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SINGULAR SEMIPOSITIVE METRICS IN NON-ARCHIMEDEAN GEOMETRY

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Abstract

Let X be a smooth projective Berkovich space over a complete discrete valuation field K of residue characteristic zero, endowed with an ample line bundle L. We introduce a general notion of (possibly singular) semipositive (or plurisubharmonic) metrics on L and prove the analogue of the following two basic results in the complex case: the set of semipositive metrics is compact modulo scaling, and each semipositive metric is a decreasing limit of smooth semipositive ones. In particular, for continuous metrics, our definition agrees with the one by S.-W. Zhang. The proofs use multiplier ideals and the construction of suitable models of X over the valuation ring of K, using toroidal techniques.

Contents

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Introduction

The notions of plurisubharmonic (psh) functions and positive currents lie at the heart of complex analysis. The study of these objects is usually referred to as *pluripotential theory*, and it has become apparent in recent years that pluripotential theory should admit an analogue in the context of non-Archimedean analytic spaces in the sense of Berkovich.

Potential theory on non-Archimedean curves is by now well established, thanks to the work of Thuillier [Thu05] (see also [FJ04, BR]). In higher dimensions, it should in principle be possible to mimic the complex case and define a plurisubharmonic function as an upper semicontinuous function whose restriction to any curve is subharmonic. While this approach is yet to be developed, a general notion of *continuous* plurisubharmonic functions was very recently[1] introduced by Chambert-Loir and Ducros in [CLD12], based on their notion of positive currents. Their definition both localizes and generalizes the previously introduced notion of a *semipositive metric* on a line bundle [Zha95, Gub98, CL06]. In this paper we propose a general (global) definition of singular (i.e. not necessarily continuous) semipositive metrics on ample line bundles and prove basic compactness and regularization results for such metrics.

In a sequel to this paper [BFJ12a] we rely on the results obtained here to adapt to the non-Archimedean case the variational approach to complex Monge-Ampère equations developed in [BBGZ13] and prove a version of the celebrated Calabi-Yau theorem.

In order to better explain our construction, let us briefly recall some facts from the complex case [Dem90]. Let X be (the analytification of) a smooth projective complex variety and let L be an *ample* line bundle on X. A smooth metric $\|\cdot\|$ on L is given in every local trivialization of L by $|\cdot|e^{-\varphi}$ for some local smooth function φ, called the local weight of the metric. The metric is said to be semipositive if its curvature, which locally is given by $dd^c\varphi$, is a semipositive $(1,1)$-form; that is, φ is psh. More generally, one defines the notion of singular semipositive metrics by allowing φ to be a general psh function, in which case the curvature is a positive closed $(1,1)$-current.

[1] In fact, [CLD12] was posted after the first version of the present paper.

It is a basic fact that every psh function is locally the decreasing limit of a sequence of smooth psh functions. The global analogue of this result for singular semipositive metrics fails for general line bundles, but a deep result of Demailly shows that every singular semipositive metric on an *ample* line bundle L is indeed a *monotone* limit of smooth semipositive metrics (cf. [Dem92] or [GZ05, Theorem 8.1], [BK07] for more recent accounts). In particular, every continuous semipositive metric on L is a *uniform* limit on X of smooth semipositive metrics, thanks to Dini's lemma.

A fundamental aspect of singular semipositive metrics is that they form a compact space modulo scaling. This fact can be conveniently understood in terms of global weights as follows. Fixing a smooth metric on L with curvature θ allows one to identify the set of singular semipositive metrics with the set $\mathrm{PSH}(X, \theta)$ of θ-psh functions. The latter are upper semicontinuous (usc) functions $\varphi : X \to [-\infty, +\infty)$ such that $\theta + dd^c\varphi$ is a positive closed $(1,1)$-current. Modulo scaling, $\mathrm{PSH}(X, \theta)$, endowed with the L^1-topology, is homeomorphic to the space of closed positive $(1,1)$-currents lying in the cohomology class $c_1(L)$ (with its weak topology) and hence is compact.

Let us now turn to the non-Archimedean case. Fix a complete, discrete valuation field K with valuation ring R and residue field k, and set $S := \operatorname{Spec} R$. We assume that k (and hence K) has characteristic zero, which means, concretely, that R is (non-uniquely) isomorphic to $k[\![t]\!]$. Let X be a smooth projective K-analytic space in the sense of Berkovich, so that X is the analytification of a smooth projective K-variety by the GAGA principle. Recall that the underlying topological space of X is compact Hausdorff. A *model* $\mathcal{X}$ of X is a normal flat projective S-scheme together with an isomorphism of the analytification of its generic fiber $\mathcal{X}_K$ with X. Each line bundle L on X is (again, by GAGA) the analytification of a line bundle on $\mathcal{X}_K$ (that we also denote by L), and a *model metric* is a metric $\|\cdot\|_{\mathcal{L}}$ on L that is naturally induced by the choice of a $\mathbf{Q}$-line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ such that $\mathcal{L}|_{\mathcal{X}_K} = L$ in $\mathrm{Pic}(\mathcal{X}_K)_{\mathbf{Q}}$. A *model function* φ on X is a function such that $e^{-\varphi}$ is a model metric on the trivial line bundle. The set $\mathcal{D}(X)$ of model functions is then dense in $C^0(X)$, a well-known consequence of the Stone-Weierstrass theorem.

Following S.-W. Zhang [Zha95, 3.1] (see also [KT, 6.2.1], [Gub98, 7.13], [CL06, 2.2]), we shall say that a model metric $\|\cdot\|_{\mathcal{L}}$ on L is *semipositive* if $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ is nef on the special fiber $\mathcal{X}_0$ of $\mathcal{X}$, i.e. $\mathcal{L} \cdot C \geq 0$ for all projective curves C in $\mathcal{X}_0$. A continuous metric on L is then *semipositive in the sense of Zhang* if it can be written as a uniform limit over X of a sequence of semipositive model metrics. The reader may consult [CL11] for a nice survey on these notions.

In order to define a general notion of singular semipositive metrics, we use the finer description of X as an inverse limit of dual complexes (called *skeletons* in Berkovich's terminology). Since the residue field k of R has characteristic zero, it follows from [Tem06] that each model of X is dominated by an *SNC model* $\mathcal{X}$, by which we understand a regular model whose special fiber $\mathcal{X}_0$ has simple normal crossing support (plus a harmless irreducibility condition that we impose for convenience). To each SNC model $\mathcal{X}$ corresponds its *dual complex* $\Delta_{\mathcal{X}}$, a compact simplicial complex which encodes the incidence properties of the irreducible components of $\mathcal{X}_0$. The dual complex $\Delta_{\mathcal{X}}$ embeds canonically in X and for the purposes of this introduction we shall view $\Delta_{\mathcal{X}}$ as a compact subset of X. There is furthermore a retraction $p_{\mathcal{X}} : X \to \Delta_{\mathcal{X}}$. These maps are compatible with respect to domination of models, and we thus get a map

$$X \to \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$$

which is known to be a homeomorphism (see, for instance, [KS06, p. 77, Theorem 10]).

Following the philosophy of [BGS95], we define the space of *closed* $(1,1)$-*forms* on X as the direct limit over all models $\mathcal{X}$ of X of the spaces $N^1(\mathcal{X}/S)$ of codimension one numerical equivalence classes. Any model metric gives rise to a *curvature form* lying in this space. The main reason for working with numerical equivalence (instead of rational equivalence as in [BGS95]) is that we can then adapt a result of [Kün96] to show that a line bundle $L \in \mathrm{Pic}(X)$ has vanishing first Chern class $c_1(L) \in N^1(X)$ iff it admits a model metric with zero curvature (cf. Corollary 4.4).

Fix a reference model metric $\|\cdot\|$ on L with curvature form θ. Any other metric can be written $\|\cdot\|e^{-\varphi}$ for some function φ on X. When $\|\cdot\|e^{-\varphi}$ is a semipositive model metric, we say that the model function φ is θ*-psh*.

Definition. Let L be an ample line bundle on a smooth projective K-analytic variety X. Fix a model metric $\|\cdot\|$ on L with curvature form θ. A θ*-plurisubharmonic function* on X is then a function $\varphi : X \to [-\infty, +\infty)$ such that:

- φ is upper semicontinuous (usc).
- $\varphi \le \varphi \circ p_{\mathcal{X}}$ for each SNC model $\mathcal{X}$.
- φ is a uniform limit, on each dual complex $\Delta_{\mathcal{X}}$, of θ-psh model functions.

A *singular semipositive metric* is a metric $\|\cdot\|e^{-\varphi}$ with φ a θ-psh function.

The consistency of the definition for model functions will be guaranteed by Theorem 5.11 below. Let φ be a θ-psh function. Since φ is usc and each $\varphi \circ p_{\mathcal{X}}$ is continuous, it follows immediately that $\varphi = \inf_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}$, so that

φ is uniquely determined by its restriction to the dense subset $\bigcup_{\mathcal{X}} \Delta_{\mathcal{X}}$ of X. We may therefore endow the set $\mathrm{PSH}(X, \theta)$ of all θ-psh functions (or, equivalently, of all singular semipositive metrics on L) with the topology of uniform convergence on dual complexes. We view this topology as an analogue of the L^1-topology in the complex case. Our first main theorem shows that this space is indeed compact modulo additive constants, something that was also announced in the unpublished manuscript by Kontsevich and Tschinkel [KT].

Theorem A. *Let L be an ample line bundle on a smooth projective K-analytic variety X endowed with a model metric with curvature form θ. Then* $\mathrm{PSH}(X, \theta)/\mathbf{R}$ *is compact.*

In other words, the set of singular semipositive metrics on L modulo scaling is compact. For curves, this result is a consequence of the work of Thuillier [Thu05] and follows from basic properties of subharmonic functions on metrized graphs.

Our second main result is the following analogue of Demailly's global regularization theorem.

Theorem B. *Let L be an ample line bundle on a smooth projective K-analytic variety X endowed with a model metric with curvature form θ. Then every θ-psh function φ is the pointwise limit on X of a decreasing net of θ-psh model functions.*

When $\dim X = 1$, Theorem B is a special case of [Thu05, Théorème 3.4.19]. Thanks to Dini's lemma, Theorem B implies a non-Archimedean version of the Demailly-Richberg theorem, stating that every continuous θ-psh function is a *uniform* limit over X of θ-psh model functions. In other words, for continuous metrics our definition of semipositivity agrees with Zhang's.

Let us briefly explain how Theorem A above is proved. The first important fact is that any dual complex $\Delta_{\mathcal{X}}$ comes equipped with a natural affine structure [KS06] such that any θ-psh function is convex on the faces of $\Delta_{\mathcal{X}}$. On this complex we put any Euclidean metric compatible with the affine structure. The statement that we actually prove, and which implies Theorem A, is

Theorem C. *For each dual complex $\Delta_{\mathcal{X}}$ there exists a constant $C > 0$ such that $\varphi|_{\Delta_{\mathcal{X}}}$ is* Lipschitz continuous *with Lipschitz constant at most C for any θ-psh model function φ.*

This result is proved in two steps. Assuming, as we may, that $\sup_{\Delta_{\mathcal{X}}} \varphi = 0$, we first bound φ from below on the vertices of $\Delta_{\mathcal{X}}$. This is done by exploiting the non-negativity of certain intersection numbers, a direct consequence of the model metric $\|\cdot\|e^{-\varphi}$ being determined by a nef line bundle on some model.

The next step is to prove the uniform Lipschitz bound. Here again, the general idea is to exploit the non-negativity of certain intersection numbers, but the argument is more subtle than in the first step. This time, the intersection numbers are computed on (possibly singular) blowups of $\mathcal{X}$ corresponding to carefully chosen combinatorial decompositions of $\Delta_{\mathcal{X}}$, in the spirit of the toroidal constructions of [KKMS]. In [BFJ12b] we adapt these techniques to prove a uniform version of Izumi's theorem [Izu85].

The proof of Theorem B is of a different nature. Using Theorem A, we first show that the usc upper envelope of any family of θ-psh functions remains θ-psh, a basic property of θ-psh functions in the complex case. As a consequence, given any continuous function $u \in C^0(X)$ the set of all θ-psh functions ψ such that $\psi \le u$ on X admits a largest element, called the *θ-psh envelope* of u and denoted by $P_\theta(u)$. On the other hand, by the density of $\mathcal{D}(X)$ in $C^0(X)$, we may write any given function $\varphi \in \mathrm{PSH}(X,\theta)$ as the pointwise limit of a decreasing family of (*a priori* not θ-psh) model functions u_j, using only the upper semicontinuity of φ. It is not difficult to see that $P_\theta(u_j)$ decreases to φ, and we are thus reduced to showing that the θ-psh envelope of any model function is the uniform limit of a sequence of θ-psh model functions. This is proved using multiplier ideals, in the spirit of [DEL00, ELS03, BFJ08]. The required properties of multiplier ideals are shown to hold on regular models in Appendix B, the key point being to show that the expected version of the Kodaira vanishing theorem holds in this context.[2]

Let us comment on our assumptions on the field K. It is expected that pluripotential theory can be developed on Berkovich spaces over arbitrary non-Archimedean fields, and as we mentioned before the first steps in that direction have been taken in [CLD12] (see also [Gub13] for a nice account on these developments), where the notions of positive forms and currents are introduced, partially building upon ideas by Lagerberg [Lag12].

Our approach is geometric, and we restrict our attention to the discretely valued case in order to avoid the use of formal models over non-Noetherian rings. We refer to [Gub98, Gub03] for related works on semipositive metrics in the general setting of a complete non-Archimedean field. More importantly, Appendix B relies on the discreteness assumption.

Furthermore, we use the assumption that K has residue characteristic zero in two ways, through the existence of SNC models and through the cohomology vanishing properties of multiplier ideals. In positive residue characteristic, the existence of SNC models is not known and it is much harder to construct retractions of X onto suitable complexes (embedded or not). We refer to Berkovich [Ber99], Hrushovski-Loeser [HL10], and Thuillier [Thu11] for important contributions to the understanding of this problem.

[2] A simpler proof of the relevant vanishing theorem appeared very recently in [MN12], which was posted after the first version of the present paper.

The paper is organized as follows. The first two sections present the necessary background on Berkovich spaces and models. The exposition is largely self-contained (hence perhaps a bit lengthy), as we feel that the easy arguments that our setting allows are worth being explained. Section 3 is devoted to dual complexes. The main technical result is Theorem 3.15, on the existence of blowups attached to decompositions of a dual complex. Section 4 deals with closed $(1,1)$-forms, defined using a numerical equivalence variant of the approach of [BGS95]. In Section 5 we prove some basic properties of θ-psh model functions. Section 6 contains the proof of Theorem A. Section 7 is devoted to the first properties of general θ-psh functions. Theorem B is proved in Section 8. Finally, Appendix A contains a technical result on Lipschitz constants of convex functions, while Appendix B establishes the expected cohomology vanishing properties of multiplier ideals in our setting.

1. Models of varieties over discrete valuation fields

1.1. S-varieties. All schemes considered in this paper are separated and Noetherian, and all ideal sheaves are coherent. Let R be a complete discrete valuation ring with fraction field K and residue field k. We shall assume that k has characteristic zero (but we don't require it to be algebraically closed). Let $\varpi \in R$ be a uniformizing parameter and normalize the corresponding absolute value on K by $\log|\varpi|^{-1} = 1$. Each choice of a field of representatives of k in R then induces an isomorphism $R \simeq k[\![\varpi]\!]$ by Cohen's structure theorem.

Write $S := \operatorname{Spec} R$. We will use the following terminology. An *S-variety* is a flat integral S-scheme $\mathcal{X}$ of finite type. We denote by $\mathcal{X}_0$ its special fiber and by $\mathcal{X}_K$ its generic fiber, and we write $\kappa(\xi)$ for the residue field of a point $\xi \in \mathcal{X}$. An ideal sheaf $\mathfrak{a}$ on $\mathcal{X}$ is *vertical* if it is co-supported on the special fiber, and a fractional ideal sheaf $\mathfrak{a}$ is vertical if $\varpi^m\mathfrak{a}$ is a vertical ideal sheaf for some positive integer m. A *vertical blowup* $\mathcal{X}' \to \mathcal{X}$ is the normalized blowup along a vertical ideal sheaf $\mathfrak{a}$; this is the same as the blowup along the integral closure of $\mathfrak{a}$. We will occasionally consider a blowup along a fractional ideal sheaf $\mathfrak{a}$, which simply means the blowup along $\varpi^m\mathfrak{a}$ for any $m \in \mathbf{N}$ such that $\varpi^m\mathfrak{a}$ is an actual ideal sheaf.

Except for Appendix B, we will use additive notation for Picard groups, and we write $\mathcal{L}+\mathcal{M} := \mathcal{L}\otimes\mathcal{M}$ and $m\mathcal{L} := \mathcal{L}^{\otimes m}$ for $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(\mathcal{X})$. We denote by $\operatorname{Div}_0(\mathcal{X})$ the group of *vertical Cartier divisors* of $\mathcal{X}$, i.e. those Cartier divisors on $\mathcal{X}$ that are supported on the special fiber. When $\mathcal{X}$ is normal, it is easy to see that $\mathrm{Div}_0(\mathcal{X})$ is a free $\mathbf{Z}$-module of finite rank and that the natural sequence

$$0 \to \mathbf{Z}\mathcal{X}_0 \to \mathrm{Div}_0(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is exact. The last arrow to the right is (by definition) surjective if $\mathcal{X}$ is semifactorial. This happens, for instance, when $\mathcal{X}$ is regular. The existence of semifactorial models is proved over any DVR by Pépin in [Pep13]. In our setting of residue characteristic zero, the existence of regular models is guaranteed by a result of Temkin [Tem06]; see below.

Given an S-variety $\mathcal{X}$ let $(E_i)_{i\in I}$ be the (finite) set of irreducible components of its special fiber $\mathcal{X}_0$. Endow each E_i with the reduced scheme structure. For each subset $J \subset I$, set $E_J := \bigcap_{j\in J} E_j$.

Definition 1.1. Let $\mathcal{X}$ be an S-variety. We say that $\mathcal{X}$ is *vertically* $\mathbf{Q}$-*factorial* if each component E_i is $\mathbf{Q}$-Cartier. We say that $\mathcal{X}$ is *SNC* if

(i) the special fiber $\mathcal{X}_0$ has *simple normal crossing support*;
(ii) E_J is irreducible (or empty) for each $J \subset I$.

Condition (i) is equivalent to the following two conditions. First, $\mathcal{X}$ is regular. Given a point $\xi \in \mathcal{X}_0$, let $I_\xi \subset I$ be the set of indices $i \in I$ for which $\xi \in E_i$, and pick a local equation $z_i \in \mathcal{O}_{\mathcal{X},\xi}$ of E_i at ξ for each $i \in I_\xi$. We then also impose that $\{z_i,\, i \in I_\xi\}$ can be completed to a regular system of parameters of $\mathcal{O}_{\mathcal{X},\xi}$.

Condition (ii) is not imposed in the usual definition of a simple normal crossing divisor, but can always be achieved from (i) by further blowing-up along components of the possibly non-connected E_J's. Since k has characteristic zero, each S-variety is a $\mathbf{Q}$-scheme, which is furthermore excellent since it has finite type over S. It therefore follows from [Tem06] that for any S-variety $\mathcal{X}$ with smooth generic fiber, there exists a vertical blowup $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is SNC.

1.2. Numerical classes and positivity. Let $\mathcal{X}$ be a normal projective S-variety.

Lemma 1.2. *Assume that* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ *is nef on* $\mathcal{X}_0$*; i.e.* $\mathcal{L} \cdot C \geq 0$ *for all* k*-proper curves* C *in* $\mathcal{X}_0$*. Then* $\mathcal{L}$ *is also nef on* $\mathcal{X}_K$*; i.e.* $\mathcal{L} \cdot C \geq 0$ *for all* K*-proper curves* C *in* $\mathcal{X}_K$ *as well.*

We will then simply say that $\mathcal{L}$ is nef. A *curve* is by definition reduced and irreducible.

Proof. Let C be a K-proper curve in $\mathcal{X}_K$ and let $\mathcal{C}$ be its closure in $\mathcal{X}$ equipped with its reduced structure. By [Har, Proposition III.9.7], $\mathcal{C}$ is flat over S. The degrees of $\mathcal{L}|_{\mathcal{C}}$ on the generic fiber and on the special fiber therefore coincide, which reads $\mathcal{L} \cdot C = \mathcal{L} \cdot \mathcal{C}_0$. Now $\mathcal{C}_0$ is an effective linear combination of vertical curves, and the result follows. $\square$

We recall the following standard notions.

Definition 1.3. Let $\mathcal{X}$ be a normal projective S-variety.

(i) The space $N^1(\mathcal{X}/S)$ of *codimension* 1 *numerical classes* is defined as the quotient of $\mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$ by the subspace spanned by *numerically trivial line bundles*, i.e. those $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ such that $\mathcal{L} \cdot C = 0$ for all projective curves contained in a fiber of $\mathcal{X} \to S$.

(ii) The *nef cone* $\mathrm{Nef}(\mathcal{X}/S) \subset N^1(\mathcal{X}/S)$ is defined as the set of numerical classes $\alpha \in N^1(\mathcal{X}/S)$ such that $\alpha \cdot C \geq 0$ for all projective curves contained in a fiber of $\mathcal{X} \to S$.

Note that the $\mathbf{R}$-vector space $N^1(\mathcal{X}/S)$ is finite dimensional. Indeed Lemma 1.2 shows that the restriction map $N^1(\mathcal{X}/S) \to N^1(\mathcal{X}_0/k)$ is injective, and the latter space is finite dimensional since $\mathcal{X}_0$ is projective over k. Observe also that $\mathrm{Nef}(\mathcal{X}/S)$ is a closed convex cone of $N^1(\mathcal{X}/S)$. Lemma 1.2 implies that $\mathrm{Nef}(\mathcal{X}/S) = \mathrm{Nef}(\mathcal{X}_0/k) \cap N^1(\mathcal{X}/S)$ under the injection $N^1(\mathcal{X}/S) \to N^1(\mathcal{X}_0/k)$.

We have the following standard fact:

Lemma 1.4. *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a vertical blowup.*

(i) *There exists a π-ample divisor $A \in \mathrm{Div}_0(\mathcal{X}')$.*

(ii) *If $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ is ample, then there exists $m \in \mathbf{N}$ such that $\pi^*(m\mathcal{L}|_{\mathcal{X}_K})$ extends to an ample line bundle $\mathcal{L}'$ on $\mathcal{X}'$.*

Proof. By definition, there exists a vertical ideal sheaf $\mathfrak{a}$ on $\mathcal{X}$ such that π is obtained as the blowup of $\mathcal{X}$ along $\mathfrak{a}$. The universal property of blowups yields a π-ample Cartier divisor A on $\mathcal{X}'$ such that $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(A)$, and A is also vertical since $\mathfrak{a}$ is, which proves (i). If $\mathcal{L}$ is ample on $\mathcal{X}$, then $m\pi^*\mathcal{L} + A$ is ample on $\mathcal{X}'$ for $m \gg 1$, and (ii) follows. $\square$

Recall that an $\mathbf{R}$-line bundle on $\mathcal{X}$ (resp. $\mathcal{X}_K$) is ample if it can be written as a positive linear combination of ample line bundles.

Corollary 1.5. *If $L \in \mathrm{Pic}(\mathcal{X}_K)_{\mathbf{R}}$ is ample, then L extends to an ample $\mathbf{R}$-line bundle $\mathcal{L}' \in \mathrm{Pic}(\mathcal{X}')_{\mathbf{R}}$ for all sufficiently high vertical blowups $\mathcal{X}' \to \mathcal{X}$.*

Proof. Write $L = \sum_i c_i L_i$ where $c_i \in \mathbf{R}_{>0}$ and $L_i \in \mathrm{Pic}(\mathcal{X}_K)$ is ample for all i. We may assume that L_i is very ample for each i, so that the linear system $|L_i|$ embeds $\mathcal{X}$ into a suitable projective space $\mathbf{P}^{r_i}_K$ over K. Let $\mathcal{X}_i$ be the normalization of the closure of $\mathcal{X}_K$ in $\mathbf{P}^{r_i}_S$ and let $\mathcal{L}_i$ be the restriction of $\mathcal{O}(1)$ to $\mathcal{X}_i$. Let $\mathcal{X}'$ be any normal S-variety dominating $\mathcal{X}$ as well as all the $\mathcal{X}_i$, and write $\pi_i : \mathcal{X}' \to \mathcal{X}_i$ for the associated vertical blowups. By Lemma 1.4(ii) we can find $m_i \in \mathbf{N}$ and ample line bundles $\mathcal{L}'_i$ on $\mathcal{X}'$ such that $\mathcal{L}'_i|_{\mathcal{X}'_K} = \pi_i^*(m_i \mathcal{L}_i|_{\mathcal{X}_{i,K}})$. We can then pick $\mathcal{L}' := \sum_i \frac{c_i}{m_i} \mathcal{L}'_i$. $\square$

We shall use the following version of the *Negativity Lemma*; cf. [KM98, Lemma 3.39]. The proof we give is a variant of the argument used in [BdFF10, Proposition 2.12].

Lemma 1.6. *Assume that $\mathcal{X}$ is vertically $\mathbf{Q}$-factorial and let $\pi : \mathcal{X}' \to \mathcal{X}$ be a vertical blowup. If $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{R}}$ is π-nef, then $\pi^*\pi_*D - D$ is effective.*

The condition that $\mathcal{X}$ is vertically $\mathbf{Q}$-factorial guarantees that the pull-back of π_*D by π is well defined.

Proof. As a first step, we reduce the assertion to the case where $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ is π-ample. Indeed, the set of vertical π-ample $\mathbf{R}$-divisors, which is an open convex cone in $\mathrm{Div}_0(\mathcal{X}')_{\mathbf{R}}$, is non-empty by (i) of Lemma 1.4. We may thus choose a basis $A_1, \dots, A_r$ of $\mathrm{Div}_0(\mathcal{X}')_{\mathbf{R}}$ made up of π-ample Cartier divisors. Let $\varepsilon = (\varepsilon_i) \in \mathbf{R}^r_+$ be such that $D_\varepsilon := D + \sum_i \varepsilon_i A_i$ is a $\mathbf{Q}$-divisor. The fact that D is π-nef means that $D \cdot C \geq 0$ for each curve C contained in a fiber of π. Since each A_i is π-ample, it follows from Kleiman's criterion [Kle66] that D_ε is π-ample on the projective k-scheme $\mathcal{X}'_0$; hence D_ε is also π-ample on $\mathcal{X}'$ by [EGA, III.4.7.1]. Upon replacing D with D_ε for ε arbitrarily small we may thus assume as desired that $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ is π-ample.

Now choose $m \gg 1$ such that $\mathcal{O}_{\mathcal{X}'}(mD)$ is π-globally generated, which means that the vertical fractional ideal sheaf $\mathfrak{a} := \pi_*\mathcal{O}_{\mathcal{X}'}(mD)$ satisfies $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(mD)$. It is obvious that $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}(m\pi_*D)$, hence $\mathcal{O}_{\mathcal{X}'}(mD) \subset \mathcal{O}_{\mathcal{X}'}(m\pi^*\pi_*D)$, and the result follows. ☐

2. Projective Berkovich spaces and model functions

2.1. Analytifications. Let Y be a proper K-scheme. As a topological space, its K-analytification Y^{an} in the sense of Berkovich is compact and can be described as follows (cf. [Ber90, Theorem 3.4.1]). Choose a finite cover of Y by Zariski open subsets of the form $U = \operatorname{Spec} A$ where A is a finitely generated K-algebra. The Berkovich space U^{an} is defined as the set of all multiplicative seminorms $|\cdot|_x : A \to \mathbf{R}_+$ extending the given absolute value of K, endowed with the topology of pointwise convergence. It is common usage to write $|f(x)| := |f|_x$ for $f \in A$ and $x \in U^{\mathrm{an}}$. The space Y^{an} is then obtained by gluing together the open sets U^{an}.

There is a canonical continuous map $s : Y^{\mathrm{an}} \to Y$, locally defined on U^{an} by setting

$$s(x) = \{f \in A \mid |f(x)| = 0\}.$$

The seminorm $|\cdot|_x$ defines a norm on the residue field $\kappa(s(x))$, extending the given absolute value on K. In particular, when Y is integral, the set of points

$x \in Y^{\mathrm{an}}$ for which $s(x)$ is the generic point of Y can be identified with the set of norms on the function field of Y extending the given norm on K.

2.2. GAGA. The Berkovich space Y^{an} naturally comes with a structure sheaf that we will not define nor explicitly use. We will also not define general K-analytic spaces [Ber90, Ber93] here. However, we will make use of the general GAGA results in [Ber90, 3.4]. For example, any projective K-analytic space X is the analytification of a projective K-scheme Y, that is, $X = Y^{\mathrm{an}}$. Further, all line bundles on projective Berkovich spaces are induced by line bundles on the underlying scheme. Similarly, morphisms between projective Berkovich spaces arise from morphisms between the underlying schemes.

2.3. Centers. Now let $\mathcal{X}$ be a proper S-variety. Its generic fiber $\mathcal{X}_K$ is, in particular, a proper K-scheme, so the discussion above applies. Write $X = \mathcal{X}_K^{\mathrm{an}}$ for the analytification of $\mathcal{X}_K$ and $s : X \to \mathcal{X}_K \subset \mathcal{X}$ for the continuous map defined in §2.1. Given $x \in X$, denote by R_x the corresponding valuation ring in the residue field $\kappa(s(x))$. By the valuative criterion of properness, the map $T_x := \operatorname{Spec} R_x \to S$ admits a unique lift $T_x \to \mathcal{X}$ mapping the generic point to $s(x)$. In line with valuative terminology [Vaq00], we call the image of the closed point of T_x in $\mathcal{X}$ the *center* of x on $\mathcal{X}$ and denote it by $c_{\mathcal{X}}(x)$. It is a specialization of $s(x)$ in $\mathcal{X}$. It also belongs to $\mathcal{X}_0$ since it maps to the closed point of S by construction. The map $c_{\mathcal{X}} : \mathcal{X}_K^{\mathrm{an}} \to \mathcal{X}_0$ so defined is anti-continuous; i.e. preimages of open sets are closed and vice versa. It is referred to as the *reduction map* in rigid geometry.

2.4. Models. From now on we let X be a given smooth connected projective K-analytic space in the sense of Berkovich. By a *model* of X we will mean a *normal and projective* S-variety $\mathcal{X}$ together with the datum of an isomorphism $\mathcal{X}_K^{\mathrm{an}} \simeq X$. By GAGA, the latter is equivalent to an isomorphism between $\mathcal{X}_K$ and the (smooth, connected) algebraic variety Y underlying $X = Y^{\mathrm{an}}$.

In particular, the set $\mathcal{M}_X$ of models of X is non-empty. Indeed, given an embedding Y into a suitable projective space $\mathbf{P}_K^m$ we can take $\mathcal{X}$ as the normalization of the closure of Y in $\mathbf{P}_S^m$. A similar construction shows that $\mathcal{M}_X$ becomes a directed set by declaring $\mathcal{X}' \geq \mathcal{X}$ if there exists a vertical blowup $\mathcal{X}' \to \mathcal{X}$ (which is then unique).

For any model $\mathcal{X}$ of X and any irreducible component E of $\mathcal{X}_0$, there exists a unique point $x_E \in X \simeq \mathcal{X}_K^{\mathrm{an}}$ whose center on $\mathcal{X}$ is the generic point of E. Such points will be called *divisorial points*.[3] Observe that the local ring of the scheme $\mathcal{X}$ at the generic point of E is precisely the valuation ring of the valuation x_E.

[3]Divisorial points are called *Shilov boundaries* in [YZ09]. They are usually referred to as Type II points when X is a curve; see [Ber90, 1.4.4]

The set X^{div} of divisorial points is dense in X; see Corollary 2.4 below and also [Poi13].

2.5. Model functions. Let $\mathcal{X}$ be a model of X. By Noetherianity, each vertical fractional ideal sheaf $\mathfrak{a}$ on $\mathcal{X}$ is locally generated by a finite set of rational functions on $\mathcal{X}$ and thus defines a continuous function $\log|\mathfrak{a}| \in C^0(X)$ by setting

$$\log|\mathfrak{a}|(x) := \max\left\{\log|f(x)| \mid f \in \mathfrak{a}_{c_{\mathcal{X}}(x)}\right\}. \tag{2.1}$$

In particular, each vertical Cartier divisor $D \in \mathrm{Div}_0(\mathcal{X})$ defines a vertical fractional ideal sheaf $\mathcal{O}_{\mathcal{X}}(D)$, hence a continuous function

$$\varphi_D := \log|\mathcal{O}_{\mathcal{X}}(D)|.$$

Note that $\varphi_{\mathcal{X}_0}$ is the constant function 1 since $\log|\varpi|^{-1} = 1$. Since models are assumed to be normal, a vertical divisor D is uniquely determined by the values $\varphi_D(x_E)$ at divisorial points x_E, and we have in particular $\varphi_D \geq 0$ iff D is effective. The map $D \mapsto \varphi_D$ extends by linearity to an injective map $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}} \to C^0(X)$.

In line with [Yua08] we introduce the following terminology.

Definition 2.1. A *model function*[4] is a function φ on X such that there exist a model $\mathcal{X}$ and a divisor $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ with $\varphi = \varphi_D$. We let $\mathcal{D}(X) = \mathcal{D}(X)_{\mathbf{Q}}$ be the space of model functions on X.

We shall also occasionally consider the similarly defined spaces $\mathcal{D}(X)_{\mathbf{Z}}$ and $\mathcal{D}(X)_{\mathbf{R}}$.

As a matter of terminology, we say that a model $\mathcal{X}$ is a *determination* of a model function φ if $\varphi = \varphi_D$ for some $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$. By the above remarks we have a natural isomorphism

$$\varinjlim_{\mathcal{X} \in \mathcal{M}_X} \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}} \simeq \mathcal{D}(X) \subset C^0(X).$$

The next result summarizes the key properties of model functions. Since our setting does not require any machinery from rigid geometry we provide direct arguments for the convenience of the reader.

Proposition 2.2. *For each model $\mathcal{X}$ of X, the subgroup of $C^0(X)$ spanned by the functions $\log|\mathfrak{a}|$, with $\mathfrak{a}$ ranging over all vertical (fractional) ideal sheaves of $\mathcal{X}$, coincides with $\mathcal{D}(X)_{\mathbf{Z}}$. It is furthermore stable under* max *and separates points of X.*

Proof. If $\mathfrak{a}$ is a vertical fractional ideal sheaf on a given model $\mathcal{X}$, then $\mathfrak{a}' := \varpi^m \mathfrak{a}$ is a vertical ideal sheaf for some $m \in \mathbf{N}$ and we have $\log|\mathfrak{a}| = \log|\mathfrak{a}'| - m$, so it is enough to consider vertical ideal sheaves.

[4] Model functions are called *algebraic* in [CL06] and *smooth* in [CL11].

Observe first that $\log|\mathfrak{a}|$ belongs to $\mathcal{D}(X)_{\mathbf{Z}}$. Indeed if $\mathcal{X}' \to \mathcal{X}$ denotes the normalization of the blowup of $\mathcal{X}$ along $\mathfrak{a}$, then the Cartier divisor D on $\mathcal{X}'$ such that $\mathfrak{a}\cdot\mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(D)$ satisfies $\varphi_D = \log|\mathfrak{a}|$. Conversely, let $\varphi \in \mathcal{D}(X)_{\mathbf{Z}}$, and let us show that φ can be written as

$$\varphi = \log|\mathfrak{a}| - \log|\mathfrak{b}|$$

with $\mathfrak{a}, \mathfrak{b}$ vertical ideal sheaves on $\mathcal{X}$. By definition, φ is determined by $D \in \mathrm{Div}_0(\mathcal{X}')$ for some vertical blowup $\pi : \mathcal{X}' \to \mathcal{X}$. By Lemma 1.4 we may choose a π-ample vertical Cartier divisor $A \in \mathrm{Div}_0(\mathcal{X}')$. Both sheaves $\mathcal{O}_{\mathcal{X}'}(mA)$ and $\mathcal{O}_{\mathcal{X}'}(D + mA)$ are then π-globally generated for $m \gg 1$. If we introduce the vertical fractional ideal sheaves $\mathfrak{a} := \pi_*\mathcal{O}_{\mathcal{X}'}(D + mA)$ and $\mathfrak{b} := \pi_*\mathcal{O}_{\mathcal{X}'}(mA)$, then the π-global generation property yields $\mathfrak{a}\cdot\mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(D + mA)$ and $\mathfrak{b}\cdot\mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(mA)$. It follows that $\varphi_{mA} = \log|\mathfrak{a}|$ and $\varphi_{D+mA} = \log|\mathfrak{b}|$, and hence $\varphi_D = \log|\mathfrak{a}| - \log|\mathfrak{b}|$. It remains to replace $\mathfrak{a}$ and $\mathfrak{b}$ with $\varpi^p\mathfrak{a}$ and $\varpi^p\mathfrak{b}$ with $p \gg 1$, so that they become actual ideal sheaves.

We next prove that $\mathcal{D}(X)_{\mathbf{Z}}$ is stable under max. Given $\varphi, \varphi' \in \mathcal{D}(X)_{\mathbf{Z}}$, choose a model $\mathcal{X}$ on which both functions are determined, say by $D, D' \in \mathrm{Div}_0(\mathcal{X})$, respectively. We then have

$$\max\{\varphi_D, \varphi_{D'}\} = \log|\mathfrak{a}|,$$

with $\mathfrak{a} := \mathcal{O}_{\mathcal{X}}(D) + \mathcal{O}_{\mathcal{X}}(D')$, which shows that $\max\{\varphi_D, \varphi_{D'}\} \in \mathcal{D}(X)_{\mathbf{Z}}$.

In order to get the separation property, we basically argue as in [Gub98, Corollary 7.7], which in turn relied on [BL93, Lemma 2.6]. Let $\mathcal{X}$ be a fixed model and pick two distinct points $x \neq y \in X$. If $\xi := c_{\mathcal{X}}(x)$ is distinct from $c_{\mathcal{X}}(y)$, then $\log|\mathfrak{m}_\xi|$ already separates x and y. Otherwise, let $\mathcal{U} = \mathrm{Spec}\,\mathcal{A}$ be an open neighborhood of ξ in $\mathcal{X}$. By the definition of $\mathcal{U}_K^{\mathrm{an}}$ there exists $f \in \mathcal{A}$ such that $|f(x)| \neq |f(y)|$. Since the scheme $\mathcal{X}$ is Noetherian, $\mathcal{O}_{\mathcal{U}} \cdot f$ extends to a (coherent) ideal sheaf $\mathfrak{a}$ on $\mathcal{X}$. For each positive integer m the ideal sheaf $\mathfrak{a}_m := \mathfrak{a} + (\varpi^m)$ is vertical on $\mathcal{X}$, and we have

$$\log|\mathfrak{a}_m| = \max\{\log|f|, -m\}$$

at x and y, so we see that $\log|\mathfrak{a}_m| \in \mathcal{D}(X)_{\mathbf{Z}}$ separates x and y for $m \gg 1$. $\square$

Thanks to the "Boolean ring version" of the Stone-Weierstrass theorem, we get as a consequence the following crucial result, which is equivalent to [Gub98, Theorem 7.12] (compare [Yua08, Lemma 3.5] and the remark following it).

Corollary 2.3. *The $\mathbf{Q}$-vector space $\mathcal{D}(X)$ is stable under* max *and separates points. As a consequence, it is dense in $C^0(X)$ for the topology of uniform convergence.*

Corollary 2.3 in turn implies the following result, which corresponds to [YZ09, Lemma 2.4]. We reproduce the short proof for completeness.

Corollary 2.4. *The set X^{div} of divisorial points is dense in X.*

Proof. Since X is a compact Hausdorff space, Urysohn's lemma applies, so it suffices to prove that any continuous function vanishing on X^{div} vanishes on all of X.

So pick $\varphi \in C^0(X)$ vanishing on X^{div} and $\varepsilon > 0$ rational. By Corollary 2.3 there exists a model $\mathcal{X}$ and a divisor $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ such that $|\varphi - \varphi_D| \le \varepsilon$ on X. Since φ vanishes on all divisorial points corresponding to irreducible components of $\mathcal{X}_0$, it follows that both divisors $\varepsilon\mathcal{X}_0 \pm D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ are effective, proving $|\varphi_D| \le \varepsilon$ and hence $|\varphi| \le 2\varepsilon$ on X. □

The collection of finite-dimensional spaces $\mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}} \simeq \mathcal{D}(\mathcal{X})^*_{\mathbf{R}}$ endowed with the transpose of pull-back morphisms on divisors and the topology of the pointwise convergence forms an inductive system, and we have:

Corollary 2.5. *For each model $\mathcal{X}$, let $\mathrm{ev}_{\mathcal{X}} : X \to \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ be the evaluation map defined by $\langle \mathrm{ev}_{\mathcal{X}}(x), D\rangle = \varphi_D(x)$. Then the induced map*

$$\mathrm{ev} : X \to \varprojlim_{\mathcal{X} \in \mathcal{M}_X} \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}} \simeq \mathcal{D}(X)^*_{\mathbf{R}}$$

is a homeomorphism onto its image.

The image of this map will be described in Corollary 3.2.

Proof. The map in question is continuous since any model function is continuous. It is injective by Corollary 2.3. Since X is compact and $\varprojlim_{\mathcal{X}\in\mathcal{M}_X} \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ is Hausdorff, we conclude that the map is a homeomorphism onto its image. □

3. Dual complexes

In this section we define, following [KS06], an embedding of the dual complex $\Delta_{\mathcal{X}}$ of an SNC model $\mathcal{X}$ into the Berkovich space X. This construction is essentially a special case of [Ber99] (see also [Thu07, ACP12]), but the present setting allows a much more elementary and explicit approach. We also explain how to construct (not necessarily SNC) models dominating $\mathcal{X}$ from suitable subdivisions of $\Delta_{\mathcal{X}}$, adapting some of the toroidal techniques of [KKMS].

3.1. The dual complex of an SNC model. Let $\mathcal{X}$ be an SNC model of X. The image of the evaluation map $\mathrm{ev}_{\mathcal{X}} : X \to \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ defined in Corollary 2.5 then admits the structure of a rational simplicial complex, defined as follows. Write the special fiber as $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$, where $b_i \in \mathbf{N}^*$ and $(E_i)_{i\in I}$ are the irreducible components. Let $x_{E_i} \in X$ be the associated divisorial points and set $e_i := \mathrm{ev}_{\mathcal{X}}(x_{E_i}) \in \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{Q}}$. Recall from Definition 1.1 that for each $J \subset I$ the intersection $E_J := \bigcap_{j\in J} E_j$ is either empty or a smooth irreducible k-variety. For each $J \subset I$ such that $E_J \neq \emptyset$, let $\hat{\sigma}_J \subset \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$

be the simplicial cone defined by $\hat{\sigma}_J := \sum_{j\in J} \mathbf{R}_+ e_j$. These cones naturally define a (regular) fan $\hat{\Delta}_{\mathcal{X}}$ in $\mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$. Slightly abusively, we shall also denote by $\hat{\Delta}_{\mathcal{X}}$ the support of this fan, that is, the union of all the cones $\hat{\sigma}_J$. We then define the *dual complex*[5] of $\mathcal{X}$ by

$$\Delta_{\mathcal{X}} := \hat{\Delta}_{\mathcal{X}} \cap \{\langle \mathcal{X}_0, \cdot \rangle = 1\}.$$

Each $J \subset I$ such that $E_J \neq \emptyset$ corresponds to a simplicial face

$$\sigma_J := \hat{\sigma}_J \cap \{\langle \mathcal{X}_0, \cdot \rangle = 1\} = \mathrm{Conv}\{e_j \mid j \in J\}$$

of dimension $|J| - 1$ in $\Delta_{\mathcal{X}}$, where Conv denotes convex hull. This endows $\Delta_{\mathcal{X}}$ with the structure of a (compact rational) simplicial complex such that σ_J is a face of σ_L iff $J \supset L$.

3.2. Embedding the dual complex in the Berkovich space.

Theorem 3.1. *Let $\mathcal{X}$ be any SNC model of X.*

(i) *The image of the evaluation map* $\mathrm{ev}_{\mathcal{X}} : X \to \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ *coincides with* $\Delta_{\mathcal{X}}$.

(ii) *There exists a unique continuous (injective) map* $\mathrm{emb}_{\mathcal{X}} : \Delta_{\mathcal{X}} \to X$ *such that:*

  (a) $\mathrm{ev}_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{X}}$ *is the identity on* $\Delta_{\mathcal{X}}$*;*

  (b) *for $s \in \Delta_{\mathcal{X}}$, the center of* $\mathrm{emb}_{\mathcal{X}}(s)$ *on $\mathcal{X}$ is the generic point ξ_J of E_J for the unique subset $J \subset I$ such that s is contained in the relative interior of σ_J.*

The proof is given in §3.3. Let us derive some consequences.

For any vertical blowup $\pi : \mathcal{X} \to \mathcal{Y}$ between models, the natural map ${}^t\pi^* : \mathrm{Div}_0(\mathcal{X})^* \to \mathrm{Div}_0(\mathcal{Y})^*$ maps $\Delta_{\mathcal{X}}$ onto $\Delta_{\mathcal{Y}}$ since ${}^t\pi^* \circ \mathrm{ev}_{\mathcal{X}} = \mathrm{ev}_{\mathcal{Y}}$ by definition. We may thus form the inverse limit $\varprojlim_{\mathcal{X}\ \mathrm{SNC}} \Delta_{\mathcal{X}}$, and we have

Corollary 3.2. *The maps* $\mathrm{ev}_{\mathcal{X}} : X \to \Delta_{\mathcal{X}} \subset \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ *induce a homeomorphism*

$$\mathrm{ev} : X \to \varprojlim_{\mathcal{X}\ SNC} \Delta_{\mathcal{X}}. \tag{3.1}$$

Remark 3.3. Corollary 3.2 can be found in [KS06, p. 77, Theorem 10] and is an example of a result exhibiting a non-Archimedean space as an inverse limit of polyhedral objects. Other examples can be found in [FJ04, Theorem 6.22], [BFJ08, Theorem 1.13], [Pay09, Theorem 1.1], [BR, Theorem 2.21], [HL10, Theorem 13.2.4], [JM12, Theorem 4.9], [HLP12, Proposition 6.1], and [BdFFU13, Theorem 2.3].

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[5] The dual complex is called the *Clemens polytope* in [KS06].

Proof of Corollary 3.2. The map ev is well-defined by Theorem 3.1(i). It is a homeomorphism onto its image by Corollary 2.5 and the fact that any model is dominated by an SNC model. As X is compact, we only need to show that $\mathrm{ev}(X)$ is dense in $\varprojlim \Delta_{\mathcal{X}}$. Pick $s = (s_{\mathcal{X}})_{\mathcal{X}} \in \varprojlim \Delta_{\mathcal{X}}$ and fix an SNC model $\mathcal{X}$. If $\mathcal{Y}$ is an SNC model dominated by $\mathcal{X}$, then $\mathrm{ev}_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{X}} = \mathrm{id}$ yields $\mathrm{ev}_{\mathcal{Y}}(\mathrm{emb}_{\mathcal{X}}(s_{\mathcal{X}})) = s_{\mathcal{Y}}$. Hence $s = \lim_{\mathcal{X}} \mathrm{ev}(\mathrm{emb}_{\mathcal{X}}(s_{\mathcal{X}})) \in \overline{\mathrm{ev}(X)}$. □

Definition 3.4. For any SNC model $\mathcal{X}$ we define a continuous map $p_{\mathcal{X}} : X \to X$ by

$$p_{\mathcal{X}} := \mathrm{emb}_{\mathcal{X}} \circ \mathrm{ev}_{\mathcal{X}} .$$

It follows from Theorem 3.1 that $p_{\mathcal{X}}$ satisfies $p_{\mathcal{X}} \circ p_{\mathcal{X}} = p_{\mathcal{X}}$ and $p_{\mathcal{X}}(x) = x$ iff $x \in \mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. Hence we view $p_{\mathcal{X}}$ as a *retraction* of X onto the image of the embedding $\mathrm{emb}_{\mathcal{X}} : \Delta_{\mathcal{X}} \to X$.

Lemma 3.5. *The retraction map $p_{\mathcal{X}}$ satisfies the following properties:*

(i) $c_{\mathcal{X}}(x) \in \overline{\{c_{\mathcal{X}}(p_{\mathcal{X}}(x))\}}$ *for all $x \in X$; more precisely, $c_{\mathcal{X}}(p_{\mathcal{X}}(x))$ is the generic point of E_J, where $J \subset I$ is the set of indices j for which $c_{\mathcal{X}}(x) \in E_j$;*

(ii) $\varphi_D \circ p_{\mathcal{X}} = \varphi_D$ *for all $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$.*

Proof. By the definition of $c_{\mathcal{X}}$ we have $c_{\mathcal{X}}(x) \in E_i$ for a given $i \in I$ iff $\langle \mathrm{ev}_{\mathcal{X}}(x), E_i \rangle > 0$, and it follows that $\mathrm{ev}_{\mathcal{X}}(x)$ lies in the relative interior of the simplex σ_J for the maximal $J \subset I$ such that $c_{\mathcal{X}}(x) \in E_J$. Property (b) in Theorem 3.1 then shows that $c_{\mathcal{X}}(p_{\mathcal{X}}(x))$ is the generic point of E_J, which proves (i).

Let us prove (ii). For each $x \in X$ we have

$$\begin{aligned}\varphi_D(p_{\mathcal{X}}(x)) &= \langle D, \mathrm{ev}_{\mathcal{X}}(p_{\mathcal{X}}(x)) \rangle = \langle D, \mathrm{ev}_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{X}} \circ \mathrm{ev}_{\mathcal{X}}(x) \rangle \\ &= \langle D, \mathrm{ev}_{\mathcal{X}}(x) \rangle = \varphi_D(x),\end{aligned}$$

using the identity $\mathrm{ev}_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{X}} = \mathrm{id}$. □

Proposition 3.6. *If $\mathcal{X} \geq \mathcal{Y}$ are two SNC models, then:*

(i) $\mathrm{ev}_{\mathcal{Y}} \circ p_{\mathcal{X}} = \mathrm{ev}_{\mathcal{Y}}$;

(ii) $p_{\mathcal{Y}} \circ p_{\mathcal{X}} = p_{\mathcal{Y}}$;

(iii) $p_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{Y}} = \mathrm{emb}_{\mathcal{Y}}$;

(iv) $p_{\mathcal{X}} \circ p_{\mathcal{Y}} = p_{\mathcal{Y}}$.

Note that (iii) says that the image in X of $\Delta_{\mathcal{Y}}$ is contained in the image of $\Delta_{\mathcal{X}}$.

Proof. Property (i) amounts to the fact that $\varphi_D \circ p_{\mathcal{X}} = \varphi_D$ for all $D \in \mathrm{Div}_0(\mathcal{Y})$, which is a special case of Lemma 3.5(ii). Postcomposing (i) with $\mathrm{emb}_{\mathcal{Y}}$ we then get (ii).

Let us now prove (iii). The map $\mathrm{emb}'_{\mathcal{Y}} := p_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{Y}} : \Delta_{\mathcal{Y}} \to X$ is continuous, and (i) implies that $\mathrm{ev}_{\mathcal{Y}} \circ \mathrm{emb}'_{\mathcal{Y}} = \mathrm{ev}_{\mathcal{Y}} \circ \mathrm{emb}_{\mathcal{Y}} = \mathrm{id}$ on $\Delta_{\mathcal{Y}}$. By the uniqueness part of Theorem 3.1 it suffices to prove that $c_{\mathcal{Y}} \circ \mathrm{emb}'_{\mathcal{Y}} = c_{\mathcal{Y}} \circ \mathrm{emb}_{\mathcal{Y}}$ on $\Delta_{\mathcal{Y}}$. Pick $s \in \Delta_{\mathcal{Y}}$ and set $x := \mathrm{emb}_{\mathcal{Y}}(s)$, $x' := \mathrm{emb}'_{\mathcal{Y}}(s)$. On the one hand, (ii) shows that

$$p_{\mathcal{Y}}(x') = p_{\mathcal{Y}} \circ p_{\mathcal{X}} \circ \mathrm{emb}_{\mathcal{Y}}(s) = p_{\mathcal{Y}} \circ \mathrm{emb}_{\mathcal{Y}}(s) = x,$$

so $c_{\mathcal{Y}}(x') \in \overline{\{c_{\mathcal{Y}}(x)\}}$ by (i) of Lemma 3.5. On the other hand $p_{\mathcal{X}}(x) = x'$ by definition, so $c_{\mathcal{X}}(x) \in \overline{\{c_{\mathcal{X}}(x')\}}$ and hence $c_{\mathcal{Y}}(x) \in \overline{\{c_{\mathcal{Y}}(x')\}}$ by continuity of the map $\mathcal{X} \to \mathcal{Y}$ for the Zariski topology.

Finally (iv) follows by postcomposing (iii) with $\mathrm{ev}_{\mathcal{Y}}$. □

Definition 3.7. We define the subset $X^{\mathrm{qm}} \subset X$ of *quasimonomial points* as

$$X^{\mathrm{qm}} := \bigcup_{\mathcal{X}} \mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}}),$$

where $\mathcal{X}$ ranges over SNC models of $\mathcal{X}$.

Remark 3.8. The set X^{qm} coincides with the set of real valuations on the function field of X whose restriction to K agrees with the given valuation and which are Abhyankar in the sense that the sum of their rational rank and their transcendence degree is equal to $\dim X + 1$; see [JM12, Proposition 3.7].[6]

Corollary 3.9. *We have* $\lim_{\mathcal{X}} p_{\mathcal{X}} = \mathrm{id}$ *pointwise on* X. *Hence* X^{qm} *is dense in* X.

Of course, we already knew from Corollary 2.4 that $X^{\mathrm{div}} \subset X^{\mathrm{qm}}$ is dense in X.

Proof. By Corollary 3.2 it suffices to show that $\lim_{\mathcal{X}} \mathrm{ev} \circ p_{\mathcal{X}} = \mathrm{ev}$, which amounts to proving $\lim_{\mathcal{X}} \mathrm{ev}_{\mathcal{Y}} \circ p_{\mathcal{X}} = \mathrm{ev}_{\mathcal{Y}}$ for each $\mathcal{Y}$. This follows from (i) of Proposition 3.6. □

3.3. Proof of Theorem 3.1. Proving the inclusion $\mathrm{ev}_{\mathcal{X}}(X) \subset \Delta_{\mathcal{X}}$ is a matter of unwinding definitions. The reverse inclusion will follow from (a). Hence (ii) implies (i).

The proof of (ii) is essentially the same as that of [JM12, Proposition 3.1]. It is also closely related to [Ber99, Lemma 5.6] and [Thu07, Corollaire 3.13]. Fix a subset $J \subset I$ with $E_J \neq \emptyset$, let ξ_J be its generic point, and let σ_J be the corresponding face of $\Delta_{\mathcal{X}}$. It will be enough to show the existence and uniqueness of a continuous map $\mathrm{emb}_{\mathcal{X}} : \sigma_J \to X$ satisfying (a) and (b) of Theorem 3.1 for $s \in \sigma_J$.

For each $j \in J$ pick a local equation $z_j \in \mathcal{O}_{\mathcal{X},\xi_J}$ of E_j, so that $(z_j)_{j \in J}$ is a regular system of parameters of $\mathcal{O}_{\mathcal{X},\xi_J}$ thanks to the SNC condition. After

[6] The rational rank of such a valuation v is defined as the dimension of the $\mathbf{Q}$-vector space generated by the value group of v. The transcendence degree of v is the transcendence degree of the field extension $k(v)/k$, where $k(v) = \{v \geq 0\}/\{v > 0\}$ is the residue field of v.

choosing a field of representatives of $\kappa(\xi_J)$ in $\mathcal{O}_{\mathcal{X},\xi_J}$, Cohen's theorem yields an isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X},\xi_J} \simeq \kappa(\xi_J)[\![t_j, j \in J]\!] \tag{3.2}$$

sending z_j to t_j.

To prove uniqueness, suppose $\mathrm{emb}_{\mathcal{X}}, \mathrm{emb}'_{\mathcal{X}} : \sigma_J \to X$ are two continuous maps satisfying (a) and (b) for $s \in \sigma_J$. Property (a) means that the valuations $\mathrm{val}_{\mathcal{X},s}$ and $\mathrm{val}'_{\mathcal{X},s}$ on $\widehat{\mathcal{O}}_{\mathcal{X},\xi_J}$ defined by

$$\mathrm{val}_{\mathcal{X},s}(f) := -\log|f(\mathrm{emb}_{\mathcal{X}}(s))| \quad \text{and} \quad \mathrm{val}'_{\mathcal{X},s}(f) := -\log|f(\mathrm{emb}'_{\mathcal{X}}(s))|$$

both take value s_j on z_j. By property (b) it follows that when s belongs to the relative interior $\mathrm{ri}(\sigma_J)$, the valuations $\mathrm{val}_{\mathcal{X},s}$, $\mathrm{val}'_{\mathcal{X},s}$ have center ξ_J on $\mathcal{X}$ and hence extend by continuity to the completion $\widehat{\mathcal{O}}_{\mathcal{X},\xi_J}$. The isomorphism (3.2) enables us to write any given $f \in \widehat{\mathcal{O}}_{\mathcal{X},\xi_J}$ as $f = \sum_{\alpha\in\mathbf{N}^J} f_\alpha z^\alpha$ with $f_\alpha \in \widehat{\mathcal{O}}_{\mathcal{X},\xi_J}$, in such a way that each non-zero f_α is a unit. For any $s \in \mathrm{ri}(\sigma_J)$ we then have

$$\mathrm{val}_{\mathcal{X},s}(f_\alpha z^\alpha) = \langle s, \alpha\rangle = \mathrm{val}'_{\mathcal{X},s}(f_\alpha z^\alpha)$$

for each $\alpha \in \mathbf{N}^J$. If $(s_j)_{j\in J}$ is $\mathbf{Q}$-linearly independent, then these numbers are furthermore mutually distinct as α ranges over $\mathbf{N}^J$, and the ultrametric property yields

$$\mathrm{val}_{\mathcal{X},s}(f) = \min_{\alpha\in\mathbf{N}^J} \langle s, \alpha\rangle = \mathrm{val}'_{\mathcal{X},s}(f). \tag{3.3}$$

We conclude that $\mathrm{emb}_{\mathcal{X}}(s) = \mathrm{emb}'_{\mathcal{X}}(s)$ on the dense set of points $s \in \mathrm{ri}(\sigma_J)$ such that $(s_j)_{j\in J}$ is $\mathbf{Q}$-linearly independent; hence $\mathrm{emb}_{\mathcal{X}} = \mathrm{emb}'_{\mathcal{X}}$ on σ_J by continuity.

Now we turn to existence. Given $s \in \sigma_J$, define v as the monomial valuation on the ring of formal power series $\kappa(\xi_J)[\![t_j,\, j \in J]\!]$ taking values $v(t_j) = s_j$, $j \in J$. In other words, the value of v on an element

$$g = \sum_{\alpha\in\mathbf{N}^J} g_\alpha t^\alpha \in \kappa(\xi_J)[\![t_j,\, j \in J]\!]$$

is given by

$$v(g) = \min\{\langle s, \alpha\rangle \mid g_\alpha \neq 0\}. \tag{3.4}$$

Using the isomorphism (3.2) we may thus define $\mathrm{val}_{\mathcal{X},s}$ as the restriction to $\mathcal{O}_{\mathcal{X},\xi_J}$ of the pull-back of v. The center of $\mathrm{val}_{\mathcal{X},s}$ is then equal to the generic point of $\bigcap_{s_j>0}\{z_j = 0\}$, i.e. the generic point of $E_{J'}$ where $\sigma_{J'}$ is the face containing s in its relative interior. The continuity of $s \mapsto \mathrm{val}_{\mathcal{X},s}(f)$ on σ_J is also easy to see using (3.4). Setting

$$\mathrm{emb}_{\mathcal{X}}(s) := \exp\left(-\mathrm{val}_{\mathcal{X},s}\right)$$

therefore concludes the proof.

Remark 3.10. For each $\xi \in \mathcal{X}_0$ let I_ξ be the set of indices $i \in I$ for which $\xi \in E_i$. Arguing as above shows that there exists a unique way to define, for each $s \in \sigma_{I_\xi}$, a valuation $\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}$ on $\widehat{\mathcal{X}}_\xi := \mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathcal{X},\xi}$, if we impose that:

- $\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}$ is centered at ξ_J for $s \in \mathrm{ri}(\sigma_J) \subset \sigma_{I_\xi}$;
- $\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}(E_i) = s_i$ for each $i \in I_\xi$;
- $s \mapsto \mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}(f)$ is continuous for each $f \in \widehat{\mathcal{O}}_{\mathcal{X},\xi}$.

Indeed, choose a regular system of parameters $(z_i)_{i \in L}$ of $\mathcal{O}_{\mathcal{X},\xi}$ such that z_i is a local equation of E_i for $i \in I_\xi \subset L$, and choose a field of representatives of $\kappa(\xi)$ in $\mathcal{O}_{\mathcal{X},\xi}$. We then have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{X},\xi} \simeq \kappa(\xi)[\![t_i,\, i \in L]\!]$ under which $\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}$ corresponds to the monomial valuation taking value s_i on t_i for $I \in I_\xi$ and 0 on t_i for $i \in L \setminus I_\xi$. Note that $\mathrm{val}_{\mathcal{X},s}$ is then the image of $\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s}$ under the natural morphism $\widehat{\mathcal{X}}_\xi \to \mathcal{X}$.

Remark 3.11. Although we shall not use it, there exists a deformation retraction of X onto the image of the dual complex $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ in X; see [Thu07, Theorem 3.26] and [NX13, Theorem 3.1.3].

3.4. Functions on dual complexes.

Proposition 3.12. *Let $\mathcal{X}$ be an SNC model of X and let $\mathfrak{a}$ be a vertical fractional ideal sheaf on $\mathcal{X}$. Then $\varphi := \log|\mathfrak{a}| \in \mathcal{D}(X)$ satisfies:*

(i) *$\varphi \circ \mathrm{emb}_{\mathcal{X}}$ is piecewise affine and convex on each face of $\Delta_{\mathcal{X}}$;*
(ii) *$\varphi \leq \varphi \circ p_{\mathcal{X}}$.*

Corollary 3.13. *Let $\mathcal{X}$ be an SNC model of X and $\psi \in \mathcal{D}(X)$ a model function. Then $\psi \circ \mathrm{emb}_{\mathcal{X}}$ is piecewise affine on the faces of $\Delta_{\mathcal{X}}$. Further, $\psi \circ \mathrm{emb}_{\mathcal{X}}$ is affine on all faces iff ψ is determined on $\mathcal{X}$.*

Proof. By Proposition 2.2, we can write $\psi = \sum_{i=1}^m c_i \log|\mathfrak{a}_i|$ for vertical ideal sheaves $\mathfrak{a}_i$ on $\mathcal{X}$ and rational numbers c_i. According to Proposition 3.12, each function $\log|\mathfrak{a}_i| \circ \mathrm{emb}_{\mathcal{X}}$ is piecewise affine on the faces of $\Delta_{\mathcal{X}}$; hence so is ψ.

The second point follows from the fact that a function on $\Delta_{\mathcal{X}}$ is affine on all the faces of $\Delta_{\mathcal{X}} \subset \mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ iff it comes from a linear form, i.e. an element of $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$. □

Proof of Proposition 3.12. Upon multiplying by ϖ^m with $m \gg 1$, we may assume that $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}$ is a vertical ideal sheaf. Pick $J \subset I$ such that E_J is non-empty, choose a point $\xi \in E_J$, and let $f_1, \dots, f_M$ be generators of $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X},\xi}$.

With the notation introduced in the proof of Theorem 3.1 we then have

$$\log|\mathfrak{a}|(\mathrm{emb}_{\mathcal{X}}(s)) = \max_{1\le m\le M} -\mathrm{val}_{\mathcal{X},s}(f_m). \tag{3.5}$$

By (3.4) each function $s \mapsto -\mathrm{val}_{\mathcal{X},s}(f_m)$ is piecewise affine and convex on σ_J, proving (i).

To prove (ii), pick any $x \in X$, set $\xi := c_{\mathcal{X}}(x)$, and let $I_\xi \subset I$ be the set of indices $i \in I$ such that $\xi \in E_i$. Arguing similarly with generators of $\mathfrak{a}\cdot\mathcal{O}_{\mathcal{X},\xi}$, it is enough to show that $|f(x)| \le |f(p_{\mathcal{X}}(x))|$ for each $f \in \mathcal{O}_{\mathcal{X},\xi}$. Note that the seminorm $f \mapsto |f(x)|$ extends by continuity to $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$ since $\xi = c_{\mathcal{X}}(x)$. Writing, in the notation of Remark 3.10, $f = \sum_{\alpha\in\mathbf{N}^L} f_\alpha z^\alpha \in \widehat{\mathcal{O}}_{\mathcal{X},\xi}$, we then have

$$|f(x)| \le \sup_{f_\alpha\ne 0} \prod_{i\in I_\xi} |z_j(x)|^{\alpha_i}$$

by the ultrametric property, using that $|f_\alpha(x)| = 1$ since each non-zero $f_\alpha \in \widehat{\mathcal{O}}_{\mathcal{X},\xi}$ is a unit. On the other hand, if we set $s_i := -\log|z_i(x)|$ for $i \in I_\xi$, then we have by definition $p_{\mathcal{X}}(x) = \mathrm{emb}_{\mathcal{X}}(s)$; hence

$$\sup_{f_\alpha\ne 0} \prod_{i\in I_\xi} |z_i(x)|^{\alpha_i} = |f(p_{\mathcal{X}}(x))|$$

and the result follows. □

Let $\mathrm{PA}(\Delta_{\mathcal{X}})_{\mathbf{Z}}$ be the set of all continuous functions $h : \Delta_{\mathcal{X}} \to \mathbf{R}$ whose restriction to each face of $\Delta_{\mathcal{X}}$ is piecewise affine, with gradients given by $\mathbf{Z}$-divisors $D \in \mathrm{Div}_0(\mathcal{X})$.

Definition 3.14. Let $h \in \mathrm{PA}(\Delta_{\mathcal{X}})_{\mathbf{Z}}$. For each $J \subset I$ such that $E_J \ne \emptyset$ we set $\mathcal{X}_J := \mathcal{X} \setminus \bigcup_{i\in I\setminus J} E_i$ and define a vertical fractional ideal sheaf $\mathfrak{a}_h$ on $\mathcal{X}$ by letting for each J

$$\mathfrak{a}_h|_{\mathcal{X}_J} := \sum \left\{\mathcal{O}_{\mathcal{X}_J}(D) \mid D \in \mathrm{Div}_0(\mathcal{X}) \text{ and } \langle D, \cdot\rangle \le h \text{ on } \sigma_J\right\}. \tag{3.6}$$

Note that these locally defined sheaves glue well together and that $\log|\mathfrak{a}_h| \circ \mathrm{emb}_{\mathcal{X}}$ is equal to the convex envelope of h on each face of $\Delta_{\mathcal{X}}$.

3.5. Subdivisions and vertical blowups. Let $\mathcal{X}$ be an SNC model. A *subdivision* Δ' of $\Delta_{\mathcal{X}}$ is a compact rational polyhedral complex of $\mathrm{Div}_0(\mathcal{X})^*_{\mathbf{R}}$ refining $\Delta_{\mathcal{X}}$. Each subdivision Δ' is thus of the form $\hat{\Delta}' \cap \{\langle \mathcal{X}_0, \cdot\rangle = 1\}$ where $\hat{\Delta}'$ is a rational fan refining $\hat{\Delta}_{\mathcal{X}}$. A subdivision Δ' is *simplicial* if its faces are simplices.

A subdivision Δ' is *projective* if it admits a *strictly convex support function*, that is, a function $h \in \mathrm{PA}(\Delta_{\mathcal{X}})_{\mathbf{Z}}$ that is convex on each face of $\Delta_{\mathcal{X}}$ and such that Δ' is the coarsest subdivision of $\Delta_{\mathcal{X}}$ on each of whose faces h is affine.

Theorem 3.15. *Let $\mathcal{X}$ be an SNC model of X and let Δ' be a simplicial projective subdivision of $\Delta_{\mathcal{X}}$. Then there exists a vertical blowup $\pi : \mathcal{X}' \to \mathcal{X}$ with the following properties:*

(i) *$\mathcal{X}'$ is normal and vertically $\mathbf{Q}$-factorial.*

(ii) *The vertices $(e'_i)_{i\in I'}$ of Δ' are in bijection with the irreducible components $(E'_i)_{i\in I'}$ of $\mathcal{X}'_0$ in such a way that $c_{\mathcal{X}'}(\mathrm{emb}_{\mathcal{X}}(e'_i))$ is the generic point of E'_i for each $i \in I'$.*

(iii) *If $J' \subset I'$, then $E'_{J'} := \bigcap_{j\in J'} E'_j$ is normal, irreducible, and non-empty iff the corresponding vertices e'_j, $j \in J'$, of Δ' span a face $\sigma'_{J'}$ of Δ'. In this case, $E'_{J'}$ has codimension $|J'|$ and its generic point is the center of $\mathrm{emb}_{\mathcal{X}}(s)$ on $\mathcal{X}'$ for all s in the relative interior of $\sigma'_{J'}$.*

(iv) *For each $D \in \mathrm{Div}_0(\mathcal{X}')$ the function $\varphi_D \circ \mathrm{emb}_{\mathcal{X}}$ is affine on the faces of Δ'.*

This result is in essence contained in the toroidal theory of [KKMS]. However, strictly speaking, these authors only deal with varieties over an algebraically closed field and with toroidal S-varieties, neither of which appears to adequately handle the case of SNC S-varieties when the special fiber is non-reduced. Since Theorem 3.15 is one of the crucial ingredients in the proof of Theorem A, we therefore provide a complete proof, mostly adapting [KKMS, pp. 76-82]. See also [Thu07] for a similar construction.

Proof.

Step 1. Given a finite set L and a field κ, we rely on basic toric geometry (cf. [KKMS, Ful93, Oda88]) to show that $Z := \mathbf{A}^L_\kappa = \mathrm{Spec}\,\kappa[t_i,\, i \in L]$ and its coordinate hyperplanes $(H_i)_{i\in L}$ satisfy an analogue of (i)-(iv). Set $T := (\mathbf{G}_{m,\kappa})^L$ to be the multiplicative split torus of dimension L over κ. The fan Σ of the toric κ-variety Z consists of the cones $\hat\sigma_J = \sum_{j\in J} \mathbf{R}_+ e_j$, $J \subset L$. For each $s \in \mathbf{R}^L_+$ let

$$\mathrm{val}_{Z,s} : \kappa[\![t_i,\, i \in L]\!] \to \mathbf{R}_+$$

be the monomial valuation with $\mathrm{val}_{Z,s}(t_i) = s_i$ for $i \in L$, so that the center of $\mathrm{val}_{Z,s}$ on Z is the generic point of $H_J := \bigcap_{j\in J} H_j$ for all s in the relative interior of $\hat\sigma_J$.

Let Σ' be a simplicial fan decomposition of Σ. The toric κ-variety Z' attached to Σ' comes with a T-equivariant proper birational morphism $\rho : Z' \to Z$ satisfying the following properties:

(a) Z' is normal (because it is toric), and all toric Weil divisors of Z' are $\mathbf{Q}$-Cartier (since Σ' is simplicial).

(b) There is a bijection between the set of rays $(R_i)_{i\in L'}$ of Σ' and the toric prime divisors $(H'_i)_{i\in L'}$ of Z' in such a way that for each $s \in R_i \setminus \{0\}$ the center of $\mathrm{val}_{Z,s}$ on Z' is the generic point of H'_i.

(c) For each $J' \subset L'$ the intersection $H'_{J'} := \bigcap_{j\in J'} H'_j$ is normal, irreducible, and non-empty iff $\hat\sigma'_{J'} = \sum_{j\in J'} R_j$ is a cone of Σ'. In this case $H'_{J'}$ has codimension $|J'|$, and its generic point is the center of $\mathrm{val}_{Z,s}$ on Z' for all s in the relative interior of $\hat\sigma'_{J'}$.

(d) For each toric divisor G of Z', the map $s \mapsto \mathrm{val}_{Z,s}(G)$ is linear on each cone of Σ'. Here we use the fact that mG is Cartier for some non-zero $m \in \mathbf{N}$ by (a) to set $\mathrm{val}_{Z,s}(G) := \frac{1}{m}\mathrm{val}_{Z,s}(f)$, with f a local equation of mG at the center of $\mathrm{val}_{Z,s}$.

With the notation of (c), assume that $H'_{J'}$ is non-empty and let $\hat\sigma_J$ be the smallest cone of Σ containing $\hat\sigma'_{J'}$. We then have $\rho(H'_{J'}) = H_J$, and we claim that

$$\rho_*\mathcal{O}_{H'_{J'}} = \mathcal{O}_{H_J}. \tag{3.7}$$

Indeed, denote by $\zeta'_{J'}$ and ζ_J the generic points of $H'_{J'}$ and H_J, respectively. Since H_J is normal, (3.7) will follow from the fact that $\kappa(\zeta_J)$ is algebraically closed in $\kappa(\zeta'_{J'})$ (cf. [EGA, III.4.3.12]). But $H'_{J'}$ is the closure of a T-orbit $(H'_{J'})^0$ in Z', mapping to the T-orbit $H^0_J := (\bigcap_{j\in J} H_j)\setminus(\bigcup_{j\notin J} H_j)$ in Z. The stabilizer of H^0_J in T is $(\mathbf{G}_{m,\kappa})^J$, so the T-equivariant morphism $(H'_{J'})^0 \to H^0_J$ has geometrically integral fibers. In particular $\zeta'_{J'}$ is the generic point of the fiber over ζ_J, and $\kappa(\zeta_J)$ is algebraically closed in $\kappa(\zeta'_{J'})$ by [EGA, IV.4.5.9].

Step 2. Let $h \in \mathrm{PA}(\Delta_{\mathcal{X}})_{\mathbf{Z}}$ be a strictly convex support function for Δ'. We define $\mathcal{X}'$ as the blowup of $\mathcal{X}$ along the fractional ideal sheaf $\mathfrak{a}_h$ given in Definition 3.14 (see §1.1). Note that $\mathcal{X}$ is normal since $\mathfrak{a}_h$ is integrally closed, being defined by valuative conditions.

Let $\xi \in \mathcal{X}_0$ be a given point and use the notation of Remark 3.10. Since $\mathcal{X}$ and $Z := \mathbf{A}^L_{\kappa(\xi)}$ are excellent we get a diagram

$$\mathcal{X} \xleftarrow{\ p\ } \widehat{\mathcal{X}}_\xi \xrightarrow{\ q\ } Z$$

where p and q are *regular*, i.e. flat and with (geometrically) regular fibers (but typically not of finite type, as opposed to a smooth morphism). By Remark 3.10 we have

$$p_*\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s} = \mathrm{val}_{\mathcal{X},s} \ \text{ and } q_*\mathrm{val}_{\widehat{\mathcal{X}}_\xi,s} = \mathrm{val}_{Z,s} \tag{3.8}$$

for all $s \in \sigma_{I_\xi}$. The subdivision of σ_{I_ξ} defined by Δ' induces a simplicial fan decomposition Σ' of $\mathbf{R}^L_+$, to which the results of Step 1 apply. Since h is a support function of Δ', the toric $\kappa(\xi)$-variety Z' attached to Σ' coincides in fact with the blowup of Z along the toric fractional ideal sheaf

$$\mathfrak{b}_h := \sum\{\mathcal{O}_Z(H_m),\, m \in \mathbf{Z}^{I_\xi},\, \langle m,\cdot\rangle \le h \text{ on } \sigma_\xi\},$$

where we have set $H_m := \sum_{i\in I_\xi} m_i H_i$. Comparing with (3.6), we see that

$$p^{-1}\mathfrak{a}_h \cdot \widehat{\mathcal{O}}_{\mathcal{X},\xi} = q^{-1}\mathfrak{b}_h \cdot \widehat{\mathcal{O}}_{\mathcal{X},\xi}.$$

Since blowups commute with flat base change (cf. [Liu, 8.1.12]), $\widetilde{\mathcal{X}}'_\xi := \mathcal{X}' \times_{\mathcal{X}} \operatorname{Spec} \widehat{\mathcal{X}}_\xi$ sits in a commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[d, "\pi"'] & \widetilde{\mathcal{X}}'_\xi \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d] & Z' \arrow[d, "\rho"] \\ \mathcal{X} & \widehat{\mathcal{X}}_\xi \arrow[l, "p"'] \arrow[r, "q"] & Z \end{tikzcd} \tag{3.9}$$

where the two squares are Cartesian. The morphisms p' and q' are also regular since the latter property is preserved under finite type base change (cf. [EGA, IV.6.8.3]).

Let $(e'_i)_{i\in I'_\xi}$ be the set of vertices of Δ' contained in σ_{I_ξ}, so that each ray $\mathbf{R}_+ e'_i$ belongs to the fan Σ'. If we let H'_i be the corresponding toric prime divisor of Z' and pick $J' \subset I'_\xi$, then $H'_{J'} = \bigcap_{j\in J'} H_j$ is normal, irreducible, and non-empty iff the e'_j, $j \in J'$, span a face $\sigma'_{J'}$ of Δ', by property (c). Since q' is regular, if follows that $q'^{-1}(H'_{J'})$ is normal and is either empty or of codimension $|J'|$. It is furthermore *irreducible* (and thus non-empty) by (3.7) and Lemma 3.16 below. In particular, $(q'^{-1}(H'_i))_{i\in I'_\xi}$ is exactly the set of irreducible components of the special fiber of $\widetilde{\mathcal{X}}'_\xi$. Using that the special fiber of $\widetilde{\mathcal{X}}'_\xi$ is precisely the union of the zero loci of the z_i's for $i \in I_\xi$, it is now easy to obtain the analogue of (i)-(iv) of Theorem 3.15 with $\widetilde{\mathcal{X}}'_\xi$, σ_{I_ξ} and $\operatorname{val}_{\widehat{\mathcal{X}}_\xi}$ in place of $\mathcal{X}'$, $\Delta_{\mathcal{X}}$ and $\operatorname{val}_{\mathcal{X}}$.

In particular, $\widetilde{\mathcal{X}}'_\xi$ is normal for each $\xi \in \mathcal{X}_0$, which shows that $\mathcal{X}'$ is normal, hence a model of X.

On the other hand, for each irreducible component E' of $\mathcal{X}'_0$ such that $\pi(E')$ contains ξ, we claim that the divisor $p'^{-1}(E')$ is irreducible. Indeed, each irreducible component of the divisor $p'^{-1}(E')$ is of the form $q'^{-1}(H'_i)$ for some $i \in I'_\xi$ since it is contained in the special fiber by construction and of codimension one by flatness. If we denote by ξ' and η'_i the generic points of E' and $p'^{-1}(H'_i)$, respectively, then we have on the one hand $p'(\eta'_i) = \xi'$ since p' is flat. On the other hand, η'_i is the center of $\operatorname{val}_{\widehat{\mathcal{X}}_\xi, e'_i}$ on $\widetilde{\mathcal{X}}'_\xi$; hence $p'(\eta'_i) = c_{\mathcal{X}'}(\operatorname{val}_{\mathcal{X},e'_i})$ thanks to (3.8). For dimensional reasons it follows that $\operatorname{emb}_{\mathcal{X}}(e'_i) = x_{E'} \in X$, and the injectivity of $\operatorname{emb}_{\mathcal{X}}$ shows that i is uniquely determined by E', which implies as desired that $p'^{-1}(E')$ is irreducible.

We may thus write the irreducible components of $\mathcal{X}'_0$ that are mapped to $\overline{\{\xi\}}$ as $(E'_i)_{i\in I'_\xi}$, with the property that

$$p'^{-1}(E'_i) = q'^{-1}(H'_i).$$

By flat descent it follows that E'_i is normal at each point of the fiber of ξ. It is also $\mathbf{Q}$-Cartier since a Weil divisor is Cartier at a point iff its restriction to the formal neighborhood of that point is Cartier (see e.g. [Sam61, Proposition 1]). It is now easy to conclude the proof of (i)-(iv) using the analogous properties for $\widehat{\mathcal{X}}'_\xi$ together with (3.8). $\square$

Lemma 3.16. *Assume that*

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ U & \longrightarrow & V \end{array}$$

is a Cartesian square of Noetherian schemes such that the vertical arrows are proper and surjective and the horizontal morphisms are regular. If U, V, and V' are irreducible, V and V' are normal, and $g_*\mathcal{O}_{V'} = \mathcal{O}_V$, then U' is normal and irreducible.

Proof. Note first that U and U' are normal by [EGA, IV.6.5.4]. Since direct images commute with flat base change we have $f_*\mathcal{O}_{U'} = \mathcal{O}_U$, which implies that f has connected fibers as a consequence of the theorem on formal functions (cf. [EGA, III.4.3.2]). Since U is connected and non-empty and since f is closed, surjective, and has connected fibers, it follows that U' is connected and non-empty, hence irreducible since it is normal. $\square$

Corollary 3.17. *For each SNC model $\mathcal{X}$, the set of rational points of $\Delta_\mathcal{X}$ coincides with the set* $\mathrm{emb}_\mathcal{X}^{-1}(X^{\mathrm{div}}) \cap \Delta_\mathcal{X}$*; hence* $\mathrm{emb}_\mathcal{X}(\Delta_\mathcal{X}) \cap X^{\mathrm{div}}$ *is dense in* $\mathrm{emb}_\mathcal{X}(\Delta_\mathcal{X})$.

Proof. If $s \in \Delta_\mathcal{X}$ is a rational point, Theorem 3.15 yields a vertical blowup $\mathcal{X}'$ such that $\mathrm{emb}_{\mathcal{X}'}(s) = x_{E'}$ for some irreducible component E' of $\mathcal{X}'_0$. Conversely, if $\mathrm{emb}_\mathcal{X}(s)$ is a divisorial point, then the corresponding valuation takes rational values on the local equations of the irreducible components of $\mathcal{X}_0$, which shows that s is a rational point of $\Delta_\mathcal{X}$. $\square$

4. Metrics on line bundles and closed $(1,1)$-forms

4.1. Metrics. We refer to [CL11] for a general discussion of metrized line bundles in a non-Archimedean context. Suffice it to say that a *continuous* metric $\|\cdot\|$ on a line bundle L on X is a way to produce a continuous function $\|s\|$ on (the Berkovich space) X from any local section s of L. Given a continuous metric $\|\cdot\|$, any other continuous metric on L is of the form $\|\cdot\|e^{-\varphi}$, with $\varphi \in C^0(X)$. If we in this expression allow an arbitrary function $\varphi : X \to [-\infty, +\infty[$, then we obtain a *singular metric* on L.

Let $\mathcal{X}$ be a model and $\mathcal{L}$ a line bundle on $\mathcal{X}$ such that $\mathcal{L}|_X = L$. To this data one can associate a unique metric $\|\cdot\|_{\mathcal{L}}$ on L with the following property: if s is a non-vanishing local section of $\mathcal{L}$ on an open set $\mathcal{U} \subset \mathcal{X}$, then $\|s\|_{\mathcal{L}} \equiv 1$ on $U := \mathcal{U} \cap X$. This makes sense since such a section s is uniquely defined up to multiplication by an element of $\Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{X}})$ and such elements have norm 1.

More generally, any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ such that $\mathcal{L}|_X = L$ in $\mathrm{Pic}(X)_{\mathbf{Q}}$ induces a metric $\|\cdot\|_{\mathcal{L}}$ on L by setting $\|s\|_{\mathcal{L}} = \|s^{\otimes m}\|^{1/m}_{m\mathcal{L}}$ for any non-zero $m \in \mathbf{N}$ such that $m\mathcal{L}$ is an actual line bundle. By definition, a *model metric*[7] on L is a metric of the form $\|\cdot\|_{\mathcal{L}}$ with $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ for some model $\mathcal{X}$ such that $\mathcal{L}|_X = L$. Model metrics are clearly continuous. If $\|\cdot\|$ is a model metric, then $\|\cdot\|e^{-\varphi}$ is a model metric iff φ is a model function.

If we denote by $\widehat{\mathrm{Pic}}(X)$ the group of isomorphism classes of line bundles on X endowed with a model metric, then it is easy to check that there is a natural isomorphism

$$\varinjlim_{\mathcal{X} \in \mathcal{M}_X} \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}} \simeq \widehat{\mathrm{Pic}}(X)_{\mathbf{Q}} \tag{4.1}$$

and that the natural sequence

$$0 \to \mathbf{Q}\mathcal{X}_0 \to \mathcal{D}(X) \to \widehat{\mathrm{Pic}}(X)_{\mathbf{Q}} \to \mathrm{Pic}(X)_{\mathbf{Q}} \to 0 \tag{4.2}$$

is exact.

4.2. Closed $(1,1)$-forms. Recall that $N^1(\mathcal{X}/S)$ is the set of $\mathbf{R}$-line bundles on a model $\mathcal{X}$ modulo those that are numerically trivial on the special fiber.

Definition 4.1. The space of *closed* $(1,1)$-*forms* on X is defined as the direct limit

$$\mathcal{Z}^{1,1}(X) := \varinjlim_{\mathcal{X} \in \mathcal{M}_X} N^1(\mathcal{X}/S).$$

As with model functions, we say that $\theta \in \mathcal{Z}^{1,1}(X)$ is *determined* on a given model $\mathcal{X}$ if it is the image of an element $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$. By definition, two classes $\alpha \in N^1(\mathcal{X}/S)$ and $\alpha' \in N^1(\mathcal{X}'/S)$ define the same element in $\mathcal{Z}^{1,1}(X)$ iff they pull-back to the same class on a model dominating both $\mathcal{X}$ and $\mathcal{X}'$.

Remark 4.2. The previous definition is directly inspired from [BGS95], where closed forms and currents are defined in the non-Archimedean setting. We choose however to work modulo numerical equivalence instead of rational equivalence. One justification for this choice is Corollary 4.4 below. The fact

[7]See Table 1 on page 34 for alternative terminology regarding metrics used in the literature.

that each space $N^1(\mathcal{X}/S)$ is endowed with a natural topology as a finite-dimensional vector space is another reason.

The isomorphism (4.1) shows that there is a natural map

$$\widehat{\mathrm{Pic}}(X) \to \mathcal{Z}^{1,1}(X).$$

The image of $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X)$ under this map is denoted by $c_1(L, \|\cdot\|) \in \mathcal{Z}^{1,1}(X)$ and is called the *curvature form* of the metrized line bundle $(L, \|\cdot\|)$.

By definition, any model function $\varphi \in \mathcal{D}(X)$ is determined on some model $\mathcal{X}$ by some divisor $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$. We set $dd^c\varphi$ to be the form determined by the numerical class of D in $N^1(\mathcal{X}/S)$. In this way, we get a natural linear map

$$dd^c : \mathcal{D}(X) \to \mathcal{Z}^{1,1}(X).$$

On the other hand, the restriction maps $N^1(\mathcal{X}/S) \to N^1(X) := N^1(\mathcal{X}_K/K)$ induce a linear map

$$\{\cdot\} : \mathcal{Z}^{1,1}(X) = \varinjlim_{\mathcal{M}_X} N^1(\mathcal{X}/S) \to N^1(X).$$

We call $\{\theta\} \in N^1(X)$ the *de Rham class* of the closed $(1,1)$-form θ. Note that

$$\{c_1(L, \|\cdot\|)\} = c_1(L)$$

for each metrized line bundle $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X)$. The next result is an analogue of the dd^c-lemma in the complex setting.

Theorem 4.3. *Let X be a smooth connected projective K-analytic variety. Then the natural sequence*

$$0 \to \mathbf{R} \to \mathcal{D}(X)_{\mathbf{R}} \xrightarrow{dd^c} \mathcal{Z}^{1,1}(X) \to N^1(X) \to 0$$

is exact.

The following equivalent reformulation is also familiar in the complex setting.

Corollary 4.4. *Let L be a line bundle on X. Then $c_1(L) \in N^1(X)$ vanishes iff L admits a model metric with zero curvature. Such a metric is then unique up to a constant.*

Theorem 4.3 is more difficult than its rather straightforward analogue (4.2), whose proof is valid without any assumption of the residue field. Here the existence of regular models is used. Exactness at $\mathcal{D}(X)$ follows from a rather standard Hodge-index type argument (compare [YZ09, Theorem 2.1] and see [YZ13a, YZ13b] for far-reaching generalizations), whereas exactness at $\mathcal{Z}^{1,1}(X)$ is essentially a reformulation of a result by Künneman [Kün96, Lemma 8.1]; see also [Gub03, Theorem 8.9]. We provide some details for the convenience of the reader.

Proof of Theorem 4.3. We are going to prove the stronger assertion that

$$0 \to \mathbf{R}\mathcal{X}_0 \to \mathrm{Div}_0(\mathcal{X})_{\mathbf{R}} \to N^1(\mathcal{X}/S) \to N^1(\mathcal{X}_K/K) \to 0$$

is exact for every *regular* model $\mathcal{X}$ of X. We first prove the exactness at $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$. Let $\mathcal{X}_0 = \sum_{i \in I} b_i E_i$ be the irreducible decomposition of the special fiber. We claim that $\mathcal{X}_0$ is connected. Since $X \simeq \mathcal{X}_K^{\mathrm{an}}$ is connected by assumption, the (easy direction of the) GAGA principle implies that $\mathcal{X}_K$ is also connected. If $\mathcal{X}_0$ were disconnected, then $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ would split as a product by the Grothendieck-Zariski theorem on formal functions [Har, Theorem 11.1], which would contradict the connectedness of $\mathcal{X}_K$. Since $\mathcal{X}$ is regular, each E_i is Cartier. Pick any ample divisor $\mathcal{A}$ on $\mathcal{X}$ and define a quadratic form q on $\mathbf{R}^I$ by setting

$$q(a) := -\left(\sum_i a_i E_i\right)^2 \cdot \mathcal{A}^{\dim X - 1}.$$

We have $q_{ij} \le 0$ for $i \ne j$, and the matrix (q_{ij}) is indecomposable since $\mathcal{X}_0$ is connected. By [BPV, Lemma 2.10] it follows that b spans the kernel of q. Now let $D = \sum_i a_i E_i$ be a vertical $\mathbf{R}$-divisor whose numerical class on $\mathcal{X}_0$ is 0. It follows that a belongs to the kernel of q, hence is proportional to b, which precisely means that $D \in \mathbf{R}\mathcal{X}_0$ as desired.

Let us now turn to exactness at $N^1(\mathcal{X}/S)$, which amounts to the following assertion: every numerically trivial $L \in \mathrm{Pic}(\mathcal{X}_K)$ admits a numerically trivial extension $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$.

Arguing as in [Kün96, Lemma 8.1], assume first that X is one-dimensional. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ be an arbitrary extension of L to the regular model $\mathcal{X}$. In the notation above we have $\sum_i b_i(\mathcal{L} \cdot E_i) = 0$ since $\mathcal{L}$ is numerically trivial on the generic fiber X. Since $b = (b_i)_{i \in I}$ spans the kernel of the intersection matrix $(E_i \cdot E_j)$, we may thus find $a \in \mathbf{Q}^I$ such that $\sum_i a_i E_i \cdot E_j = \mathcal{L} \cdot E_j$ for $j \in I$, which shows that $\mathcal{L} - \sum_i a_i E_i$ is a numerically trivial extension of L to $\mathcal{X}$.

We now consider the general case, again following [Kün96, Lemma 8.1]. Given any S-scheme Y we write $\mathcal{X}_Y := \mathcal{X} \times_S Y$. Since L is numerically trivial on $\mathcal{X}_K$, some multiple mL belongs to $\mathrm{Pic}^0(\mathcal{X}_{\bar{K}})$ by [Mat57], and hence there exists a finite extension K'/K such that the pull-back of mL to $\mathcal{X}_{K'}$ is algebraically equivalent to 0. This implies that there exists a smooth projective K'-curve T, a numerically trivial $\mathbf{Q}$-line bundle M on T, and a (Cartier) divisor D on $\mathcal{X}_T$ such that

$$L = q_* \left(p^* M \cdot D\right)$$

in $\mathrm{Pic}(\mathcal{X}_K)_{\mathbf{Q}}$, where $p : \mathcal{X}_T \to T$ and $q : \mathcal{X}_T \to \mathcal{X}_K$ are the natural morphisms. Now let $\mathcal{T}$ be a regular model of T over the integral closure S' of S in K', and consider the commutative diagram

$$\begin{tikzcd} T \arrow[d] & \mathcal{X}_T \arrow[l, "p"'] \arrow[r, "q"] \arrow[d] & \mathcal{X}_K \arrow[d] \\ \mathcal{T} & \mathcal{X}_{\mathcal{T}} \arrow[l, "p"'] \arrow[r, "q"] & \mathcal{X} \end{tikzcd} \tag{4.3}$$

where we also use for simplicity p and q to denote the natural projections $\mathcal{X}_{\mathcal{T}} \to \mathcal{T}$ and $\mathcal{X}_{\mathcal{T}} \to \mathcal{X}$. By the one-dimensional case, M extends to a numerically trivial $\mathbf{Q}$-line bundle $\mathcal{M} \in \mathrm{Pic}(\mathcal{T})_{\mathbf{Q}}$. Also, let $\mathcal{D}$ be the closure of D in $\mathcal{X}_{\mathcal{T}}$, which is *a priori* merely a Weil divisor. We may then set

$$\mathcal{L} := q_* \left(p^* \mathcal{M} \cdot \mathcal{D}\right).$$

Note that $\mathcal{L}$ belongs to $\mathrm{CH}^1(\mathcal{X})_{\mathbf{Q}} = \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ since $\mathcal{X}$ is regular. It is clear that $\mathcal{L}$ extends L, and it remains to show that $\deg(\mathcal{L} \cdot C) = 0$ for each vertical projective curve C on $\mathcal{X}$. Since $\mathcal{X}$ is regular, $\mathrm{CH}(\mathcal{X})_{\mathbf{Q}}$ is a graded commutative algebra with respect to cup-product, by [GS87, §8.3]. As in [GS92, §2.3] one can then define the cap-product $\alpha \cdot_q \beta$ of $\alpha \in \mathrm{CH}(\mathcal{X})_{\mathbf{Q}}$ and $\beta \in \mathrm{CH}(\mathcal{X}_{\mathcal{T}})_{\mathbf{Q}}$, which turns $\mathrm{CH}(\mathcal{X}_{\mathcal{T}})_{\mathbf{Q}}$ into a graded $\mathrm{CH}(\mathcal{X})_{\mathbf{Q}}$-module such that both $q_* : \mathrm{CH}(\mathcal{X}_{\mathcal{T}})_{\mathbf{Q}} \to \mathrm{CH}(\mathcal{X})_{\mathbf{Q}}$ and multiplication with $\beta' \in \mathrm{Pic}(\mathcal{X}_{\mathcal{T}})_{\mathbf{Q}}$ are maps of $\mathrm{CH}(\mathcal{X})_{\mathbf{Q}}$-modules. Applying this with $\beta' = p^*\mathcal{M} \in \mathrm{Pic}(\mathcal{X}_{\mathcal{T}})_{\mathbf{Q}}$, which is numerically trivial on the special fiber of $\mathcal{X}_{\mathcal{T}}$, we get

$$\deg\left(C \cdot \mathcal{L}\right) = \deg\left(C \cdot_q (\beta' \cdot \mathcal{D})\right) = \deg\left(\beta' \cdot (C \cdot_q \mathcal{D})\right) = 0.$$

Finally, the surjectivity of $N^1(\mathcal{X}/S) \to N^1(\mathcal{X}_K/K)$ is clear since $N^1(\mathcal{X}_K/K)$ is spanned by classes of Cartier divisors on X, of which the closures in $\mathcal{X}$ are also Cartier since $\mathcal{X}$ is regular. □

5. Positivity of forms and metrics

5.1. Positive closed $(1,1)$-forms and metrics. The following definition extends the ones in [Zha95, Gub98, CL06].

Definition 5.1. A closed $(1,1)$-form θ is said to be:

(i) *semipositive* if $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$ is nef for some (or, equivalently, any) determination $\mathcal{X}$ of θ;

(ii) $\mathcal{X}$*-positive* if $\mathcal{X} \in \mathcal{M}_X$ is a determination of θ and $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$ is ample.

A model metric $\|\cdot\|$ on a line bundle L is said to be semipositive if the curvature form $c_1(L, \|\cdot\|)$ is semipositive.

The equivalence in (i) follows from the following standard fact: if $\alpha \in N^1(\mathcal{X}/S)$ is a numerical class and $\pi : \mathcal{X}' \to \mathcal{X}$ is a vertical blowup, then $\pi^*\alpha$ is nef iff α is nef. On the other hand, the analogous result is obviously wrong for ample classes, so that it is indeed necessary to specify the model in (ii). If ω is $\mathcal{X}$-positive and $\theta \in \mathcal{Z}^{1,1}(X)$ is determined on $\mathcal{X}$, then $\omega + \varepsilon\theta$ is also $\mathcal{X}$-positive for all $0 < \varepsilon \ll 1$.

Note that if a closed $(1,1)$-form θ is semipositive, then its de Rham class $\{\theta\} \in N^1(X)$ is automatically nef. See Remark 5.4 for a more precise statement.

The set of all semipositive closed $(1,1)$-forms is a convex cone $\mathcal{Z}^{1,1}_+(X)$ of $\mathcal{Z}^{1,1}(X)$ that can be equivalently defined as

$$\mathcal{Z}^{1,1}_+(X) := \varinjlim_{\mathcal{X}} \mathrm{Nef}(\mathcal{X}/S).$$

Proposition 5.2. *Let θ be a closed $(1,1)$-form whose de Rham class $\{\theta\} \in N^1(X)$ is ample. For every sufficiently high model $\mathcal{X}$, we may then find a model function φ such that $\theta + dd^c\varphi$ is $\mathcal{X}$-positive. If θ is furthermore semipositive, then we may also arrange that $-\varepsilon \le \varphi \le 0$ for any given $\varepsilon > 0$.*

Proof. Let $\mathcal{X}'$ be a determination of θ and let $\mathcal{L}' \in \mathrm{Pic}(\mathcal{X}')_{\mathbf{R}}$ be a representative of θ. The assumption implies that the $\mathbf{R}$-line bundle $L := \mathcal{L}'|_{\mathcal{X}_K}$ is ample. By Corollary 1.5 we may thus assume that $\mathcal{X}'$ has been chosen so that L admits an ample extension $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$ for each model $\mathcal{X}$ dominating $\mathcal{X}'$. If $\pi : \mathcal{X} \to \mathcal{X}'$ denotes the corresponding vertical blowup, then $\mathcal{L} - \pi^*\mathcal{L}' = D$ for some $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$, and $\varphi = \varphi_D$ is a model function such that $\theta + dd^c\varphi$ is $\mathcal{X}$-positive.

Now suppose θ is semipositive and pick $\mathcal{X}$, φ as above. Upon replacing φ by $\varphi - \sup_X \varphi$ we may assume that $\varphi \le 0$. Then the closed $(1,1)$-form

$$\theta + dd^c(\varepsilon\varphi) = \varepsilon(\theta + dd^c\varphi) + (1-\varepsilon)\theta$$

is also $\mathcal{X}$-positive for each $0 < \varepsilon < 1$, completing the proof since φ is bounded. □

Since the nef cone of $N^1(X)$ is the closure of the ample cone, we get as a consequence:

Corollary 5.3. *The closure of the image of $\mathcal{Z}^{1,1}_+(X)$ in $N^1(X)$ coincides with the nef cone of $N^1(X)$.*

Remark 5.4. In the complex case, it is not always possible to find a smooth semipositive form in a nef class, so the image of $\mathcal{Z}^{1,1}_+(X)$ in $N^1(X)$ is *strictly* contained in $\mathrm{Nef}(X)$ in general; see [DPS94, Example 1.7]. In the non-Archimedean setting, the situation is unclear.

5.2. θ-psh model functions. By analogy with the complex case, we introduce:

Definition 5.5. Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form. A model function $\varphi \in \mathcal{D}(X)$ is said to be θ-plurisubharmonic (θ-psh for short) if the closed $(1,1)$-form $\theta + dd^c\varphi$ is semipositive.

Note that constant functions are θ-psh model functions iff θ is semipositive. Moreover, the existence of a θ-psh model function implies that the de Rham class $\{\theta\} \in N^1(X)$ is nef. Also note that if $\psi \in \mathcal{D}(X)$, then φ is a θ-psh model function iff $\varphi - \psi$ is $(\theta + dd^c\psi)$-psh.

We will need two technical results relating θ-psh model functions to fractional ideal sheaves.

Lemma 5.6. *Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ and let $\|\cdot\|$ be the corresponding model metric on $L := \mathcal{L}|_{\mathcal{X}_K}$. If $\mathfrak{a}$ is a vertical fractional ideal sheaf on $\mathcal{X}$ such that $\mathcal{L} \otimes \mathfrak{a}$ is generated by its global sections, then $\log|\mathfrak{a}|$ is a $c_1(L, \|\cdot\|)$-psh model function.*

Since S is affine, the direct image on S of a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is always generated by its global sections; hence $\mathcal{F}$ is globally generated in the absolute sense iff it is globally generated in the relative sense.

Proof. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the normalization of the blowup of $\mathcal{X}$ along $\mathfrak{a}$ and let $D \in \mathrm{Div}_0(\mathcal{X}')$ be the vertical Cartier divisor such that $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(D)$. The assumption implies that $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'}(D)$ is also generated by its global sections, so that $\pi^*\mathcal{L} + D$ is nef. The result follows since the model function $\log|\mathfrak{a}|$ is determined on $\mathcal{X}'$ by D. □

Lemma 5.7. *Let θ be a closed $(1,1)$-form and let $\mathcal{X}$ be a determination of θ. Then each θ-psh model function $\varphi \in \mathcal{D}(X)$ is a uniform limit on X of functions of the form $\frac{1}{m}\log|\mathfrak{a}|$ with $m \in \mathbf{N}^*$ and $\mathfrak{a}$ a vertical fractional ideal sheaf on $\mathcal{X}$.*

Proof. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a (normalized) vertical blowup such that $\varphi = \varphi_D$ for some $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$. Since θ is determined by $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$, the assumption that φ is θ-psh implies that D is π-nef. By Lemma 1.4 and Kleiman's criterion [Kle66], we may find a vertical π-ample $\mathbf{Q}$-divisor $A \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ arbitrarily close to D. It is then clear that φ_A is uniformly close to $\varphi = \varphi_D$ on X (see the proof of Corollary 2.4). Since A is π-ample we may find $m \gg 1$ such that $\mathcal{O}_{\mathcal{X}'}(mA)$ is π-globally generated. If we set $\mathfrak{a} := \pi_*\mathcal{O}_{\mathcal{X}'}(mA)$, we then have $\varphi_A = \frac{1}{m}\log|\mathfrak{a}|$, which concludes the proof. □

We are now in a position to establish the first properties of θ-psh model functions.

Proposition 5.8. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form. Then the set of θ-psh model functions $\varphi \in \mathcal{D}(X)$ is $(\mathbf{Q})$-convex and stable under* max.

Proof. Convexity is clear from the definition. To prove stability under maxima, let $\varphi_1, \varphi_2 \in \mathcal{D}(X)$ be θ-psh, pick a common determination $\mathcal{X}$ of θ and the φ_i's, and let $D_i \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ be a representative of φ_i for $i = 1, 2$.

Since the ample cone of $N^1(\mathcal{X}/S)$ is open, we may find ample line bundles $\mathcal{A}_1, \ldots, \mathcal{A}_r \in \mathrm{Pic}(\mathcal{X})$ whose numerical classes $\alpha_1, \ldots, \alpha_r$ form a basis of $N^1(\mathcal{X}/S)$. We may thus pick $t_1, \ldots, t_r \in \mathbf{R}$ such that $\mathcal{L} := \sum_j t_j \mathcal{A}_j$ is a representative of θ in $\mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$. Let $\varepsilon_1, \ldots, \varepsilon_r > 0$ be (small) positive numbers such that $t_j + \varepsilon_j \in \mathbf{Q}$ for each j and set $\mathcal{L}_\varepsilon := \sum_j (t_j + \varepsilon_j)\mathcal{A}_j$. Since φ_i is θ-psh it follows that $\mathcal{L}_\varepsilon + D_i$ is an ample $\mathbf{Q}$-divisor on $\mathcal{X}$ for $i = 1, 2$. We may thus find a positive integer m such that $m\mathcal{L}_\varepsilon \in \mathrm{Pic}(\mathcal{X})$, $mD_i \in \mathrm{Div}_0(\mathcal{X})$, and both sheaves $\mathcal{O}_{\mathcal{X}}\left(m\left(\mathcal{L}_\varepsilon + D_i\right)\right)$, $i = 1, 2$, are generated by their global sections on $\mathcal{X}$. If we introduce the vertical fractional ideal sheaf

$$\mathfrak{a}_m := \mathcal{O}_{\mathcal{X}}(mD_1) + \mathcal{O}_{\mathcal{X}}(mD_2),$$

then it follows that $\mathcal{O}_{\mathcal{X}}(m\mathcal{L}_\varepsilon) \otimes \mathfrak{a}_m$ is also generated by its global sections. By Lemma 5.6, $\log|\mathfrak{a}_m| = m\max\{\varphi_1, \varphi_2\}$ is thus psh with respect to $m(\theta + \sum_j \varepsilon_j \alpha_j)$, that is,

$$\theta + \sum_j \varepsilon_j \alpha_j + dd^c \max\{\varphi_1, \varphi_2\} \geq 0.$$

Letting $\varepsilon_j \to 0$, we conclude as desired that $\theta + dd^c \max\{\varphi_1, \varphi_2\} \geq 0$. □

Proposition 5.9. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form and let $\mathcal{X}$ be an SNC model on which θ is determined. Then each θ-psh model function $\varphi \in \mathcal{D}(X)$ satisfies:*

(i) *$\varphi \circ \mathrm{emb}_{\mathcal{X}}$ is piecewise affine and convex on each face of $\Delta_{\mathcal{X}}$;*
(ii) *$\varphi \leq \varphi \circ p_{\mathcal{X}}$ with equality if φ is determined on $\mathcal{X}$.*

Proof. This follows directly from Lemma 3.5(ii), Proposition 3.12 and Lemma 5.7 □

Finally we show that θ-psh model functions are plentiful as soon as $\{\theta\}$ is ample.

Proposition 5.10. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form whose de Rham class $\{\theta\} \in N^1(X)$ is ample. Then $\mathcal{D}(X)$ is spanned by θ-psh model functions.*

Proof. Let $\varphi \in \mathcal{D}(X)$. By Proposition 5.2 we may find a model $\mathcal{X}$ and a model function ψ such that θ, φ, and ψ are all determined on $\mathcal{X}$ and such that $\theta + dd^c\psi$ is $\mathcal{X}$-positive. Since the closed $(1,1)$-form $dd^c\varphi$ is determined on $\mathcal{X}$ we may thus find a rational number $0 < \varepsilon \ll 1$ such that $\theta + dd^c(\psi + \varepsilon\varphi) \geq 0$. It follows that $\varepsilon\varphi = (\psi + \varepsilon\varphi) - \psi$ is a difference of θ-psh model functions, and the result follows. □

5.3. Closedness of θ-psh model functions. The next result will be used to show that the definition of θ-psh functions in Section 7 below extends the one for model functions.

Theorem 5.11. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form. Then the set of θ-psh model functions is closed in $\mathcal{D}(X)$ with respect to the topology of pointwise convergence on X^{div}.*

This theorem in particular implies that S.-W. Zhang's definition of continuous semipositive metrics as uniform limits of semipositive model metrics (cf. [Zha95, 3.1]) is consistent when applied to model metrics. Another argument for this, valid in arbitrary residue characteristic, has been communicated to the authors by A. Thuillier. This argument uses a theorem of Tate to reduce to the case of curves.

We start the proof with the following special case.

Lemma 5.12. *Let $\mathcal{X}$ be an SNC model and pick $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ such that $L := \mathcal{L}|_{\mathcal{X}_K}$ is ample. Assume that the model metric $\|\cdot\|_{\mathcal{L}}$ is a pointwise limit over X^{div} of semipositive model metrics on L. Then $\|\cdot\|_{\mathcal{L}}$ itself is semipositive; i.e. $\mathcal{L}$ is nef.*

Proof.

Step 1. For each $m \geq 0$ let $\mathfrak{a}_m \subset \mathcal{O}_{\mathcal{X}}$ be the base-ideal of $\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$, i.e. the image of the evaluation map

$$H^0(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})) \otimes \mathcal{O}_{\mathcal{X}}(-m\mathcal{L}) \to \mathcal{O}_{\mathcal{X}}.$$

We are going to show that $\frac{1}{m}\log|\mathfrak{a}_m|$ converges pointwise to 0 on X^{div}. Note that $\mathfrak{a}_m$ is vertical for $m \gg 1$ since $\mathcal{L}$ is ample on the generic fiber of $\mathcal{X}$. The sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\geq 0}$ is a graded sequence of ideals; i.e. we have $\mathfrak{a}_m \cdot \mathfrak{a}_l \subset \mathfrak{a}_{m+l}$ for all m, l. It follows that $(\log|\mathfrak{a}_m|)_m$ is a super-additive sequence, which implies that

$$\lim_{m\to\infty} \frac{1}{m}\log|\mathfrak{a}_m| = \sup_m \frac{1}{m}\log|\mathfrak{a}_m| \leq 0 \tag{5.1}$$

pointwise on X. Pick a rational number $\varepsilon > 0$ and $x \in X^{\mathrm{div}}$. Let θ be the curvature form of $\|\cdot\|_{\mathcal{L}}$. Since 0 is by assumption a pointwise limit of θ-psh model functions, there exists a vertical blowup $\pi : \mathcal{X}' \to \mathcal{X}$ and $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ such that φ_D is θ-psh, $\varphi_D(x) \geq -\varepsilon$ and $\varphi_D(x_{E_i}) \leq \varepsilon$ for each irreducible component E_i of our given model $\mathcal{X}$. By Proposition 5.9 the latter condition yields $\varphi_D \leq \varepsilon$ on X, so that $D' := D + \varepsilon\mathcal{X}'_0 \in \mathrm{Div}_0(\mathcal{X}')$ satisfies $D' \leq 0$ and $\varphi_{D'}(x) \geq -2\varepsilon$. On the other hand, we may assume that $\mathcal{X}'$ has been chosen high enough to apply Proposition 5.2 and get $D'' \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ with $D'' \leq 0$, $\varphi_{D''} \geq -\varepsilon$ on X, and $\pi^*\mathcal{L} + D' + D''$ ample. Since $D' + D'' \leq 0$ we then have

$$\mathcal{O}_{\mathcal{X}'}(m\,(\pi^*\mathcal{L} + D' + D'')) \subset \mathcal{O}_{\mathcal{X}'}(m\pi^*\mathcal{L})$$

for all $m \in \mathbf{N}$. Now the left-hand side is globally generated for some m. Since $\pi_* \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$, all sections of $m\pi^*\mathcal{L}$ are pull-backs of sections of $m\mathcal{L}$, and the projection formula therefore yields

$$\mathcal{O}_{\mathcal{X}'}(m(D' + D'')) \subset \mathcal{O}_{\mathcal{X}'} \cdot \mathfrak{a}_m,$$

hence

$$-3\varepsilon \leq \varphi_{D'+D''}(x) \leq \frac{1}{m} \log |\mathfrak{a}_m|(x).$$

We have thus shown that $\sup_m \frac{1}{m} \log |\mathfrak{a}_m| \geq 0$ at each $x \in X^{\mathrm{div}}$, which implies as desired that $\frac{1}{m} \log |\mathfrak{a}_m|$ converges to 0 pointwise on X^{div} thanks to (5.1).

Step 2. Let us now show that $\mathcal{L}$ is nef. For each $c > 0$ let $\mathcal{J}(\mathfrak{a}_\bullet^c) \subset \mathcal{O}_{\mathcal{X}}$ be the multiplier ideal attached to the graded sequence $\mathfrak{a}_\bullet$ (cf. Appendix B). We have the elementary inclusion $\mathfrak{a}_m \subset \mathcal{J}(\mathfrak{a}_\bullet^m)$ for all $m \in \mathbf{N}$, whereas the subadditivity property (cf. Theorem B.7) implies $\mathcal{J}(\mathfrak{a}_\bullet^{ml}) \subset \mathcal{J}(\mathfrak{a}_\bullet^m)^l$ for all $l, m \in \mathbf{N}$. We infer that $\mathfrak{a}_{ml} \subset \mathcal{J}(\mathfrak{a}_\bullet^m)^l$ for any m, l and hence

$$\sup_l \tfrac{1}{l} \log |\mathfrak{a}_{ml}| \leq \log |\mathcal{J}(\mathfrak{a}_\bullet^m)| \leq 0.$$

By Step 1 we conclude that $\log |\mathcal{J}(\mathfrak{a}_\bullet^m)| = 0$; i.e. $\mathcal{J}(\mathfrak{a}_\bullet^m) = \mathcal{O}_{\mathcal{X}}$ since multiplier ideals are integrally closed by definition. The uniform global generation property of multiplier ideals (Theorem B.8) now yields an (ample) line bundle $\mathcal{A} \in \mathrm{Pic}(\mathcal{X})$ independent of m such that $m\mathcal{L} + \mathcal{A}$ is globally generated (and hence nef) for all $m \in \mathbf{N}$. This immediately shows that $\mathcal{L}$ is nef since the latter is a closed condition. □

Proof of Theorem 5.11. Suppose that $\varphi \in \mathcal{D}(X)$ is a pointwise limit of θ-psh model functions. Our goal is to show that φ is θ-psh. Upon replacing θ with $\theta + dd^c\varphi$ we may assume that $\varphi = 0$. Note that the existence of at least one θ-psh model function implies that the de Rham class $\{\theta\} \in N^1(X)$ is nef. Let $\mathcal{X}$ be a determination of θ. Thus $\theta_{\mathcal{X}}|_{\mathcal{X}_K}$ is nef. As in Proposition 5.8 we can choose finitely many ample line bundles $\mathcal{A}_i \in \mathrm{Pic}(\mathcal{X})$ such that their numerical classes $\alpha_i \in N^1(\mathcal{X}/S)$ form a basis of $N^1(\mathcal{X}/S)$. There exist arbitrarily small positive numbers ε_i such that $\theta_{\mathcal{X}} + \sum_i \varepsilon_i \alpha_i$ is a rational class, hence the class of a $\mathbf{Q}$-line bundle $\mathcal{L}_\varepsilon$ on $\mathcal{X}$, whose restriction to $\mathcal{X}_K$ is ample. Since 0 is a pointwise limit of θ-psh model functions and since $\| \cdot \|_{\mathcal{L}_\varepsilon} e^{-\psi}$ is semipositive for each θ-psh model function ψ, we may now apply Lemma 5.12 to conclude that $\mathcal{L}_\varepsilon$ is nef. It follows that $\theta_{\mathcal{X}} \in \mathrm{Nef}(\mathcal{X}/S)$ by closedness of the nef cone. □

Remark 5.13. The use of multiplier ideals in Step 2 is similar to [ELMNP06, Proposition 2.8] and very much in the spirit of the arguments we shall use to prove Theorem B. It would be interesting to have a proof along the lines of [Goo69, p. 178, Proposition 8].

5.4. Comparison of terminology. The terminology for (semipositive) model metrics is unfortunately not uniform across the literature. Here is a tentative summary.

TABLE 1. Terminology for metrics on line bundles.

Model metric: [YZ09]	*Semipositive continuous metric*: [CL06, CL11]
Algebraic metric: [BPS11, CL06, Liu11]	Approachable metric: [BPS11]
Smooth metric: [CL11]	Semipositive metric: [YZ09, YZ13a, YZ13b, Liu11]
Root of an algebraic metric: [Gub98]	Semipositive admissible metric: [Gub98]

6. Equicontinuity

The following result is the key to the compactness property in Theorem A.

Theorem 6.1. *Let X be a smooth connected projective K-analytic space. Let $\mathcal{X}$ be an SNC model of X and $\theta \in \mathcal{Z}^{1,1}(X)$ a closed $(1,1)$-form determined on $\mathcal{X}$. Then there exists a constant $C = C(\mathcal{X}, \theta) > 0$ such that for every θ-psh model function φ, the composition $\varphi \circ \mathrm{emb}_{\mathcal{X}}$ is convex, piecewise affine and C-Lipschitz continuous on each face of $\Delta_{\mathcal{X}}$.*

Corollary 6.2. *With the same notation, the family*

$$\{\varphi \circ \mathrm{emb}_{\mathcal{X}} \mid \varphi \text{ a } \theta\text{-psh model function}\} \subset C^0(\Delta_{\mathcal{X}})$$

is equicontinuous on $\Delta_{\mathcal{X}}$.

The rest of this section is devoted to the proof of Theorem 6.1. For the sake of notational simplicity we will (in this section only) ignore the map $\mathrm{emb}_{\mathcal{X}}$ and simply view $\Delta := \Delta_{\mathcal{X}}$ as a subset of X.

Let us first set some notation. Let $\mathcal{X}_0 = \sum_{i \in I} b_i E_i$ be the irreducible decomposition of the special fiber of $\mathcal{X}$ and let $e_i = \mathrm{ev}_{\mathcal{X}}(x_{E_i})$ be the vertex of Δ corresponding to E_i. Recall that the faces σ_J of the simplical complex Δ correspond to subsets $J \subset I$ such that $E_J = \bigcap_{j \in J} E_j$ is non-empty, in such a way that σ_J is a simplex with $\{e_j,\, j \in J\}$ as its vertices. The *star* $\mathrm{Star}(\sigma)$ of a face σ of Δ is defined as usual as the union of all faces of Δ containing σ. An irreducible component E_i intersects E_J iff the corresponding vertex e_i of Δ belongs to $\mathrm{Star}(\sigma_J)$; the intersection is proper iff $e_i \notin \sigma_J$.

Fix an ample line bundle $\mathcal{A}$ on $\mathcal{X}$. In what follows we denote by $C > 0$ a dummy constant, which may vary from line to line but only depends on $\mathcal{X}$, θ and $\mathcal{A}$.

Let $\varphi \in \mathcal{D}(X)$ be a θ-psh model function and $\pi : \mathcal{Y} \to \mathcal{X}$ a vertical blowup such that $\varphi = \varphi_G$ for some $G \in \mathrm{Div}_0(\mathcal{Y})_{\mathbf{Q}}$. By Proposition 5.9 we have $\sup_X \varphi = \max_{i \in I} \varphi(e_i)$. Upon replacing G with $G - (\max_{i \in I} \varphi(e_i))\mathcal{Y}_0$ we may thus assume that φ is normalized by $\sup_X \varphi = 0$.

6.1. Bounding the values on vertices. We first prove

$$\max_{i \in I} |\varphi(e_i)| \leq C. \tag{6.1}$$

Recall that we have normalized φ so that $\max_{i \in I} \varphi(e_i) = 0$. We may therefore assume that $\mathcal{X}_0$ has at least two irreducible components. Observe that the push-forward of the $\mathbf{Q}$-Weil divisor G is given by $\pi_* G = \sum_j b_j \varphi(e_j) E_j$. For each $i \in I$, the projection formula shows that

$$(\theta_{\mathcal{X}} + \pi_* G) \cdot E_i \cdot \mathcal{A}^{n-1} = (\pi^* \theta_{\mathcal{X}} + G) \cdot \pi^* E_i \cdot (\pi^* \mathcal{A})^{n-1},$$

which is non-negative since $\pi^* E_i \in \mathrm{Div}_0(\mathcal{Y})$ is effective, and both classes $\pi^* \mathcal{A}$ and $\pi^* \theta_{\mathcal{X}} + G$ are nef (the latter because φ is θ-psh). It follows that there exists $C = C(\mathcal{X}, \theta, \mathcal{A})$ such that

$$\sum_j b_j \varphi(e_j)(E_i \cdot E_j \cdot \mathcal{A}^{n-1}) + C \geq 0 \tag{6.2}$$

for all i. Note that $E_i \cdot E_j \cdot \mathcal{A}^{n-1} \geq 0$ for all $i \neq j$, with strict inequality if $E_i \cap E_j \neq \emptyset$. Thus

$$b_i E_i \cdot E_i \cdot \mathcal{A}^{n-1} = E_i \cdot (b_i E_i - \mathcal{X}_0) \cdot \mathcal{A}^{n-1} = -\sum_{j \neq i} b_j \, E_i \cdot E_j \cdot \mathcal{A}^{n-1} \leq -1$$

for all i since $\mathcal{X}_0$ has connected support and contains at least two irreducible components.

Now pick $i_0, \dots, i_M$ such that $\varphi(e_{i_0}) = 0$, $\varphi(e_{i_M}) = \min_{i \in I} \varphi(e_i)$, and e_{i_m} and $e_{i_{m+1}}$ are connected by a one-dimensional face, so that $E_{i_m} \cdot E_{i_{m+1}} \cdot \mathcal{A}^{n-1} \geq 1$. Write

$$\lambda := \max_{i \in I} \left\{ -b_i E_i^2 \cdot \mathcal{A}^{n-1} \right\} \geq 1.$$

Applying (6.2) to $i = i_m$, $0 \leq m < M$, we get

$$\begin{aligned}\lambda \varphi(e_{i_m}) \leq -b_{i_m} \varphi(e_{i_m})(E_{i_m}^2 \cdot \mathcal{A}^{n-1}) &\leq C + \sum_{j \neq i_m} b_j \varphi(e_j)(E_{i_m} \cdot E_j \cdot \mathcal{A}^{n-1}) \\ &\leq C + b_{i_{m+1}} \varphi(e_{i_{m+1}})(E_{i_m} \cdot E_{i_{m+1}} \cdot \mathcal{A}^{n-1}) \leq C + \varphi(e_{i_{m+1}}),\end{aligned}$$

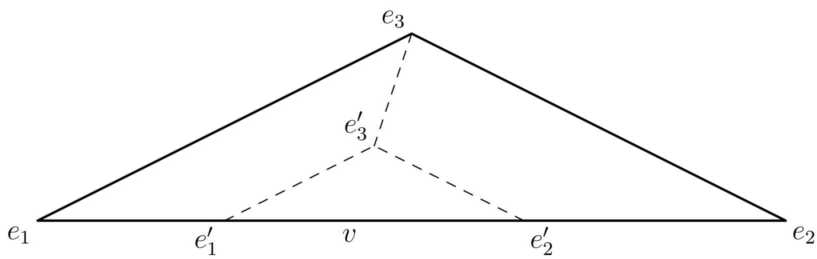

FIGURE 1. The subdivision of §6.2. Here v lies in the relative interior of the simplex σ of $\Delta_{\mathcal{X}}$ with vertices e_1 and e_2. The picture shows the intermediate subdivision Δ^ε, where v lies in the relative interior of the simplex σ' with vertices e_1' and e_2'. The final subdivision Δ' is obtained from Δ^ε by barycentric subdivision of the quadrilaterals $\mathrm{Conv}(e_1, e_3, e_1', e_3')$ and $\mathrm{Conv}(e_2, e_3, e_2', e_3')$.

so that

$$0 \geq \varphi(e_{i_M}) \geq -C + \lambda\varphi(e_{i_{M-1}}) \geq -C - C\lambda + \lambda^2\varphi(e_{i_{M-2}}) \\ \geq \cdots \geq -C\sum_{m=0}^{M-1}\lambda^m + \lambda^M\varphi(e_{i_0}) = -C\sum_{m=0}^{M-1}\lambda^m,$$

which proves (6.1).

6.2. Special subdivisions. We shall need the following construction; see Figure 1. Let $\sigma = \sigma_J$ be a face of Δ and $L \subset I$ the set of vertices of Δ contained in $\mathrm{Star}_\Delta(\sigma)$. Consider a rational point v in the relative interior of σ. Given $0 < \varepsilon < 1$ rational and $j \in L$ set $e_j^\varepsilon := \varepsilon e_j + (1-\varepsilon)v$. We shall define a projective simplicial subdivision $\Delta' = \Delta'(\varepsilon, v)$ of Δ.

To define Δ', we first introduce a polyhedral subdivision $\Delta^\varepsilon = \Delta^\varepsilon(v)$ of Δ leaving the complement of $\mathrm{Star}_\Delta(\sigma)$ unchanged, as follows. The set of vertices of Δ^ε is precisely $(e_i)_{i\in I} \cup (e_j^\varepsilon)_{j\in L}$ and the faces of Δ^ε contained in $\mathrm{Star}(\sigma)$ are of one of the following types:

- if the convex hull $\mathrm{Conv}(e_{j_1}, \dots, e_{j_m})$ is a face of Δ containing σ, then $\mathrm{Conv}(e_{j_1}^\varepsilon, \dots, e_{j_m}^\varepsilon)$ is a face of Δ^ε;
- if $\mathrm{Conv}(e_{j_1}, \dots, e_{j_m})$ is a face of Δ contained in $\mathrm{Star}(\sigma)$ but not containing σ, then both $\mathrm{Conv}(e_{j_1}, \dots, e_{j_m})$ and $\mathrm{Conv}(e_{j_1}, \dots, e_{j_m}, e_{j_1}^\varepsilon, \dots, e_{j_m}^\varepsilon)$ are faces of Δ^ε.

In a neighborhood of v, note that the subdivision Δ^ε is obtained by scaling Δ by a factor ε. More precisely, consider the affine map $\psi^\varepsilon : \mathrm{Star}(\sigma) \to \mathrm{Star}(\sigma)$ defined by $\psi^\varepsilon(w) = \varepsilon w + (1-\varepsilon)v$. Then $\sigma^\varepsilon := \psi^\varepsilon(\sigma)$ is the face of Δ^ε containing v in its relative interior, and $\psi^\varepsilon(\mathrm{Star}_\Delta(\sigma)) = \mathrm{Star}_{\Delta^\varepsilon}(\sigma^\varepsilon)$. In particular, even though Δ^ε is not simplicial in general, all polytopes of Δ^ε containing σ^ε are simplicial.

We claim that Δ^ε is projective. To see this, write $v = \sum_{j\in J} s_j e_j$, with $s_j > 0$ rational and $\sum s_j = 1$. For $j \in J$, define a linear function ℓ_j on $\sum_{i\in I} \mathbf{R}_+ e_i \supset \Delta$ by $\ell_j(\sum t_i e_i) = -t_j/s_j$ and set

$$h = \max\{\max_{j\in J} \ell_j, -(1-\varepsilon)\}.$$

A suitable integer multiple of h is then a strictly convex support function for Δ^ε in the sense of §3.5.

Now define $\Delta' = \Delta'(\varepsilon)$ as a simplicial subdivision of Δ^ε obtained using repeated barycentric subdivision in a way that leaves $\mathrm{Star}_{\Delta^\varepsilon}(\sigma^\varepsilon)$ unchanged. By [KKMS, pp. 115–117], Δ' is still projective.

Note that $\sigma' := \sigma^\varepsilon$ is the face of Δ' containing v in its relative interior. For $j \in L$ set $e'_j = e^\varepsilon_j$. These are the vertices of Δ' contained in $\mathrm{Star}_{\Delta'}(\sigma')$.

6.3. Bounding Lipschitz constants. Let τ be a face of Δ. Our aim is to prove by induction on $\dim \tau$ that the $C^{0,1}$-norm of φ on τ is bounded by C. Recall that the $C^{0,1}$-norm is defined as the sum of the sup-norm and the Lipschitz norm; see Appendix A.

The case $\dim \tau = 0$ is settled by (6.1), so let us assume that $\dim \tau > 0$. By Proposition 5.9, the restriction of φ to τ is piecewise affine and convex. It therefore admits directional derivatives, and we set as in Appendix A

$$D_v\varphi(w) := \left.\frac{d}{dt}\right|_{t=0+} \varphi\left((1-t)v + tw\right)$$

for $v, w \in \tau$.

Let us say that a codimension 1 face of τ is *opposite* a vertex when it is the convex hull of the remaining vertices of τ. This notion is well-defined since τ is a simplex.

Proposition 6.3. *There exists a constant $C > 0$ such that*

$$D_v\varphi(e) \geq -C$$

for any vertex e of τ and any rational point v in the relative interior of the face σ of τ opposite e such that $\varphi|_\sigma$ is affine near v.

Note that the assumptions of the proposition are automatically satisfied when τ has dimension 1, with boundary consisting of e and v.

Granting this result, let us explain how to conclude the proof. By induction we have $\sup_{\partial\tau} |\varphi| \leq C$. The convexity of φ and the inductive assumption imply $D_v\varphi(e) \le \varphi(e) - \varphi(v) \le C$ for any $e, v \in \partial\tau$. As for a lower bound, Proposition 6.3 gives $D_v\varphi(e) \ge -C$ for any vertex e of τ and any rational point v in the relative interior of the face σ opposite e such that $\varphi|_\sigma$ is affine near v. Now, by elementary properties of convex functions, $v \mapsto D_v\varphi(e)$ is upper semicontinuous on the relative interior of σ, so since φ is piecewise affine, the lower bound $D_v\varphi(e) \ge -C$ holds for any v in the relative interior of σ. We conclude by Proposition A.1 that the $C^{0,1}$-norm of $\varphi|_\tau$ is bounded by C, completing the proof of Theorem 6.1.

Proof of Proposition 6.3. Let I be the set of vertices in Δ, let $L \subset I$ be the set of vertices contained in $\mathrm{Star}_\Delta(\sigma)$, and let $J \subset L$ be the set of vertices of σ. Thus $\sigma = \sigma_J$.

Consider the simplicial projective subdivision $\Delta' = \Delta'(\varepsilon)$ constructed in §6.2. For $j \in L$, $e'_j := \varepsilon e_j + (1-\varepsilon)v$ is a vertex of Δ'. Recall that $\sigma' = \sigma'_J$ is the face of Δ' containing v in its relative interior. Since $\varphi|_\sigma$ is assumed affine in a neighborhood of v, we may choose $\varepsilon > 0$ small enough that:

- φ is affine on $\sigma' \subset \sigma$;
- φ is affine on each segment $[v, e'_j]$, $j \in L$.

Let $\rho : \mathcal{X}' \to \mathcal{X}$ be the vertical blowup corresponding to the subdivision Δ' of Δ as in Theorem 3.15. Note that ρ induces a generically finite map $E'_J \to E_J$ of projective k-varieties. Indeed, E_J (resp. E'_J) is the closure of the center of v on $\mathcal{X}$ (resp. $\mathcal{X}'$), and both have codimension $|J|$ by Theorem 3.15.

Recall that $\varphi = \varphi_G$ for some $G \in \mathrm{Div}_0(\mathcal{Y})$. We may assume that the determination $\mathcal{Y}$ of φ dominates $\mathcal{X}'$, so that π factors as $\pi = \rho \circ \mu$ with $\mu : \mathcal{Y} \to \mathcal{X}'$. Note that $\mu_* G$ is $\mathbf{Q}$-Cartier since $\mathcal{X}'$ is vertically $\mathbf{Q}$-factorial.

As we shall see shortly, a first computation shows:

Lemma 6.4. *We have*

$$\rho^* \left(\sum_{j \in L} D_v\varphi(e_j)\, b_j E_j \right)\Bigg|_{E'_J} = (\mu_* G)|_{E'_J}$$

in $\mathrm{Pic}(E'_J)_{\mathbf{Q}}$.

The key observation is now the following positivity property:

Lemma 6.5. *If* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}')$ *is nef, then* $E'_J \cdot (\rho^*\theta_{\mathcal{X}} + \mu_* G) \cdot \mathcal{L}^{n-|J|-1} \ge 0$.

Grant this result for the moment. Lemma 6.4 and the projection formula yield

$$\deg(\rho|_{E'_J}) E_J \cdot \left(\theta_{\mathcal{X}} + \sum_{j \in L} D_v\varphi(e_j)\, b_j E_j \right) \cdot \mathcal{A}^{n-|J|-1} = E'_J \cdot (\rho^*\theta_{\mathcal{X}} + \mu_* G) \cdot \rho^* \mathcal{A}^{n-|J|-1}.$$

Here the right-hand side is non-negative by Lemma 6.5 since $\rho^*\mathcal{A}$ is nef, and we get

$$\sum_{j\in L} D_v\varphi(e_j)\, b_j \left(E_J \cdot E_j \cdot \mathcal{A}^{n-|J|-1}\right) \geq -(E_J \cdot \theta_{\mathcal{X}} \cdot \mathcal{A}^{n-|J|-1}). \tag{6.3}$$

By induction, the $C^{0,1}$-norm of $\varphi|_\sigma$ is under control. Since v belongs to $\sigma = \sigma_J$, this gives

$$|\varphi(v)| \leq C \text{ and } \max_{j\in J} |D_v\varphi(e_j)| \leq C,$$

and (6.3) yields a lower bound

$$\sum_{j\in L\setminus J} D_v\varphi(e_j)\, b_j \left(E_J \cdot E_j \cdot \mathcal{A}^{n-|J|-1}\right) \geq -C. \tag{6.4}$$

Now the convexity of φ and the normalization $\sup_X \varphi = 0$ show that

$$\max_{j\in L\setminus J} D_v\varphi(e_j) \leq \max_{j\in L\setminus J}(\varphi(e_j) - \varphi(v)) \leq -\varphi(v) \leq C.$$

Here $E_j|_{E_J}$ is a non-zero effective divisor for $j \notin J$; hence $E_J \cdot E_j \cdot \mathcal{A}^{n-|J|-1} \geq 1$ since $\mathcal{A}$ is ample. From (6.4) we therefore obtain, as desired, that $D_v\varphi(e) \geq -C$ since $e = e_j$ for some $j \in L \setminus J$. $\square$

Proof of Lemma 6.4. We write $v = \sum_{j\in J} s_j e_j$ with $s_j > 0$ rational and $\sum_{j\in J} s_j = 1$. Set $s_i = 0$ for $i \in I \setminus J$. For $i \in I$ let φ_i be the model function induced by the vertical divisor $b_i E_i \in \mathrm{Div}_0(\mathcal{X})$. This function is affine on each face of Δ and satisfies $\varphi_i(e_j) = \delta_{ij}$ for all $j \in I$. Since $e'_j = \varepsilon e_j + (1-\varepsilon)v$ for $j \in L$ we get

$$\varphi_i(e'_j) = \begin{cases} \varepsilon + (1-\varepsilon)s_i & \text{if } i = j \in J, \\ (1-\varepsilon)s_i & \text{if } i \neq j \in J, \\ \varepsilon & \text{if } i = j \in L \setminus J, \\ 0 & \text{if } i \neq j \in L \setminus J. \end{cases}$$

By Theorem 3.15, E'_j intersects E'_J iff $j \in L$. We thus have

$$\rho^*(b_i E_i)|_{E'_J} = \sum_{j\in L} \varphi_i(e'_j) b'_j\, E'_j|_{E'_J} \text{ for all } i \in I$$

and

$$(\mu_* G - \varphi(v)\rho^*\mathcal{X}_0)|_{E'_J} = \sum_{j\in L} (\varphi(e'_j) - \varphi(v)) b'_j\, E'_j|_{E'_J}$$

in $\mathrm{Pic}(E'_J)_{\mathbf{Q}}$, where we have set $b'_j := \mathrm{ord}_{E'_j}(\varpi)$.

Recall also that φ is affine on each segment $[v, e_i']$, so that $D_v\varphi(e_i) = \varepsilon^{-1}(\varphi(e_i') - \varphi(v))$ for $i \in L$. We can now compute in $\operatorname{Pic}(E_J')_{\mathbf{Q}}$:

$$\rho^*\left(\sum_{i\in L} D_v\varphi(e_i)\, b_i E_i\right)\Bigg|_{E_J'} = \sum_{i\in L}\varepsilon^{-1}\left(\varphi(e_i') - \varphi(v)\right)\left(\sum_{j\in L}\varphi_i(e_j')\, b_j'\, E_j'|_{E_J'}\right)$$

$$= \sum_{i\in J}\varepsilon^{-1}(\varphi(e_i') - \varphi(v))\left(\varepsilon\, b_i'\, E_i'|_{E_J'} + s_i\sum_{j\in J}(1-\varepsilon)\, b_j'\, E_j'|_{E_J'}\right)$$

$$+ \sum_{i\in L\setminus J}\varepsilon^{-1}(\varphi(e_i') - \varphi(v))\,\varepsilon\, b_i'\, E_i'|_{E_J'}$$

$$= \sum_{i\in L}(\varphi(e_i') - \varphi(v))\, b_i'\, E_i'|_{E_J'} + \varepsilon^{-1}(1-\varepsilon)\left(\sum_{i\in J}s_i(\varphi(e_i') - \varphi(v))\right)\left(\sum_{j\in J}b_j'\, E_j'|_{E_J'}\right)$$

$$= (\mu_*G - \varphi(v)\rho^*\mathcal{X}_0)|_{E_J'} = \mu_*G|_{E_J'}.$$

The second to last equality follows from the fact that φ is affine on the simplex σ_J' of Δ' so that $\sum_{i\in J}s_i\varphi(e_i') = \varphi(v) = \sum_{i\in J}s_i\varphi(v)$. This concludes the proof. $\square$

Proof of Lemma 6.5. Set $F := \mu^*\mu_*G - G \in \operatorname{Div}_0(\mathcal{Y})_{\mathbf{Q}}$. The divisor G is μ-nef since $\mu^*(\rho^*\theta_{\mathcal{X}}) + G$ is nef by assumption, and Lemma 1.6 therefore implies that F is effective.

Let W be the closure of the center of v on $\mathcal{Y}$. Since the center of v on $\mathcal{X}'$ is the generic point of E_J', we must have $\mu(W) = E_J'$. Note, however, that we do not claim $\dim W = \dim E_J'$.

By Theorem 3.15, the function φ_{μ_*G} is affine on the face σ_J' of Δ'. But φ_G is also affine on σ_J' by assumption, and we have

$$\varphi_G(e_j') = \frac{1}{b_j'}\operatorname{ord}_{E_j'}(G) = \varphi_{\mu_*G}(e_j') \quad \text{for all } j \in J.$$

It follows that $\varphi_F \equiv 0$ on σ_J', and in particular $v(F) = 0$ (here we view v as an element of the dual of the space of Cartier divisors supported on the special fiber). But this means precisely that W is not contained in $\operatorname{Supp} F$, so that $F|_W$ is an effective $\mathbf{Q}$-Cartier divisor. Hence

$$\mu^*(\rho^*\theta_{\mathcal{X}} + \mu_*G)|_{E_J'} = (\pi^*\theta_{\mathcal{X}} + \mu^*\mu_*G)|_W = (\pi^*\theta_{\mathcal{X}} + G)|_W + F|_W$$

is the sum of a nef class and an effective class. We conclude by Lemma 6.6 below. $\square$

Lemma 6.6. *Let $\mu : W \to V$ be a surjective morphism between projective varieties over a field k and let $\alpha \in N^1(V)$. If $\mu^*\alpha = \gamma + F$ where γ is nef and F is effective, then $\left(\alpha \cdot \beta^{\dim V - 1}\right)_V \geq 0$ for every nef class $\beta \in N^1(V)$.*

Proof. Let k^a be an algebraic closure of k and let V' be an irreducible component of (the reduced scheme associated to) $V^a := V \otimes k^a$. There exists a component W' of W^a dominating V'. Upon replacing W and V by W' and V' we are reduced to the case where k is algebraically closed. Upon taking successive hyperplane sections of W not containing any component of F and choosing an irreducible component dominating V we may then assume that μ is generically finite. In that case we have

$$(\deg \mu)\left(\alpha \cdot \beta^{\dim V-1}\right)_V = \left(\mu^*\alpha \cdot \mu^*\beta^{\dim V-1}\right)_W$$

and the result follows from Kleiman's theorem (see e.g. [Laz, Theorem 1.4.9]) since $\mu^*\beta$ is nef. □

7. General θ-psh functions and semipositive singular metrics

We are now ready to introduce the class of general θ-psh functions and their cousins: semipositive singular metrics. The equicontinuity result in Corollary 6.2 will be used to show Theorem A, asserting that the space of θ-psh functions is compact up to translation.

Throughout this section we let X be as before a smooth connected projective K-analytic variety and fix a closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ whose de Rham class $\{\theta\} \in N^1(X)$ is *ample*. As before we write "psh" as shorthand for "plurisubharmonic". Similarly, "usc" and "lsc" will mean "upper semicontinuous" and "lower semicontinuous", respectively.

Definition 7.1. Let θ be as above. A *θ-psh function* $\varphi : X \to [-\infty, +\infty[$ is a usc function such that for each SNC model $\mathcal{X}$ of X on which θ is determined we have

(i) $\varphi \leq \varphi \circ p_{\mathcal{X}}$ on X;
(ii) the restriction of φ to $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ is a uniform limit of restrictions of θ-psh model functions.

We write $\mathrm{PSH}(X, \theta)$ for the set of θ-psh functions on X.

We say that $\varphi : X \to [-\infty, +\infty[$ is *quasi-psh* if φ is θ-psh for some θ as above. Thanks to Theorem 5.11, the previous definition is consistent with Definition 5.5 when φ is a model function. In particular, constant functions are θ-psh iff θ is semipositive.

Remark 7.2. A function on a compact (complex) Kähler manifold X is quasi-psh if it is locally the sum of a psh function and a smooth function. Given a closed $(1,1)$-form θ, a θ-psh function φ is a quasi-psh function such that $\theta + dd^c\varphi \geq 0$ in the sense of currents. When the de Rham class $\{\theta\} \in H^{1,1}(X)$ is a Kähler class, we have a global characterization: θ-psh functions are decreasing limits of sequences of smooth θ-psh functions; see [Dem92, Theorem 1.1]. In our current non-Archimedean setting, a general local theory of psh functions is still to be developed (although the first steps are taken in [CLD12]). For this reason, we work globally and assume that $\{\theta\}$ is ample.

Recall the definitions from §4 of singular and model metrics on line bundles.

Definition 7.3. Let L be an ample line bundle on X. A singular metric on L is *semipositive* if it is of the form $\|\cdot\|e^{-\varphi}$, where $\|\cdot\|$ is a model metric and φ is $c_1(L,\|\cdot\|)$-psh.

One checks that this definition does not depend on the choice of reference metric $\|\cdot\|$. Below we shall state various properties of θ-psh functions. We leave it to the reader to formulate analogous assertions about semipositive singular metrics on ample line bundles.

7.1. Basic properties. Fix a closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ as above. From Propositions 5.8 and 5.9 we obtain

Proposition 7.4. *The set* $\mathrm{PSH}(X,\theta)$ *is convex. If* φ,ψ *are* θ*-psh and* $c \in \mathbf{R}$*, then* $\varphi + c$ *and* $\max\{\varphi,\psi\}$ *are* θ*-psh.*

Proposition 7.5. *If* $\varphi \in \mathrm{PSH}(X,\theta)$ *and* $\mathcal{X}$ *is an SNC model on which* θ *is determined, then:*

(i) $\varphi \circ \mathrm{emb}_{\mathcal{X}}$ *is continuous (and finite) on* $\Delta_{\mathcal{X}}$ *and convex on each face;*
(ii) $\varphi \circ p_{\mathcal{X}}$ *is continuous on* X *and* $\varphi \le \varphi \circ p_{\mathcal{X}}$.

The next result shows how to reconstruct a θ-psh function from its values on quasimonomial points.

Proposition 7.6. *Let* $\varphi \in \mathrm{PSH}(X,\theta)$*. Then, as* $\mathcal{X}$ *runs through the directed set of SNC models on which* θ *is determined,* $(\varphi \circ p_{\mathcal{X}})_{\mathcal{X}}$ *forms a decreasing net of continuous functions on* X*, converging pointwise to* φ*.*

Proof. Let $\mathcal{X}' \ge \mathcal{X}$ be two SNC models on which θ is determined. Then $p_{\mathcal{X}} \circ p_{\mathcal{X}'} = p_{\mathcal{X}}$. By Proposition 7.5(ii) this implies $\varphi \le \varphi \circ p_{\mathcal{X}'} \le \varphi \circ p_{\mathcal{X}} \circ p_{\mathcal{X}'} = \varphi \circ p_{\mathcal{X}}$, with equality on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. Set $\tilde{\varphi} := \lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}$. Then $\tilde{\varphi} \ge \varphi$. On the other hand, by Corollary 3.9, $p_{\mathcal{X}}$ converges to the identity on X, so by upper semicontinuity of φ we have $\varphi \ge \tilde{\varphi}$. □

Corollary 7.7. *If* $\varphi,\psi \in \mathrm{PSH}(X,\theta)$ *and* $\varphi = \psi$ *on* X^{div}*, then* $\varphi = \psi$*.*

Proof. Let $\mathcal{X}$ be an SNC model on which θ is determined. By Corollary 3.17, X^{div} is dense in $\mathrm{emb}(\Delta_{\mathcal{X}})$ and by Proposition 7.5, φ and ψ are continuous on $\mathrm{emb}(\Delta_{\mathcal{X}})$, so our assumptions imply that $\varphi = \psi$ on $\mathrm{emb}(\Delta_{\mathcal{X}})$. Thus $\varphi \circ p_{\mathcal{X}} = \psi \circ p_{\mathcal{X}}$. Letting $\mathcal{X}$ run though all sufficiently large models, we conclude using Proposition 7.6. □

7.2. Equicontinuity. The Lipschitz estimates in Theorem 6.1 carry over to general θ-psh functions. As a consequence we have

Corollary 7.8. *For any SNC model on which* θ *is determined, the family*

$$\{\varphi \circ \mathrm{emb}_{\mathcal{X}} \mid \varphi \in \mathrm{PSH}(X,\theta)\}$$

is an equicontinuous family of functions on $\Delta_{\mathcal{X}}$*.*

Corollary 7.9. *If $(\varphi_i)_i$ is a decreasing net in* $\mathrm{PSH}(X, \theta)$ *and* $\varphi = \lim_i \varphi_i$, *then either* $\varphi \equiv -\infty$ *or* $\varphi \in \mathrm{PSH}(X, \theta)$.

Proof. Assume $\varphi \not\equiv -\infty$ and the upper semicontinuity of φ_i implies that of φ. Let $\mathcal{X}$ be any SNC model on which θ is determined. The inequality $\varphi_i \le \varphi_i \circ p_{\mathcal{X}}$ implies $\varphi \le \varphi \circ p_{\mathcal{X}}$, and hence

$$\sup_{\Delta_{\mathcal{X}}}(\varphi \circ \mathrm{emb}_{\mathcal{X}}) = \sup_X \varphi$$

is finite. By Proposition 7.5, the supremum of each φ_i is attained on the finite set of divisorial valuations associated to vertices of $\mathcal{D}_X$. Hence

$$\sup_{\Delta_{\mathcal{X}}}(\varphi_i \circ \mathrm{emb}_{\mathcal{X}}) \ge \sup_{\Delta_{\mathcal{X}}}(\varphi \circ \mathrm{emb}_{\mathcal{X}}) > -\infty$$

for all i. It then follows from Corollary 7.8 that $\varphi_i \circ \mathrm{emb}_{\mathcal{X}}$ converges uniformly to $\varphi \circ \mathrm{emb}_{\mathcal{X}}$ and that $\varphi \circ \mathrm{emb}_{\mathcal{X}}$ is continuous. □

7.3. Compactness. We endow the set $\mathrm{PSH}(X, \theta)$ of all θ-psh functions with the topology of uniform convergence on dual complexes. A basis of open neighborhoods of a fixed θ-psh function φ_0 is then given by

$$\{\varphi \mid \sup_{\Delta_{\mathcal{X}}} |\varphi - \varphi_0| \le \varepsilon\}$$

where $\mathcal{X}$ ranges over SNC models on which θ is determined and where $\varepsilon > 0$. Thanks to Proposition 7.6, the natural map

$$\mathrm{PSH}(X, \theta) \to \prod_{\mathcal{X}} C^0(\Delta_{\mathcal{X}})$$

is a homeomorphism onto its image. Note also that $\mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ is dense in $\mathrm{PSH}(X, \theta)$ by definition. The following result is a more precise version of Theorem A.

Theorem 7.10. *If $\psi \in \mathcal{D}(X)$ is any model function, then the map* $\mathrm{PSH}(X, \theta) \to \mathbf{R}$ *defined by* $\varphi \mapsto \sup_X(\varphi - \psi)$ *is continuous and proper. Hence* $\mathrm{PSH}(X, \theta)/\mathbf{R}$ *is compact. Furthermore, the topology on* $\mathrm{PSH}(X, \theta)$ *is equivalent to the topology of pointwise convergence on either* X^{qm} *or* X^{div}.

Proof. Let $\mathcal{X}$ be an SNC model on which θ and ψ are determined. Then $\psi = \psi \circ p_{\mathcal{X}}$, so Proposition 7.5 implies that $(\varphi - \psi) \circ p_{\mathcal{X}}$ is continuous and that $(\varphi - \psi) \circ p_{\mathcal{X}} \ge (\varphi - \psi)$ for any $\varphi \in \mathrm{PSH}(X, \theta)$. Hence the supremum of $\varphi - \psi$ is attained on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$, which implies the continuity of $\varphi \mapsto \sup_X(\varphi - \psi)$.

To prove properness, we need to show that

$$\mathcal{F}_C := \{\varphi \in \mathrm{PSH}(X, \theta) \mid |\sup_X(\varphi - \psi)| \le C\}$$

is compact for any $C > 0$. Recall that $\mathrm{PSH}(X, \theta)$ embeds in $\prod_{\mathcal{X}} C^0(\Delta_{\mathcal{X}})$. By Tychonoff's theorem, the compactness of $\mathcal{F}_C$ is therefore equivalent to the compactness in $C^0(\Delta_{\mathcal{X}})$ of the closure of the image of $\mathcal{F}_C$ in $C^0(\Delta_{\mathcal{X}})$, for each SNC model $\mathcal{X}$ on which θ and ψ are determined. But this is a direct consequence of Corollary 7.8 and Ascoli's theorem.

Picking $\psi = 0$, we see that $\varphi \mapsto \sup_X \varphi$ is proper, which implies the compactness of $\mathrm{PSH}(X, \theta)$.

For the last statement, it is clear that convergence in $\mathrm{PSH}(X, \theta)$ implies pointwise convergence on X^{qm} which in turn implies pointwise convergence on X^{div}. Now let $(\varphi_\alpha)_{\alpha\in A}$ be a net of θ-psh functions converging pointwise to $\varphi \in \mathrm{PSH}(X, \theta)$ on X^{div}. Fix any SNC model $\mathcal{X}$ on which θ is determined. We must show that φ_α converges uniformly to φ on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. But $X^{\mathrm{div}} \cap \mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ is dense in $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ by Corollary 3.17, so this follows from the equicontinuity statement in Corollary 7.8. □

7.4. Upper envelopes. Finally we shall prove the following result, whose complex analogue serves as a basic ingredient of pluripotential theory. While we will not go deeper into pluripotential theory here, we will use the result below in §8. The proof will in fact not use the compactness result in §7.3.

Theorem 7.11. *Let $(\varphi_\alpha)_{\alpha\in A}$ be an arbitrary family of θ-psh functions on X, and assume that (φ_α) is uniformly bounded from above. If we set $\varphi(x) := \sup_{\alpha\in A} \varphi_\alpha(x)$ for each $x \in X$, then the usc regularization φ^* of φ is θ-psh and coincides with φ on $X^{\mathrm{qm}} = \bigcup_{\mathcal{X}} \mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. Further, we have $\varphi^* = \lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} = \inf_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}$.*

Recall that the *usc regularization* of a function u on a topological space X is the smallest usc function $u^* \geq u$, given by $u^*(x) = \limsup_{y\to x} u(y)$.

Proof of Theorem 7.11. Upon considering the new family $\varphi_I = \max_{\alpha\in I} \varphi_\alpha$ with I ranging over all finite subsets of A, we may assume that A is a directed set and (φ_α) is an increasing net. We must show that conditions (i)–(ii) of Definition 7.1 hold for φ^*.

For each SNC model $\mathcal{X}$ on which θ is determined, $\varphi_\alpha \circ \mathrm{emb}_{\mathcal{X}}$ is convex and continuous on $\Delta_{\mathcal{X}}$ for each α. The increasing limit $\varphi \circ \mathrm{emb}_{\mathcal{X}} = \lim_\alpha \varphi_\alpha \circ \mathrm{emb}_{\mathcal{X}}$ is therefore convex and lsc on $\Delta_{\mathcal{X}}$. On the other hand, any convex function on a convex polytope is usc; see [GKR68]. Thus $\varphi \circ \mathrm{emb}_{\mathcal{X}}$ is continuous, and, by Dini, $\varphi_\alpha \circ \mathrm{emb}_{\mathcal{X}}$ converges uniformly on $\Delta_{\mathcal{X}}$ to $\varphi \circ \mathrm{emb}_{\mathcal{X}}$. Equivalently, $\varphi \circ p_{\mathcal{X}}$ is continuous and $\varphi_\alpha \circ p_{\mathcal{X}}$ converges uniformly to $\varphi \circ p_{\mathcal{X}}$ on X.

Since $\mathcal{X} \mapsto \varphi_\alpha \circ p_{\mathcal{X}}$ is decreasing, the same is true for $\mathcal{X} \mapsto \varphi \circ p_{\mathcal{X}}$. We claim that $\lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} = \varphi^*$. To see this, note that $\psi := \lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}$ is a decreasing limit of continuous functions and hence usc. On the one hand, $\varphi \circ p_{\mathcal{X}} \geq \varphi$ for all $\mathcal{X}$ implies $\psi \geq \varphi$, so that $\psi \geq \varphi^*$ by the definition of the usc regularization. On the other hand, Corollary 3.9 and the upper semicontinuity of φ^* imply

$$\varphi^* \geq \varlimsup_{\mathcal{X}} \varphi^* \circ p_{\mathcal{X}} \geq \varlimsup_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} = \psi,$$

so that $\varphi^* = \lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} = \inf_{\mathcal{X}} \varphi \circ p_{\mathcal{X}}$, hence the claim.

If $\mathcal{Y} \leq \mathcal{X}$ are SNC models on which θ is determined, then $p_{\mathcal{X}} \circ p_{\mathcal{Y}} = p_{\mathcal{Y}}$ by Proposition 3.6(iv). By what precedes, this implies $\varphi^* \circ p_{\mathcal{Y}} = \lim_{\mathcal{X}} \varphi \circ p_{\mathcal{X}} \circ p_{\mathcal{Y}} = \varphi \circ p_{\mathcal{Y}}$. In particular, $\varphi^* = \varphi$ on $\mathrm{emb}_{\mathcal{Y}}(\Delta_{\mathcal{Y}})$. Since $\mathcal{Y}$ was arbitrary, we get $\varphi^* = \varphi$ on X^{qm}.

It remains to prove that φ^* is θ-psh in the sense of Definition 7.1. First, $\varphi_\alpha \leq \varphi_\alpha \circ p_{\mathcal{X}}$ for all α implies $\varphi \leq \varphi \circ p_{\mathcal{X}}$, which gives $\varphi^* \leq (\varphi \circ p_{\mathcal{X}})^* = \varphi \circ p_{\mathcal{X}} = \varphi^* \circ p_{\mathcal{X}}$, where the first equality follows from the continuity of $\varphi \circ p_{\mathcal{X}}$. Thus (i) holds. Second, the restriction of each φ_α to $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ is, by assumption, a uniform limit of θ-psh model functions. Since φ_α converges uniformly to $\varphi = \varphi^*$ on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$, we see that the restriction of φ^* to $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ is also a uniform limit of θ-psh model functions. Thus (ii) also holds and φ^* is θ-psh. □

8. Envelopes and regularization

We continue to assume that $\theta \in \mathcal{Z}^{1,1}(X)$ is a closed $(1,1)$-form whose de Rham class $\{\theta\} \in N^1(X)$ is ample. This positivity property will be crucial in the arguments to follow.

8.1. Regularity of envelopes. As a tool to prove our regularization theorem, we rely on the following envelope construction, whose complex analogue is widely used.

Definition 8.1. The *θ-psh envelope* $P_\theta(u)$ of a continuous function $u \in C^0(X)$ is defined by setting, for each $x \in X$,

$$P_\theta(u)(x) := \sup \{\varphi(x) \mid \varphi \in \mathrm{PSH}(X, \theta),\ \varphi \leq u \text{ on } X\}.$$

Here are a few easy properties of the envelope operator.

Proposition 8.2. *Let $u, u' \in C^0(X)$.*

(i) *$P_\theta(u)$ is θ-psh and is the largest θ-psh function dominated by u on X.*
(ii) *P_θ is non-decreasing, i.e. $u \leq v \Rightarrow P_\theta(u) \leq P_\theta(v)$.*
(iii) *$P_\theta(u)$ is concave in both arguments in the sense that*

$$P_{t\theta+(1-t)\theta'}(tu + (1-t)u') \geq tP_\theta(u) + (1-t)P'_\theta(u')$$

for $0 \leq t \leq 1$.
(iv) *For each $c \in \mathbf{R}$ we have $P_\theta(u + c) = P_\theta(u) + c$.*
(v) *For each $v \in \mathcal{D}(X)$ we have $P_\theta(u) = P_{\theta+dd^c v}(u - v) + v$.*
(vi) *P_θ is 1-Lipschitz continuous with respect to the sup-norm, i.e. $\sup_X |P_\theta(u) - P_\theta(v)| \leq \sup_X |u - v|$.*
(vii) *Given a determination $\mathcal{X}$ of θ and a convergent sequence $\theta_m \to \theta$ in $N^1(\mathcal{X}/S)$, we have $P_{\theta_m}(u) \to P_\theta(u)$ uniformly on X.*

Proof. (i) The only thing to show is that $P_\theta(u)$ is θ-psh. Since $P_\theta(u) \le u$ and u is continuous, it follows that the usc regularization satisfies $P_\theta(u)^* \le u$. Now, $P_\theta(u)^*$ is θ-psh by Theorem 7.11, and is hence a competitor in the definition of $P_\theta(u)$. Thus $P_\theta(u) = P_\theta(u)^*$ is indeed θ-psh.

(ii) is trivial.

(iii) follows from the fact that given $\varphi \in \mathrm{PSH}(X, \theta)$, $\varphi' \in \mathrm{PSH}(X, \theta')$ with $\varphi \le u$ and $\varphi' \le u'$, $t\varphi + (1-t)\varphi'$ belongs to $\mathrm{PSH}(X, t\theta + (1-t)\theta')$ and is dominated by $tu + (1-t)u'$.

(iv) and (v) are seen similarly.

(vi) is a formal consequence of (ii) and (iv).

(vii) By Proposition 5.2 we may assume, after perhaps passing to a higher model, that there exists a model function v determined on $\mathcal{X}$ such that $\theta + dd^c v$ is $\mathcal{X}$-positive, i.e. determined by an ample class in $N^1(\mathcal{X}/S)$. As a consequence, there exists an open neighborhood $V \subset N^1(\mathcal{X}/S)$ of θ such that $\tau + dd^c v$ is $\mathcal{X}$-positive for all $\tau \in V$.

We claim that $P_\tau(u)$ is uniformly bounded on X for $\tau \in V$. Indeed for each $\tau \in V$ we have $\mathbf{R} \subset \mathrm{PSH}(X, \tau + dd^c v)$; hence $P_{\tau + dd^c v}(u - v) \ge \inf_X u - \sup_X |v|$. By (v) it follows that

$$\inf_X u - 2 \sup_X |v| \le P_\tau(u) \le \sup_X u,$$

which proves the claim.

Now for each $x \in X$ the function $\tau \mapsto P_\tau(u)(x)$ is concave on V, hence locally Lipschitz continuous on V, with local Lipschitz constant only depending on $\sup_{\tau \in V} |P_\tau(u)(x)|$, which is in turn bounded independently of $x \in X$, and the result follows. $\square$

Our main result in this section is the following regularity property of envelopes. As we shall see, it is in fact equivalent to the monotone regularization theorem.

Theorem 8.3. *For any $u \in C^0(X)$ the θ-psh envelope $P_\theta(u)$ is a uniform limit on X of θ-psh model functions. In particular, $P_\theta(u)$ is continuous.*

Before attacking Theorem 8.3 we shall prove the following weaker statement.

Lemma 8.4. *Let $\widetilde{P}_\theta(u)$ be the pointwise supremum of all θ-psh* model *functions φ such that $\varphi \le u$. Then $\widetilde{P}_\theta(u) \le P_\theta(u)$ and equality holds on X^{qm}.*

Proof. The inequality $\widetilde{P}_\theta(u) \le P_\theta(u)$ is trivial. To prove that equality holds on X^{qm}, pick $\varepsilon > 0$ and $x \in \mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ for some SNC model $\mathcal{X}$ on which θ is determined. By construction, there exists $\psi \in \mathrm{PSH}(X, \theta)$ such that $\psi \le u$ and $\psi(x) \ge P_\theta(u)(x) - \varepsilon$. By the definition of $\mathrm{PSH}(X, \theta)$, there then exists a θ-psh model function φ such that $|\varphi - (\psi - \varepsilon)| \le \varepsilon$ on $\mathrm{emb}_{\mathcal{X}}(\Delta_{\mathcal{X}})$. Thus

$\varphi \le \varphi \circ p_{\mathcal{X}} \le \psi \circ p_{\mathcal{X}} \le u$ on X and $\varphi(x) \ge \psi(x) - 2\varepsilon \ge P_\theta(u)(x) - 3\varepsilon$. We conclude that $\widetilde{P}_\theta(u) = P_\theta(u)$ on X^{qm}. □

Proof of Theorem 8.3. We shall reduce the statement to a geometric assertion that can be proved using asymptotic multiplier ideals.

First, we may assume that $u \in \mathcal{D}(X)$, thanks to Corollary 2.3 and (vi) of Proposition 8.2.

Second, we can reduce to the case when $\theta \in N^1(\mathcal{X}/S)_{\mathbf{Q}}$ is a rational class, using (vii) of Proposition 8.2.

Third, we may further reduce to the case $u = 0$, after replacing θ with $\theta + dd^c u$, using (v) of Proposition 8.2.

After scaling, we may finally assume that θ is the curvature form of a model metric determined by a line bundle $\mathcal{L}$ on some model $\mathcal{X}$. Now we conclude the proof using the following result. □

Theorem 8.5. *Let L be an ample line bundle on X and $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ an extension of L to an SNC model $\mathcal{X}$. Let $\theta \in \mathcal{Z}^{1,1}(X)$ be the curvature form of the corresponding model metric on L. For $m \gg 1$ let $\mathfrak{a}_m \subset \mathcal{O}_{\mathcal{X}}$ be the (vertical) base-ideal of $m\mathcal{L}$ and set $\varphi_m := \frac{1}{m}\log|\mathfrak{a}_m|$. Then φ_m is a θ-psh model function and $\varphi_m \to P_\theta(0)$ uniformly on X as $m \to \infty$.*

Proof of Theorem 8.5. For $m \gg 1$, $m\mathcal{L}|_{\mathcal{X}_K}$ is globally generated, which shows that the ideal sheaf $\mathfrak{a}_m$ is vertical. Since $\mathcal{O}_{\mathcal{X}}(m\mathcal{L}) \otimes \mathfrak{a}_m$ is globally generated by the definition of $\mathfrak{a}_m$, it follows that $\varphi_m \in \mathcal{D}(X)$ is θ-psh by Lemma 5.6. Note that $\mathfrak{a}_m \cdot \mathfrak{a}_l \subset \mathfrak{a}_{m+l}$ for all m, l. This yields the superadditivity property $m\varphi_m + l\varphi_l \le (m+l)\varphi_{m+l}$. As a consequence, the pointwise limit $\lim_m \varphi_m$ exists and coincides with $\sup_m \varphi_m$.

Step 1. Let us first prove that $P_\theta(0) = \sup_m \varphi_m$ on X^{qm}. The argument is similar to Step 2 of the proof of Theorem 5.11. Since φ_m is θ-psh and $\varphi_m \le 0$ for all m, we have $\sup_m \varphi_m \le P_\theta(0)$ on X. To see that equality holds on X^{qm}, pick $\varepsilon > 0$ and $x \in \mathrm{emb}_{\mathcal{X}'}(\Delta_{\mathcal{X}'})$ for some SNC model $\mathcal{X}'$ dominating $\mathcal{X}$. By Lemma 8.4 there exists a θ-psh *model function* φ such that $\varphi \le 0$ and $\varphi(x) \ge P_\theta(0)(x) - \varepsilon$. Replacing $\mathcal{X}'$ by a higher model, we may assume that $\varphi = \varphi_D$ is determined by some divisor $D \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$. Invoking Proposition 5.2 we may also assume that there exists $D' \in \mathrm{Div}_0(\mathcal{X}')_{\mathbf{Q}}$ with $-\varepsilon \le \varphi_{D'} \le 0$ on X and $\pi^*\mathcal{L} + D + D'$ ample. Since $D + D' \le 0$ we then have

$$\mathcal{O}_{\mathcal{X}'}(m\pi^*\mathcal{L} + m(D + D')) \subset \mathcal{O}_{\mathcal{X}'}(m\pi^*\mathcal{L}).$$

Now the left-hand side is globally generated for some m, and we conclude that

$$\mathcal{O}_{\mathcal{X}'}(m(D + D')) \subset \mathcal{O}_{\mathcal{X}'} \cdot \mathfrak{a}_m;$$

hence

$$P_\theta(0)(x) \le \varphi_D(x) + \varepsilon \le \varphi_{D+D'}(x) + 2\varepsilon \le \frac{1}{m}\log|\mathfrak{a}_m|(x) + 2\varepsilon \le \sup_l \varphi_l(x) + 2\varepsilon.$$

Step 2. Introduce, for each $m \in \mathbf{N}$, the asymptotic multiplier ideal $\mathfrak{b}_m := \mathcal{J}(\mathfrak{a}_\bullet^m) \subset \mathcal{O}_\mathcal{X}$ associated to the graded sequence $\mathfrak{a}_\bullet$. We refer to Appendix B for the definition and the proof of the fundamental properties of multiplier ideals in our present setting. We shall use the following results. First, we have the elementary inclusion $\mathfrak{a}_m \subset \mathfrak{b}_m$ for all m. Second, the subadditivity property (cf. Theorem B.7) implies $\mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for any l, m. We infer that $\mathfrak{a}_{ml} \subset \mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for any m, l and hence

$$\frac{1}{m} \log |\mathfrak{b}_m| \geq \sup_l \frac{1}{ml} \log |\mathfrak{a}_{ml}| = \sup_l \varphi_{ml} = P_\theta(0) \tag{8.1}$$

on X^{qm} for all m, where the last equality follows from Step 1.

Since both $P_\theta(0)$ and φ_m remain unchanged when $\mathcal{X}$ is replaced with a higher model, we may assume that there exists an effective divisor $E \in \mathrm{Div}_0(\mathcal{X})_\mathbf{Q}$ such that $\mathcal{A} := \mathcal{L} - E$ is ample on $\mathcal{X}$. By the uniform global generation property of multiplier ideals (Theorem B.8 and Remark B.9) we may then choose $m_0 \in \mathbf{N}$ such that $m_0\mathcal{A}$ is ample enough to guarantee that $\mathcal{O}_\mathcal{X}(m\mathcal{L} + m_0\mathcal{A}) \otimes \mathfrak{b}_m$ is globally generated for all m. Since $\mathcal{O}_\mathcal{X}(m\mathcal{L} + m_0\mathcal{A})$ injects in $\mathcal{O}_\mathcal{X}((m + m_0)\mathcal{L})$ by multiplying with the canonical section of $\mathcal{O}_\mathcal{X}(m_0 E)$, it follows that

$$\log |\mathfrak{b}_m| \leq \log |\mathfrak{a}_{m+m_0}| + m_0\varphi_E.$$

Replacing m with $m - m_0$ and using (8.1) we infer $(m - m_0)P_\theta(0) \leq m\varphi_m + m_0\varphi_E$, so that

$$\varphi_m \leq P_\theta(0) \leq \frac{m}{m-m_0}\varphi_m + \frac{m_0}{m-m_0}\varphi_E$$

on X^{qm} for $m \gg 1$. As φ_m, $P_\theta(0)$ and φ_E are all θ-psh, Proposition 7.6 shows that this inequality extends to all of X. Now φ_E is bounded and φ_m is uniformly bounded, as follows from $\varphi_1 \leq \varphi_m \leq 0$, so φ_m converges uniformly on X to $P_\theta(0)$, as was to be shown. □

Let us end this subsection with a result that will be used in [BFJ12a].

Corollary 8.6. *If $\varphi \in \mathrm{PSH}(X, \theta)$ and $v \in C^0(X)$ are such that $\varphi \leq v$, then for every $\varepsilon > 0$ there exists a θ-psh model function ψ such that $\varphi \leq \psi \leq v + \varepsilon$.*

Proof. We may assume $\varphi = P_\theta(v)$, in which case the result follows from Theorem 8.3. □

8.2. Monotone regularization of θ-psh functions. By our definition, the set $\mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ of θ-psh model functions is dense in $\mathrm{PSH}(X, \theta)$ with respect to its topology of uniform convergence on dual complexes. This property may be seen as an analogue of the fact that every θ-psh function is a L^1-limit of smooth θ-psh functions in the complex case, which follows from the much more useful fact that every θ-psh function is a *decreasing* limit of smooth θ-psh functions [Dem92]. The next result gives an analogue of this monotone regularization theorem in our context.

Theorem 8.7. *For each θ-psh function φ, there exists a decreasing net $(\varphi_i)_{i\in I}$ of θ-psh model functions that converges pointwise on X to φ.*

One may hope that there is in fact a decreasing *sequence* $(\varphi_m)_{m=1}^{\infty}$ of θ-psh model functions converging to φ. In the companion paper [BFJ12a], we will prove, using Theorem 8.7 and capacity estimates, that this is indeed the case.

As a consequence of Theorem 8.7, we get at any rate the following version of the Demailly-Richberg regularization theorem.

Corollary 8.8. *Every continuous θ-psh function φ is the uniform limit on X of a sequence $(\varphi_m)_{m\in\mathbf{N}}$ of θ-psh model functions.*

Proof. By Theorem 8.7 there exists a decreasing net (ψ_j) of θ-psh model functions converging pointwise to φ. For each $\varepsilon > 0$ the compact set X is the increasing union of the open sets $\{\psi_j < \varphi + \varepsilon\}$; hence $\psi_j < \varphi + \varepsilon$ for some j (Dini's lemma). It follows that φ lies in the closure of $\mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ in $C^0(X)$ with respect to the topology of uniform convergence. Since the latter is defined by a norm, the result follows. □

The proof of Theorem 8.7 reduces immediately to Theorem 8.3, in view of the following elementary result.

Lemma 8.9. *The following properties are equivalent.*

(i) *Every θ-psh function φ is the pointwise limit of a decreasing net of θ-psh model functions.*

(ii) *For each $u \in C^0(X)$ we have*

$$P_\theta(u) = \sup \{\varphi \mid \varphi \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta),\ \varphi \le u \text{ on } X\}.$$

(iii) *For each $u \in C^0(X)$, $P_\theta(u)$ is a uniform limit of θ-psh model functions.*

Proof. (i)$\Longrightarrow$(ii). Let $u \in C^0(X)$. By (i) there exists a decreasing net (φ_j) of θ-psh model functions converging pointwise to $P_\theta(u)$. Since $P_\theta(u) \le u$, we see that the compact set X is for each $\varepsilon > 0$ the increasing union of the open sets $\{\varphi_j < u + \varepsilon\}$; hence $\varphi_j < u + \varepsilon$ for some j. Since $\varphi_j - \varepsilon$ is θ-psh and dominated by u, we get $\varphi_j - \varepsilon \le P_\theta(u)$ by the definition of the envelope, which proves (ii).

(ii)$\Longrightarrow$(iii). Since the set of $\varphi \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ such that $\varphi \le u$ is stable by max, (ii) shows that we can construct an increasing family $\varphi_j \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ converging pointwise to $P_\theta(u)$. But $P_\theta(u) - \varphi_j$ is usc for each j, and Dini's lemma therefore shows that the convergence is uniform on X.

(iii)$\Longrightarrow$(i). Let φ be a θ-psh function. We first claim that for each $x \in X$ we have

$$\varphi(x) = \inf \{\psi(x) \mid \psi \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta), \psi \ge \varphi\}. \tag{8.2}$$

Indeed, given $\varepsilon > 0$ there exists $u \in C^0(X)$ such that $u \geq \varphi$ and $u(x) \leq \varphi(x) + \varepsilon$, simply because φ is usc. Since φ is θ-psh, the maximality property of envelopes implies $\varphi \leq P_\theta(u)$. By (iii) we may then find $\psi \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ such that $P_\theta(u) \leq \psi \leq P_\theta(u) + \varepsilon$. We thus have $\psi \geq \varphi$ and $\psi(x) \leq \varphi(x) + 2\varepsilon$, and the claim follows.

Now consider the set I of all $\psi \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ such that $\psi > \varphi$ on X. Note that the latter condition implies that $\psi \geq \varphi + \varepsilon$ for some $\varepsilon > 0$ since $\varphi - \psi$ is usc. We claim that I is a directed set, which will conclude the proof, since

$$\varphi = \inf_{\psi \in I} \psi = \lim_{\psi \in I} \psi$$

pointwise on X, thanks to (8.2). To get the claim, let $\psi_1, \psi_2 \in I$ and choose $\varepsilon > 0$ such that $\min\{\psi_1, \psi_2\} \geq \varphi + 3\varepsilon$. We then also have $P_\theta(\min\{\psi_1, \psi_2\}) \geq \varphi + 3\varepsilon$. By (iii) we find $\psi_3 \in \mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ such that $|\psi_3 - P_\theta(\min\{\psi_1, \psi_2\})| \leq \varepsilon$. Then $\varphi + \varepsilon \leq \psi_3 - \varepsilon \leq \min\{\psi_1, \psi_2\}$, which concludes the proof. $\square$

Appendix A. Lipschitz constants of convex functions

Let V be a finite-dimensional real vector space and let $K \subset V$ be a convex body, i.e. a compact convex set with non-empty interior. Denote by $\mathcal{E}(K)$ the set of extremal points of K. Given a norm $\|\cdot\|$ on V the Lipschitz constant of a continuous function $\varphi : K \to \mathbf{R}$ is defined as usual as

$$\mathrm{Lip}_K(\varphi) := \sup_{v \neq v'} \frac{|\varphi(v) - \varphi(v')|}{\|v - v'\|} \in [0, +\infty]$$

and its $C^{0,1}$-norm is then

$$\|\varphi\|_{C^{0,1}(K)} := \|\varphi\|_{C^0(K)} + \mathrm{Lip}_K(\varphi).$$

This quantity of course depends on the choice of $\|\cdot\|$, but since all norms on V are equivalent, choosing another norm only affects the estimates to follow by an overall multiplicative constant.

Let $\varphi : K \to \mathbf{R}$ be a continuous convex function. Our goal is to estimate the $C^{0,1}$-norm of φ on K in terms of $\|\varphi\|_{C^0(\partial K)}$ and certain directional derivatives of φ at boundary points. Let us first introduce some notation. First, for $v, w \in K$ we define the directional derivative of φ at v towards w as

$$D_v\varphi(w) := \left.\frac{d}{dt}\right|_{t=0_+} \varphi((1-t)v + tw); \tag{A.1}$$

this limit exists by convexity of φ. Second, given a point $e \in K$ we define a projection $\pi_e : K \setminus \{e\} \to \partial K$ by setting

$$t_e(v) := \sup\{t \in \mathbf{R},\, e + t(v-e) \in K\}$$

and

$$\pi_e(v) := e + t_e(v)(v-e),$$

so that $\pi_e(v) \in \partial K$ is the unique point such that $v \in [e, \pi_e(v)]$.

Proposition A.1. *There exists $C > 0$ such that every Lipschitz continuous convex function $\varphi : K \to \mathbf{R}$ satisfies*

$$C^{-1}\|\varphi\|_{C^{0,1}(K)} \le \|\varphi\|_{C^0(\partial K)} + \sup_{e\in\mathcal{E}(K), v\in\mathrm{int}(K)} \left|D_{\pi_e(v)}\varphi(e)\right| \le C\,\|\varphi\|_{C^{0,1}(K)}.$$

Proof. The right-hand inequality is clear, so we focus on the left-hand one. Given $v \in \mathrm{int}(K)$ and $e \in \mathcal{E}(K)$ we may write $v = \pi_e(v) + t_0\,(e - \pi_e(v))$ for some $0 < t_0 < 1$. Consider the restriction of φ to the segment $[\pi_e(v), e]$; i.e. set $\theta(t) := \varphi\left(\pi_e(v) + t(e - \pi_e(v))\right)$, $t \in [0,1]$. If we denote by $\theta'(t)$ the right-derivative of θ at t, then the convexity of θ yields

$$\theta'(0) \le \left(\theta(t_0) - \theta(0)\right)/t_0 \tag{A.2}$$

and

$$\theta'(0) \le \theta'(t_0) \le \left(\theta(1) - \theta(t_0)\right)/(1-t_0). \tag{A.3}$$

Now, by definition, $\theta'(0) = D_{\pi_e(v)}\varphi(e)$, $t_0\theta'(0) = D_{\pi_e(v)}\varphi(v)$, and $(1-t_0)\theta'(t_0) = D_v\varphi(e)$, so that (A.2) reads

$$t_0 D_{\pi_e(v)}\varphi(e) \le \varphi(v) - \varphi(\pi_e(v)).$$

Since we also have $\sup_K \varphi = \sup_{\partial K} \varphi$ by convexity, this shows that

$$\|\varphi\|_{C^0(K)} \le \|\varphi\|_{C^0(\partial K)} + \sup_{e\in\mathcal{E}(K), v\in\mathrm{int}(K)} \left|D_{\pi_e(v)}\varphi(e)\right|.$$

On the other hand, (A.2) combined with (A.3) yields

$$(1-t_0)D_{\pi_e(v)}\varphi(e) \le D_v\varphi(e) \le \varphi(e) - \varphi\left(\pi_e(v)\right) - t_0 D_{\pi_e(v)}\varphi(e)$$

and we conclude by Lemma A.2 below. $\square$

Lemma A.2. *There exists a constant $C > 0$ such that every Lipschitz continuous function $\varphi : K \to \mathbf{R}$ satisfies*

$$C^{-1}\,\mathrm{Lip}_K(\varphi) \le \sup_{e\in\mathcal{E}(K), v\in A} |D_v\varphi(e)| \le C\,\mathrm{Lip}_K(\varphi)$$

where $A \subset \mathring{K}$ denotes the set of points at which φ is differentiable.

Proof. It is clear that $|D_v\varphi(e)| \le \operatorname{diam}(K)\operatorname{Lip}_K(\varphi)$ for all e, v. Conversely it is a standard consequence of Rademacher's theorem that $\operatorname{Lip}_K(\varphi) = \sup_{v\in A}\|\nabla\varphi(v)\|$. For each $v \in A$ we also have $D_v\varphi(e) = \langle\nabla\varphi(v), e - v\rangle$. We now claim that there exists $C > 0$ such that

$$\|\lambda\| \le C \sup_{e\in\mathcal{E}(K)} |\langle\lambda, v - e\rangle|$$

for all $\lambda \in V^*$ and all $v \in K$, which will conclude the proof. Indeed the supremum in the right-hand side is a lower semicontinuous function of $(\lambda, v) \in V^* \times K$. As a consequence it achieves its infimum on the compact set $\{\lambda \in V^*,\ \|\lambda\| = 1\} \times K$, and this infimum cannot be zero since $\{v - e,\ e \in \mathcal{E}(K)\}$ spans V for each $v \in K$. The claim follows by homogeneity. □

Appendix B. Multiplier ideals on S-varieties

The purpose of this section is to define multiplier ideals on regular S-varieties and establish their basic properties. We are grateful to Osamu Fujino, János Kollár and Mircea Mustaţă for their helpful suggestions.

It should be noted that most results in this appendix are also obtained, with simpler arguments and in a slightly more general setting, in [MN12], which appeared after a first version of the present paper was completed. We felt, however, that the alternative arguments presented here might still be of some interest to the reader.

In this appendix, and as opposed to the main body of the article, it will be more convenient to use multiplicative notation for Picard groups. We also fix the choice of an isomorphism $R \simeq k[\![t]\!]$.

B.1. Kodaira vanishing. The usual compactification argument that reduces the relative version of Kodaira (or Kawamata-Viehweg) vanishing to its global projective version over k cannot be applied for S-varieties. Following suggestions of János Kollár and Mircea Mustaţă, we rely instead on a Kodaira type vanishing theorem on the (possibly reducible) special fiber.

Theorem B.1 (Kodaira vanishing)**.** *Let $\mathcal{X}$ be an SNC S-variety, denote by $\omega_{\mathcal{X}}$ its dualizing sheaf, and let $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ be an ample line bundle. Then we have*

$$H^q(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L}) = 0 \quad \textit{for all } q \ge 1.$$

Proof. By flat base change we may assume that k is algebraically closed. The desired result is equivalent to $R^q f_*(\omega_{\mathcal{X}} \otimes \mathcal{L}) = 0$ for $q \ge 1$ since S is affine, and all fibers of $f : \mathcal{X} \to S$ are Cohen-Macaulay since $\mathcal{X}$ is in particular Cohen-Macaulay. We may therefore use relative duality for f, which shows that the desired result is equivalent to $R^q f_*\mathcal{L}^{-1} = 0$ for $q < n = \dim X$.

Let $d \in \mathbf{N}^*$ be a common multiple of the multiplicities of $\mathcal{X}_0$, set $S_d := \operatorname{Spec}[\![t^{1/d}]\!]$, and let $\mathcal{Y}$ be the normalization of $\mathcal{X} \times_S S_d$, with structure map $g : \mathcal{Y} \to S_d$. The pull-back $\mathcal{M}$ of $\mathcal{L}$ to $\mathcal{Y}$ is still ample since $\mathcal{Y} \to \mathcal{X}$ is finite. By [KKMS, pp. 200–201] the S_d-scheme $\mathcal{Y}$ is toroidal and its special fiber $\mathcal{Y}_0$ is *reduced*.

The relative trace $\operatorname{Tr}_{\mathcal{Y}/\mathcal{X}}$ shows that $R^q g_* \mathcal{M}^{-1}$ contains $R^q f_* \mathcal{L}^{-1}$ as a direct summand, and it is therefore enough to show by semicontinuity that $H^q(\mathcal{Y}_0, \mathcal{M}^{-1}) = 0$ for $q < n$. Like any toroidal S-scheme, $\mathcal{Y}$ is Cohen-Macaulay. As a consequence, the Cartier divisor $\mathcal{Y}_0$ is Cohen-Macaulay as well. By another application of duality, this time on $\mathcal{Y}_0$, we are reduced to showing that $H^q(\mathcal{Y}_0, \omega_{\mathcal{Y}_0} \otimes \mathcal{M}) = 0$ for $q \geq 1$.

By [KKMS] we may choose a toroidal vertical blowup $\pi : \mathcal{Y}' \to \mathcal{Y}$ such that $\mathcal{Y}'_0$ has simple normal crossing support (and hence $\mathcal{Y}'$ is regular). A toric computation (compare [Kol97, Proposition 3.7]) shows that

$$\omega_{\mathcal{Y}'} \otimes \mathcal{O}_{\mathcal{Y}'}(\mathcal{Y}'_{0,\mathrm{red}}) \simeq \pi^* \left(\omega_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_0)\right).$$

Since $\mathcal{Y}_0$ and $\mathcal{Y}'_{0,\mathrm{red}}$ are Cartier divisors on $\mathcal{Y}$ and $\mathcal{Y}'$, respectively, adjunction applies (see for instance [KM98, Proposition 5.73]) and we get $\omega_{\mathcal{Y}'_{0,\mathrm{red}}} \simeq \pi^* \omega_{\mathcal{Y}_0}$. On the other hand, the projective and reduced (but *a priori* reducible) k-scheme $\mathcal{Y}'_{0,\mathrm{red}}$ has *embedded* SNC singularities. It is indeed an SNC divisor in $\mathcal{Y}'$, and [Art69] implies the existence of an algebraic k-variety containing $\mathcal{Y}'_0$ as a divisor. Since $\pi : \mathcal{Y}'_{0,\mathrm{red}} \to \mathcal{Y}_0$ is projective and $\mathcal{M}$ is ample on $\mathcal{Y}_0$, we may therefore apply a vanishing theorem originally due to Kawamata and Ambro and corrected by Fujino ([Kaw85, Theorem 4.4], [Amb03, Theorem 3.2], and [Fuj09, Theorem 2.39]) to get that $\pi_* \left(\omega_{\mathcal{Y}'_{0,\mathrm{red}}} \otimes \pi^* \mathcal{M}\right)$ is acyclic on $\mathcal{Y}'$. Now we have $\pi_* \omega_{\mathcal{Y}'_{0,\mathrm{red}}} = \omega_{\mathcal{Y}_0}$; hence

$$\pi_* \left(\omega_{\mathcal{Y}'_{0,\mathrm{red}}} \otimes \pi^* \mathcal{M}\right) \simeq \omega_{\mathcal{Y}_0} \otimes \mathcal{M}$$

by the projection formula, and we conclude as desired that $\omega_{\mathcal{Y}_0} \otimes \mathcal{M}$ is acyclic. □

B.2. Kawamata-Viehweg vanishing. We next explain how to infer from Theorem B.1 a version of the Kawamata-Viehweg vanishing theorem on SNC models. We rely as usual on the "covering trick" and basically follow the proof of [KM98, Theorem 2.64] but provide some details for the convenience of the reader.

Lemma B.2 (Covering trick)**.** *Assume that k is algebraically closed. Let $\mathcal{X}$ be an SNC S-variety and denote by $(E_i)_{i \in I}$ the set of irreducible components of $\mathcal{X}_0$. Let also $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ and $m \in \mathbf{N}^*$. Then there exists an SNC S-scheme $\mathcal{X}'$ and a finite surjective morphism $\rho : \mathcal{X}' \to \mathcal{X}$ such that $\rho^* \mathcal{L}$ is divisible by*

m in $\mathrm{Pic}(\mathcal{X}')$ *and $\rho^* E_i$ is smooth over k (but possibly disconnected) for all $i \in I$.*

We emphasize that the generic fiber of $\mathcal{X}'$ is a finite cover of the generic fiber of $\mathcal{X}$.

Proof. Writing $\mathcal{L} = (\mathcal{A} + \mathcal{L}) - \mathcal{A}$ for some sufficiently ample $\mathcal{A} \in \mathrm{Pic}(\mathcal{X})$ reduces us to the case where $\mathcal{L}$ is very ample. We then get a closed embedding $i : \mathcal{X} \hookrightarrow \mathbf{P}^N_k \times_k S$ over S such that $\mathcal{L}$ coincides with the restriction of $\mathcal{O}(1)$. Let $\pi : \mathbf{P}^N_k \to \mathbf{P}^N_k$ be the morphism $[X_0 : \cdots : X_N] \mapsto [X_0^m : \cdots : X_N^m]$, which satisfies $\pi^* \mathcal{O}(1) = \mathcal{O}(m)$. For each $\sigma \in \mathrm{PGL}(N+1, k)$ set $i_\sigma := \sigma \circ i$ and consider $\mathcal{X}' := \mathcal{X} \times_{i_\sigma} \pi$ with the finite surjective morphism $\rho : \mathcal{X}' \to \mathcal{X}$, so that $\rho^* \mathcal{L} = \mathcal{O}(m)|_{\mathcal{X}'}$ is divisible by m in $\mathrm{Pic}(\mathcal{X}')$.

Applying Kleiman's Bertini type theorem (cf. [Har, III.10.8]) to the smooth k-varieties $E_J = \bigcap_{j \in J} E_j$ for all subsets $J \subset I$ shows that we may choose $\sigma \in \mathrm{PGL}(N+1, k)$ such that each $\rho^* E_i$ is smooth over k and $\sum_i \rho^* E_i$ has simple normal crossings. This implies in particular that $\mathcal{X}'$ is an SNC model. □

Theorem B.3 (Kawamata-Viehweg vanishing)**.** *Let $\mathcal{X}$ be an SNC model of X. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ be a line bundle whose restriction to the generic fiber X is ample and such that $\mathcal{L} - D$ is nef for some $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ with coefficients in $[0, 1[$. Then we have*

$$H^q(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L}) = 0 \quad \textit{for all } q \geq 1.$$

Proof. As in Theorem B.1 the desired result is equivalent to $H^q(\mathcal{X}, \mathcal{L}^{-1}) = 0$ for $q < n$ by relative duality. By flat base change we may assume that k is algebraically closed.

Step 1. Assume first that $\mathcal{L} - D$ is ample. Let $E_1, \dots, E_N$ be the components of $\mathcal{X}_0$ and set $a_i := \mathrm{ord}_{E_i} D$. Choose $m \in \mathbf{N}^*$ such that $b := m\, a_1 \in \mathbf{N}$. By Lemma B.2 there exists an SNC S-variety $\mathcal{X}'$ with a finite surjective morphism $\rho : \mathcal{X}' \to \mathcal{X}$ such that $\rho^* E_i$ is smooth (possibly disconnected) for each i, $\sum_i \rho^* E_i$ is SNC, and $\rho^* E_1$ is given as the zero divisor of a section $s \in H^0(\mathcal{X}', \mathcal{M}^m)$ for some $\mathcal{M} \in \mathrm{Pic}(\mathcal{X}')$. Note that $H^q(\mathcal{X}, \mathcal{L}^{-1})$ is a direct summand of $H^q(\mathcal{X}', \rho^* \mathcal{L}^{-1})$ thanks to the trace map. Now let

$$\mathcal{X}_1 := \mathrm{Spec}_{\mathcal{X}'} \left(\bigoplus_{0 \leq j < m} \mathcal{M}^{-j} \right)$$

be the cyclic cover associated with $s \in H^0(\mathcal{X}', \mathcal{M}^m)$, where $\bigoplus_{0 \leq j < m} \mathcal{M}^{-j}$ is endowed with the $\mathcal{O}_{\mathcal{X}'}$-algebra structure induced by s. By definition there is a finite surjective morphism $\tau : \mathcal{X}_1 \to \mathcal{X}'$ which satisfies

$$\tau_* \mathcal{O}_{\mathcal{X}_1} = \bigoplus_{0 \leq j < m} \mathcal{M}^{-j}.$$

If we set

$$\mathcal{L}_1 := \tau^* \left(\rho^* \mathcal{L} \otimes \mathcal{M}^{-b}\right)$$

we thus have

$$H^q \left(\mathcal{X}_1, \mathcal{L}_1^{-1}\right) \simeq \bigoplus_{0 \le j < m} H^q \left(\mathcal{X}', \rho^* \mathcal{L}^{-1} \otimes \mathcal{M}^{b-j}\right)$$

by Leray's spectral sequence (since the higher direct images vanish, τ being finite).

But $b/m = a_1$ is less than 1 by assumption, and we thus see that $H^q(\mathcal{X}_1, \mathcal{L}_1^{-1})$ contains $H^q(\mathcal{X}', \rho^* \mathcal{L}^{-1})$, hence also $H^q(\mathcal{X}, \mathcal{L}^{-1})$, as a direct summand.

Since $\rho^* E_i$ is smooth for each i and $\sum_{i=2}^N \rho^* E_i$ has normal crossings with $\operatorname{div}(s) = \rho^* E_1$, one sees as in [KM98, Claim 2.65] that $E_i^{(1)} := \tau^* \rho^* E_i$ is smooth for each i and $\mathcal{X}_{1,0}$ has SNC support, so that $\mathcal{X}_1$ is an SNC S-scheme. Finally $\mathcal{L}_1 - \sum_{i=2}^N a_i E_i^{(1)}$ is $\mathbf{Q}$-linearly equivalent to $\tau^* \rho^* (\mathcal{L} - D)$, hence is ample.

We now use Lemma B.2 to find $c_1 : \mathcal{X}_1' \to \mathcal{X}_1$ such that $c_1^* E_i^{(1)}$ is smooth for all i, $\sum_i c_1^* E_i^{(1)}$ is SNC, and $c_1^* E_2^{(1)}$ is divisible in $\operatorname{Pic}(\mathcal{X}_1')$ by the denominator of a_2. We then perform the same cyclic cover construction as above. Iterating the whole process finally yields an SNC S-variety $\mathcal{X}_N$ with an *ample* line bundle $\mathcal{L}_N$ such that $H^q(\mathcal{X}, \mathcal{L}^{-1})$ is a direct summand of $H^q(\mathcal{X}_N, \mathcal{L}_N^{-1})$, and we conclude by Theorem B.1.

Step 2. We now consider the general case where $\mathcal{L} - D$ is merely nef. Since $\mathcal{L}|_X$ is ample by assumption there exists a vertical blowup $\pi : \mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ SNC and a vertical π-exceptional effective $\mathbf{Q}$-divisor $E \in \operatorname{Div}_0(\mathcal{Y})_{\mathbf{Q}}$ such that $\pi^* \mathcal{L} - E$ is ample. This condition implies in particular that $-E$ is π-ample. If we fix $0 < \varepsilon \ll 1$ rational so that εE has coefficients < 1, then $\pi^* \mathcal{L} - \varepsilon E = (1 - \varepsilon) \pi^* \mathcal{L} + \varepsilon (\pi^* \mathcal{L} - E)$ is also ample since $\pi^* \mathcal{L}$ is nef, and we get

$$H^q(\mathcal{Y}, \omega_{\mathcal{Y}} \otimes \pi^* \mathcal{L}) = 0 \ \text{ for all } q \ge 1$$

by Step 1.

We are next going to show that $R^q \pi_* (\omega_{\mathcal{Y}} \otimes \pi^* \mathcal{L}) = 0$ for each $q \ge 1$. Since we have $\pi_* \omega_{\mathcal{Y}} = \omega_{\mathcal{X}}$ (the relative canonical bundle $K_{\mathcal{Y}/\mathcal{X}}$ is π-exceptional and effective since $\mathcal{X}$ is regular), the degeneration of the Leray spectral sequence of π will then yield as desired

$$H^q(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L}) \simeq H^q(\mathcal{Y}, \omega_{\mathcal{Y}} \otimes \pi^* \mathcal{L}) = 0$$

for $q \ge 1$. Let us now prove the claim. Given $q \ge 1$ choose $\mathcal{A} \in \operatorname{Pic}(\mathcal{X})$ sufficiently ample to guarantee that $\mathcal{A} \otimes R^q \pi_* (\omega_{\mathcal{Y}} \otimes \pi^* \mathcal{L})$ is globally generated on $\mathcal{X}$ and

$$H^p \left(\mathcal{X}, \mathcal{A} \otimes R^m \pi_* (\omega_{\mathcal{Y}} \otimes \pi^* \mathcal{L})\right) = 0 \ \text{ for all } p \ge 1 \text{ and } m \ge 0$$

(note that we are only imposing finitely many non-trivial conditions). The degeneration of the Leray spectral sequence yields

$$H^0\left(\mathcal{X}, \mathcal{A} \otimes R^q\pi_*(\omega_{\mathcal{Y}} \otimes \pi^*\mathcal{L})\right) \simeq H^q\left(\mathcal{Y}, \omega_{\mathcal{Y}} \otimes \pi^*(\mathcal{L} \otimes \mathcal{A})\right) = 0$$

for $q \geq 1$ by Step 1 again, since $\pi^*(\mathcal{L} \otimes \mathcal{A}) - \varepsilon E$ is also ample. It follows that $\mathcal{A} \otimes R^q\pi_*(\omega_{\mathcal{Y}} \otimes \pi^*\mathcal{L}) = 0$ by global generation, which proves the claim since $\mathcal{A}$ is invertible. □

B.3. Multiplier ideals. Let us first give the definition of multiplier ideals in our setting:

Definition B.4. Let $\mathcal{X}$ be a regular model and let $\mathfrak{a}$ be a vertical ideal sheaf on $\mathcal{X}$. For each rational number $c > 0$ the *multiplier ideal* of $\mathfrak{a}^c$ is the vertical ideal sheaf of $\mathcal{X}$ defined as

$$\mathcal{J}(\mathfrak{a}^c) := \pi_*\mathcal{O}_{\mathcal{X}'}\left(K_{\mathcal{X}'/\mathcal{X}} - \lfloor c\, D \rfloor\right)$$

where $\pi : \mathcal{X}' \to \mathcal{X}$ is a vertical blowup with $\mathcal{X}'$ SNC such that $\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'}$ is locally principal and $D \in \mathrm{Div}_0(\mathcal{X}')$ is the corresponding effective Cartier divisor.

This definition only depends on the model function $c \log|\mathfrak{a}|$ (cf. [JM12]) and would in fact make sense for an arbitrary non-positive model function $\varphi \in \mathcal{D}(X)$.

If $\mathfrak{a}_\bullet$ is a graded sequence of vertical ideals, then $\mathcal{J}(\mathfrak{a}_\bullet^c)$ is defined as the largest element of the family of coherent ideals $\mathcal{J}(\mathfrak{a}_m^{c/m})$, $m \geq 1$; see [Laz, Definition 11.1.5].

As a matter of terminology, if $\mathcal{L}$ is a line bundle on a model $\mathcal{X}$, $\mathfrak{a}$ is a vertical coherent ideal sheaf and $c > 0$, then we shall say that $\mathcal{L} \otimes \mathfrak{a}^c$ is nef if $\pi^*\mathcal{L} - cD$ is nef, where $\pi : \mathcal{X}' \to \mathcal{X}$ is the normalization of the blowup of $\mathcal{X}$ along $\mathfrak{a}$ and $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(-D)$. In other words, the model function $c \log|\mathfrak{a}|$ is required to be θ-psh, where θ is the curvature form of the model metric on L induced by $\mathcal{L}$.

Using Theorem B.3 we may follow the usual line of arguments to prove the following basic vanishing property of multiplier ideals:

Theorem B.5 (Nadel vanishing)**.** *Let $\mathcal{X}$ be a regular model of X and $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ a line bundle whose restriction to X is ample. If $\mathfrak{a}$ is a vertical coherent ideal sheaf on $\mathcal{X}$ and $c > 0$ is a rational number such that $\mathcal{L} \otimes \mathfrak{a}^c$ is nef, then we have*

$$H^q\left(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L} \otimes \mathcal{J}(\mathfrak{a}^c)\right) = 0 \quad \textit{for all } q \geq 1.$$

In particular, if $\mathfrak{a}_\bullet$ is a graded sequence of vertical coherent ideal sheaves on $\mathcal{X}$ such that $\mathcal{L}^m \otimes \mathfrak{a}_m$ is globally generated for all sufficiently divisible m, then

$$H^q\left(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L} \otimes \mathcal{J}(\mathfrak{a}_\bullet)\right) = 0 \quad \textit{for all } q \geq 1.$$

Proof. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be an SNC model dominating the blowup of $\mathcal{X}$ along $\mathfrak{a}$, so that we have $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(-D)$ for some effective divisor $D \in \mathrm{Div}_0(\mathcal{X}')$. By the projection formula we have

$$\omega_{\mathcal{X}} \otimes \mathcal{L} \otimes \mathcal{J}(\mathfrak{a}^c) = \pi_* \left(\omega_{\mathcal{X}'} \otimes \pi^* \mathcal{L}(-\lfloor c\, D \rfloor)\right).$$

Now $\pi^*\mathcal{L} - c\, D$ is nef and $c\, D - \lfloor c\, D \rfloor$ has coefficients in $[0, 1[$. Lemma B.6 below together with the projection formula yields

$$R^q \pi_* \left(\omega_{\mathcal{X}'} \otimes \pi^* \mathcal{L}\left(-\lfloor c\, D \rfloor\right)\right) = 0 \quad \text{for all } q \geq 1.$$

The Leray spectral sequence is thus degenerate and we conclude using Theorem B.3. The second point follows since $\mathcal{J}(\mathfrak{a}_\bullet) = \mathcal{J}(\mathfrak{a}_m^{1/m})$ for some m by construction, while the global generation assumption implies that $\mathcal{L} \otimes \mathfrak{a}_m^{1/m}$ is nef. □

Lemma B.6 (Local vanishing)**.** *Let $\mathcal{X}$ be a regular model, let $\mathfrak{a}$ be a vertical ideal sheaf on $\mathcal{X}$, and let $\pi : \mathcal{X}' \to \mathcal{X}$ be an SNC model such that $\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(-D)$ with $D \in \mathrm{Div}_0(\mathcal{X}')$. Then we have*

$$R^q \pi_* \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right) = 0 \quad \textit{for all } q \geq 1.$$

Proof. We argue as in the last part of the proof of Theorem B.3. Let $\mathcal{A} \in \mathrm{Pic}(\mathcal{X})$ be sufficiently ample to guarantee:

(i) $\pi^*\mathcal{A} - cD$ is nef.
(ii) $\mathcal{A} \otimes R^q \pi_* \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right)$ is globally generated on $\mathcal{X}$.
(iii) $H^p \left(\mathcal{X}, \mathcal{A} \otimes R^m \pi_* \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right)\right) = 0$ for all $p \geq 1$ and $m \geq 0$.

Note that the first condition can be achieved since $-D$ is π-globally generated. The degeneration of the Leray spectral sequence shows that

$$H^0 \left(\mathcal{X}, \mathcal{A} \otimes R^q \pi_* \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right)\right) = H^q \left(\mathcal{X}', \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right) \otimes \pi^* \mathcal{A}\right),$$

which vanishes by Theorem B.3. It follows that $\mathcal{A} \otimes R^q \pi_* \omega_{\mathcal{X}'} \left(-\lfloor c\, D \rfloor\right) = 0$ by global generation, whence the result. □

We may now deduce from the above results the following two consequences that we need in the proof of Theorem B.

Theorem B.7 (Subadditivity)**.** *Let $\mathcal{X}$ be a regular model, $\mathfrak{a}, \mathfrak{b}$ vertical coherent ideal sheaves on $\mathcal{X}$, and $c, d > 0$. Then we have*

$$\mathcal{J}(\mathfrak{a}^c \cdot \mathfrak{b}^d) \subset \mathcal{J}(\mathfrak{a}^c) \cdot \mathcal{J}(\mathfrak{b}^d).$$

Proof. This is proved exactly as in [Laz, Theorem 9.5.20] using local vanishing. (See also [JM12, Theorem A.2] for a different proof.) □

Theorem B.8 (Uniform generation property)**.** *Let $\mathcal{X}$ be a regular model. Then there exists an ample line bundle $\mathcal{A}$ on $\mathcal{X}$ such that the following holds. Given $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, a vertical ideal sheaf $\mathfrak{a}$, and a rational number $c > 0$ such that $\mathcal{L} \otimes \mathfrak{a}^c$ is nef, the sheaf*

$$\mathcal{A} \otimes \mathcal{L} \otimes \mathcal{J}(\mathfrak{a}^c)$$

is globally generated. In particular, if $\mathfrak{a}_\bullet$ is a graded sequence of vertical coherent ideal sheaves on $\mathcal{X}$ such that $\mathcal{L}^m \otimes \mathfrak{a}_m$ is globally generated for all sufficiently divisible m, then

$$\mathcal{A} \otimes \mathcal{L}^m \otimes \mathcal{J}(\mathfrak{a}_\bullet^m)$$

is globally generated for all m.

Remark B.9. Fix any ample line bundle $\mathcal{B}$ on $\mathcal{X}$. Since, for any line bundle $\mathcal{A}'$, the line bundle $\mathcal{B}^n \otimes (\mathcal{A}')^{-1}$ is globally generated for $n \gg 0$, we may take $\mathcal{A} := \mathcal{B}^n$ with n sufficiently large in the previous statement.

Proof. Let $\mathcal{B}$ be a given very ample line bundle such that $\mathcal{A} := \omega_{\mathcal{X}} \otimes \mathcal{B}^{n+1}$ is ample. By the Castelnuovo-Mumford criterion it is enough to check that

$$H^q\left(\mathcal{X}, \mathcal{A} \otimes \mathcal{L} \otimes \mathcal{B}^{-q} \otimes \mathcal{J}(\mathfrak{a}^c)\right) = 0$$

for $q = 1, \dots, n$, and this is a consequence of Theorem B.5. $\square$

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