbb{Q}$-models $\mathscr{L}$ and $\mathscr{M}_i$ of L and M, determined on a model $\mathscr{X}$ of X, such that $\phi = \phi_{\mathscr{L}}$ and $\psi_i = \phi_{\mathscr{M}_i}$. Since K is algebraically closed, we can further assume after passing to a higher model that $\mathscr{X}$ has reduced special fiber, and that L and M admit ample $\mathbb{Q}$-models $\mathscr{L}'$, $\mathscr{M}'$ on $\mathscr{X}$, by Lemma 1.1. Replacing $\mathscr{L}$ and $\mathscr{M}_i$ with $(1-\varepsilon)\mathscr{L} + \varepsilon\mathscr{L}'$ and $(1-\varepsilon)\mathscr{M}_i + \varepsilon\mathscr{M}'$, $0 < \varepsilon \ll 1$, we are thus reduced to the case where $\mathscr{L}$ and the $\mathscr{M}_i$ themselves are ample $\mathbb{Q}$-line bundles, using again the continuity of Monge–Ampère measures and relative volumes with respect to uniform convergence. Replacing L and M with large enough multiples and using the homogeneity property of relative volumes, we can finally assume that $\mathscr{L}$ and the $\mathscr{M}_i$ are honest ample line bundles on $\mathscr{X}$ such that each admits a relatively regular section, using [6, Proposition A.15].

Observe that adding to f a constant $a \in \mathbb{R}$ translates the quantity

$$\int_{X^{\mathrm{an}}} f\,(dd^c\phi + dd^c\psi_1)^n - \mathrm{vol}(L, \phi + f, \phi)$$

by aC. In order to prove the left-hand inequality in (3.1), we may thus replace f with $f - \inf_{X^{\mathrm{an}}} f$ and assume $\inf_{X^{\mathrm{an}}} f = 0$. The unique vertical Cartier divisor E on $\mathscr{X}$ such that $\mathscr{M}_1 - \mathscr{M}_2 = \mathcal{O}(E)$ satisfies $\phi_E = f \geq 0$, and E is thus effective by Lemma 1.3, since $\mathscr{X}$ has reduced special fiber. Pick integers $1 \leq j \leq m$. The restriction exact sequence

$$0 \to H^0\left(\mathscr{X}, m\mathscr{L} + (j-1)E\right) \to H^0\left(\mathscr{X}, m\mathscr{L} + jE)\right) \to H^0\left(E, (m\mathscr{L} + jE)|_E\right)$$

yields

$$\begin{aligned}\mathrm{vol}\left(\|\cdot\|_{m\phi+jf}, \|\cdot\|_{m\phi+(j-1)f}\right) &= \mathrm{vol}\left(\|\cdot\|_{H^0(\mathscr{X}, m\mathscr{L}+jE)}, \|\cdot\|_{H^0(\mathscr{X}, m\mathscr{L}+(j-1)E)}\right) \\ &\leq h^0\left(E, (m\mathscr{L} + jE)|_E\right),\end{aligned}$$

where the 1st equality follows from Lemma 1.3 and the inequality follows from (1.1). Summing up over j and using the cocycle property of relative volumes, we infer

$$\mathrm{vol}\left(\|\cdot\|_{m(\phi+f)}, \|\cdot\|_{m\phi}\right) \leq \sum_{j=1}^{m} h^0\left(E, (m\mathscr{L} + jE)|_E\right).$$

Since $\mathscr{M}_1$ and $\mathscr{M}_2$ admit relatively regular sections, their restrictions to E admit regular sections as well, by Lemma 1.2. For $j = 1, \ldots, m$ we thus have

$$h^0\left(E, (m\mathscr{L} + jE)|_E\right) = h^0\left(E, (m\mathscr{L} + j\mathscr{M}_1 - j\mathscr{M}_2)|_E\right) \leq h^0\left(E, m(\mathscr{L} + \mathscr{M}_1)|_E\right),$$

and hence

$$\mathrm{vol}(\|\cdot\|_{m(\phi+f)}, \|\cdot\|_{m\phi}) \leq m\, h^0\left(E, m(\mathscr{L} + \mathscr{M}_1)|_E\right).$$

As a result,

$$\begin{aligned}\mathrm{vol}\,(L, \phi + f, \phi) &= \lim_{m\to\infty} \frac{n!}{m^{n+1}} \mathrm{vol}\left(\|\cdot\|_{m(\phi+f)}, \|\cdot\|_{m\phi}\right) \\ &\leq \lim_{m\to\infty} \frac{n!}{m^n} h^0\left(E, m(\mathscr{L} + \mathscr{M}_1)|_E\right) = \int_{X^{\mathrm{an}}} f\,(dd^c\phi + dd^c\psi_1)^n,\end{aligned}$$

where the last equality follows from Theorem 2.4. This concludes the proof of the left-hand inequality in (3.1).

The proof of the right-hand inequality is very similar. In that case, we may replace f with $f - \sup_{X^{\mathrm{an}}} f$ and assume $\sup_{X^{\mathrm{an}}} f = 0$. As a result, the vertical Cartier divisor D with $\mathcal{O}(D) = \mathcal{M}_2 - \mathcal{M}_1$ is effective, using $\phi_D = -f \geq 0$. The restriction exact sequence

$$0 \to H^0\left(\mathscr{X}, m\mathscr{L} - (j+1)D\right) \to H^0\left(\mathscr{X}, m\mathscr{L} - jD\right) \to H^0\left(D, (m\mathscr{L} - jD)|_D\right)$$

then shows that

$$\operatorname{vol}\left(\|\cdot\|_{m\phi}, \|\cdot\|_{m(\phi+f)}\right) \leq \sum_{j=0}^{m-1} h^0\left(D, (m\mathscr{L} - jD)|_D\right) \leq m\, h^0\left(D, m(\mathscr{L} + \mathscr{M}_1)|_D\right),$$

which yields

$$-\operatorname{vol}(L, \phi + f, \phi) = \operatorname{vol}(L, \phi, \phi + f) \leq \int_{X^{\mathrm{an}}} \phi_D \left(dd^c(\phi + \psi_1)\right)^n = -\int_{X^{\mathrm{an}}} f \left(dd^c(\phi + \psi_1)\right)^n$$

proving the right-hand inequality and hence the claim. ■

Proof of Theorem 3.1. Let ϕ be a continuous psh metric on L and f be a continuous function on X^{an}. Assume first that there exist continuous psh metrics ψ_1, ψ_2 on an ample line bundle M such that $f = \psi_1 - \psi_2$. Pick $m \in \mathbb{Z}_{>0}$, $t \in (0, m^{-1}]$, and observe that $mtf = \psi_1 - \psi_{2,t}$ where

$$\psi_{2,t} := \psi_1 - mtf = (1 - mt)\psi_1 + mt\psi_2$$

is a continuous psh metric on M, as a convex combination of such metrics. By Lemma 3.2, we thus have

$$tmC_m \inf_{x \in X^{\mathrm{an}}} f(x) \leq tm \int_{X^{\mathrm{an}}} f\,(mdd^c\phi + dd^c\psi_1)^n - \operatorname{vol}(mL, m\phi + mtf, m\phi) \leq tmC_m \sup_{x \in X^{\mathrm{an}}} f(x)$$

with

$$C_m := ((mL + M)^n) - ((mL)^n).$$

By homogeneity of relative volumes, $\operatorname{vol}(mL, m\phi + mtf, m\phi) = m^{n+1}\operatorname{vol}(L, \phi + tf, \phi)$, thus

$$m^{-n}C_m \inf_{x \in X^{\mathrm{an}}} f(x) \leq \int_{X^{\mathrm{an}}} f\,(dd^c\phi + m^{-1}dd^c\psi_1)^n - t^{-1}\operatorname{vol}(L, \phi + tf, \phi) \leq m^{-n}C_m \sup_{x \in X^{\mathrm{an}}} f(x),$$

and hence

$$\int_{X^{\mathrm{an}}} f\,(dd^c\phi + m^{-1}dd^c\psi_1)^n - m^{-n}C_m \sup_{x\in X^{\mathrm{an}}} f(x) \le \liminf_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi)$$

$$\le \limsup_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi) \le \int_{X^{\mathrm{an}}} f\,(dd^c\phi + m^{-1}dd^c\psi_1)^n - m^{-n}C_m \inf_{x\in X^{\mathrm{an}}} f(x).$$

Now $m^{-n}C_m \to 0$ as $m\to\infty$, and we conclude as desired

$$\lim_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi) = \int_{X^{\mathrm{an}}} f\,(dd^c\phi)^n.$$

Let now f be an arbitrary continuous function on X^{an}. By density of model functions in $C^0(X^{\mathrm{an}})$, we can pick a sequence $(f_i)_{i\in\mathbb{N}}$ of model functions on X^{an} such that

$$\varepsilon_i := \sup_{x\in X^{\mathrm{an}}} |f(x) - f_i(x)| \to 0.$$

Pick any ample line bundle M on X. Since M admits ample $\mathbb{Q}$-models on arbitrarily high models [22, Proposition 4.11, Lemma 4.12], each model function f_i can be written as $f_i = \psi_{i1} - \psi_{i2}$ where ψ_{i1},ψ_{i2} are model metrics on a_iM for some non-zero $a_i \in \mathbb{N}$, determined by ample $\mathbb{Q}$-models $\mathscr{M}_{i1},\mathscr{M}_{i2}$ of a_iM.

Since $f_i - \varepsilon_i \le f \le f_i + \varepsilon_i$, the monotonicity of relative volumes yields for each $t > 0$

$$\mathrm{vol}(L,\phi+tf_i,\phi) - tV\varepsilon_i \le \mathrm{vol}(L,\phi+tf,\phi) \le \mathrm{vol}(L,\phi+tf_i,\phi) + tV\varepsilon_i$$

with $V := (L^n)$. By the first part of the proof, we infer

$$\int_{X^{\mathrm{an}}} f_i\,(dd^c\phi)^n - V\varepsilon_i \le \liminf_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi)$$

$$\le \limsup_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi) \le \int_{X^{\mathrm{an}}} f_i\,(dd^c\phi)^n + V\varepsilon_i,$$

and letting $i\to\infty$ yields as desired

$$\lim_{t\to 0_+} t^{-1}\mathrm{vol}(L,\phi+tf,\phi) = \int_{X^{\mathrm{an}}} f\,(dd^c\phi)^n.$$

Replacing f by $-f$, we conclude that the above holds also for t negative, and Theorem 3.1 follows. ■

3.2 Differentiability and orthogonality

In this subsection, we assume that continuity of envelopes holds for (X, L). The psh envelope $\mathrm{P}(\phi)$ of a continuous metric ϕ on L is thus the greatest continuous psh metric on L such that $\mathrm{P}(\phi) \leq \phi$, see §1.5. Note that $\phi \mapsto \mathrm{P}(\phi)$ is monotone increasing, and satisfies $\mathrm{P}(\phi + c) = \mathrm{P}(\phi) + c$ for $c \in \mathbb{R}$, two properties that formally imply

$$|\mathrm{P}(\phi) - \mathrm{P}(\psi)| \leq \sup_{x \in X^{\mathrm{an}}} |\phi(x) - \psi(x)| \tag{3.2}$$

for all continuous metrics ϕ, ψ on L.

To ease notation, we fix in what follows a reference continuous psh metric ϕ_0 on L, and denote by

$$\mathrm{E}(\phi) := \mathrm{E}(\phi, \phi_0)$$

the relative energy of a continuous psh metric ϕ on L with respect to ϕ_0. By Theorem 1.7, we have

$$\mathrm{E}(\mathrm{P}(\phi)) = \mathrm{vol}(L, \phi, \phi_0) \tag{3.3}$$

for all continuous metrics ϕ on L.

Definition 3.3. Given a continuous metric ϕ on L, we say that

- $\mathrm{E} \circ \mathrm{P}$ *is differentiable at* ϕ if

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{E}(\mathrm{P}(\phi + tf)) = \int_{X^{\mathrm{an}}} f\,(dd^c \mathrm{P}(\phi))^n, \tag{3.4}$$

  for all $f \in C^0(X^{\mathrm{an}})$;
- *orthogonality holds for* ϕ if the Monge–Ampère measure $(dd^c\mathrm{P}(\phi))^n$ is supported in the contact locus $\{\mathrm{P}(\phi) = \phi\}$, that is,

$$\int_{X^{\mathrm{an}}} (\phi - \mathrm{P}(\phi))\,(dd^c \mathrm{P}(\phi))^n = 0. \tag{3.5}$$

Theorem 3.4. Assume that continuity of envelopes holds for (X, L). Then $\mathrm{E} \circ \mathrm{P}$ is differentiable at each continuous metric ϕ on L, and orthogonality holds for ϕ.

Lemma 3.5. The following properties are equivalent:

(i) $\mathrm{E}\circ\mathrm{P}$ is differentiable at all continuous metrics on L;

(ii) $\mathrm{E}\circ\mathrm{P}$ is differentiable at all continuous psh metrics on L;

(iii) orthogonality holds for all continuous metrics on L.

Proof. (i)$\Longrightarrow$(ii) is trivial. We reproduce the simple argument for (ii)$\Longrightarrow$(iii) given in [11, Theorem 6.3.2]. Pick a continuous metric ϕ, and set $\psi := \mathrm{P}(\phi)$ and $f := \phi - \psi$. For each $t \in [0,1]$, $\psi + tf = (1-t)\mathrm{P}(\phi) + t\phi$ satisfies $\mathrm{P}(\phi) \le \psi + tf \le \phi$, and hence $\mathrm{P}(\phi) = \mathrm{P}(\psi + tf)$. Differentiability of $\mathrm{E}\circ\mathrm{P}$ at ψ thus yields

$$0 = \frac{d}{dt}\bigg|_{t=0} \mathrm{E}(\mathrm{P}(\psi + tf)) = \int_{X^{\mathrm{an}}} f\,(dd^c\mathrm{P}(\phi))^n = \int (\phi - \mathrm{P}(\phi))(dd^c\mathrm{P}(\phi))^n,$$

which proves that ϕ satisfies the orthogonality property. Finally, the following simple argument for (iii)$\Longrightarrow$(i) is similar to the proof of [28, Lemma 6.13]. Pick a continuous metric ϕ and a continuous function f. By concavity of E (see (1.4)), we have

$$\int_{X^{\mathrm{an}}} (\mathrm{P}(\phi + tf) - \mathrm{P}(\phi))\,(dd^c\mathrm{P}(\phi + tf))^n \le \mathrm{E}(\mathrm{P}(\phi + tf)) - \mathrm{E}(\mathrm{P}(\phi))$$
$$\le \int_{X^{\mathrm{an}}} (\mathrm{P}(\phi + tf) - \mathrm{P}(\phi))\,(dd^c\mathrm{P}(\phi))^n.$$

Using the orthogonality property at $\phi + tf$ and ϕ together with $\mathrm{P}(\phi) \le \phi$ and $\mathrm{P}(\phi + tf) \le \phi + tf$, this yields for $t > 0$

$$\int_{X^{\mathrm{an}}} f\,(dd^c\mathrm{P}(\phi + tf))^n \le \frac{\mathrm{E}(\mathrm{P}(\phi + tf)) - \mathrm{E}(\mathrm{P}(\phi))}{t} \le \int_{X^{\mathrm{an}}} f\,(dd^c\mathrm{P}(\phi))^n,$$

and hence

$$\lim_{t\to 0_+} \frac{\mathrm{E}(\mathrm{P}(\phi + tf)) - \mathrm{E}(\mathrm{P}(\phi))}{t} = \int_{X^{\mathrm{an}}} f\,(dd^c\mathrm{P}(\phi))^n,$$

by uniform convergence of $\mathrm{P}(\phi + tf)$ to $\mathrm{P}(\phi)$, cf. (3.2). Replacing f by $-f$ proves (iii)$\Longrightarrow$(i). ■

Proof of Theorem 3.4. Taking into account (3.3), Theorem 3.1 precisely says that $\mathrm{E}\circ\mathrm{P}$ is differentiable at every continuous psh metric on L, and Theorem 3.4 thus follows from Lemma 3.5. ■

3.3 An application to Monge–Ampère equations

In this subsection we still assume that continuity of envelopes holds for (X, L). As in [10], we define a (possibly singular) psh metric on L as a decreasing limit of continuous psh metrics, not identically $-\infty$ on any component of X. A subset $E \subset X^{\mathrm{an}}$ is *pluripolar* if there exists a psh metric ϕ with $\phi \equiv -\infty$ on E, this condition being easily seen to be independent of the choice of ample line bundle L. If E is nonpluripolar, one proves exactly as in [9, Proposition 5.2(ii)] that for each continuous metric ϕ on L there exists a constant $C > 0$ such that

$$\sup_{x \in X^{\mathrm{an}}} (\psi(x) - \phi(x)) \le \sup_{x \in E} (\psi(x) - \phi(x)) + C \tag{3.6}$$

for all psh metrics ψ on L. Given a nonpluripolar compact $E \subset X^{\mathrm{an}}$ and a continuous metric ϕ on L, we can thus define the *equilibrium metric* of the pair (E, ϕ) as

$$\mathrm{P}(E, \phi) := \sup\{\psi \text{ psh metric on } L \mid \psi \le \phi \text{ on } E\}.$$

Since every psh metric ψ on L is a decreasing limit of continuous psh metrics, Dini's lemma easily yields

$$\mathrm{P}(E, \phi) = \sup\{\psi \text{ continuous psh metric on } L \mid \psi \le \phi \text{ on } E\},$$

(see [6, Proposition 7.26]) which is thus lsc. By (3.6), the family of metrics ψ in the definition of $\mathrm{P}(E, \phi)$ is uniformly bounded from above, the usc regularization $\mathrm{P}(E, \phi)^\star$ is thus psh, since we assume continuity of envelopes (see [6, Lemma 7.30]). As a result, $\mathrm{P}(E, \phi)^\star \le \phi$ holds on E if and only if $\mathrm{P}(E, \phi) = \mathrm{P}(E, \phi)^\star$ is continuous. Following classical terminology in pluripotential theory, we then say that (E, ϕ) is *L-regular*.

For a nonpluripolar point $x \in X^{\mathrm{an}}$, L-regularity of $(\{x\}, \phi)$ is independent of the continuous metric ϕ, as the latter only appears through its value at x, and we then simply say that x is *L-regular*.

Example 3.6. By [10, Lemma 2.20, Theorem 2.21], every quasimonomial point of X^{an} is nonpluripolar.

Conjecturally, every nonpluripolar point should be L-regular; this has been shown in [9, Theorem 5.13] when X is smooth, K has residue characteristic 0, and is trivially or discretely valued.

Relying on the variational argument developed in [5, 7], we prove the following result, which corresponds to Corollary C in the introduction.

Theorem 3.7. Assume that continuity of envelopes holds for (X, L). Let $x \in X^{\mathrm{an}}$ be a nonpluripolar point, ϕ a continuous metric on L, and assume that x is L-regular, so that

$$\phi_x := \mathrm{P}(\{x\}, \phi) = \sup\{\psi \text{ psh metric on } L \mid \psi(x) \le \phi(x)\}$$

is continuous and psh. Then $V^{-1}(dd^c\phi_x)^n = \delta_x$ with $V := (L^n)$.

Proof. Pick $f \in C^0(X^{\mathrm{an}})$. Since $\mathrm{P}(\phi_x + f) - f(x)$ is a continuous psh metric on L and satisfies

$$\mathrm{P}(\phi_x + f)(x) - f(x) \le \phi_x(x) = \phi(x),$$

we have $\mathrm{P}(\phi_x + f) - f(x) \le \phi_x$ by definition of the latter, and hence $\mathrm{E}(\mathrm{P}(\phi_x + f)) - Vf(x) \le \mathrm{E}(\phi_x)$. Applying this to tf, $t > 0$, we infer

$$t^{-1}\left(\mathrm{E}(\mathrm{P}(\phi_x + tf)) - \mathrm{E}(\phi_x)\right) \le Vf(x),$$

and Theorem 3.4 thus yields

$$\int_{X^{\mathrm{an}}} f\,(dd^c\phi_x)^n \le Vf(x).$$

Replacing f with $-f$ concludes the proof. ■

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