J. ALGEBRAIC GEOMETRY
18 (2009) 279–308
S 1056-3911(08)00490-6
Article electronically published on April 23, 2008

DIFFERENTIABILITY OF VOLUMES OF DIVISORS AND A PROBLEM OF TEISSIER

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Abstract

We give an algebraic construction of the positive intersection products of pseudo-effective classes and use them to prove that the volume function on the Néron–Severi space of a projective variety is $\mathcal{C}^1$-differentiable, expressing its differential as a positive intersection product. We also relate the differential to the restricted volumes. We then apply our differentiability result to prove an algebro-geometric version of the Diskant inequality in convex geometry, allowing us to characterize the equality case of the Khovanskii–Teissier inequalities for nef and big classes.

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Introduction

The *volume* of a line bundle L on a projective variety X of dimension n is a nonnegative real number measuring the positivity of L from the point of view of birational geometry. It is defined as the growth rate of sections of

Received December 21, 2006 and, in revised form, April 16, 2007. The second author was supported by the Japanese Society for the Promotion of Science. The third author was supported by NSF Grant No. DMS-0449465, the Swedish Science Council, and the Gustafsson Foundation.

multiples of L:

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n}\, h^0(X, kL)$$

and is positive iff the linear system $|kL|$ embeds X birationally in a projective space for k large enough; L is then said to be *big*.

The volume has been studied by several authors, and the general theory is presented in detail and with full references in [L, §2.2.C]. In particular, it is known that the volume only depends on the numerical class of L in the real Néron–Severi space $N^1(X)$, and that it uniquely extends to a continuous function on this whole space, such that $\mathrm{vol}^{1/n}$ is homogeneous of degree 1, concave on the open convex cone of big classes, and zero outside.

Given its fundamental nature, it is quite natural to ask what kind of regularity besides continuity the volume function exhibits in general. In the nice survey [ELMNP1], many specific examples were investigated, leading the authors to conjecture that the volume function is always real analytic on a "large" open subset of the big cone. Our concern here will be the differentiability of the volume function. The simple example of $\mathbf{P}^2$ blown-up in one point already shows that the volume function is not twice differentiable on the entire big cone in general. Our main result is as follows.

Theorem A. *The volume function is $\mathcal{C}^1$-differentiable on the big cone of $N^1(X)$. If $\alpha \in N^1(X)$ is big and $\gamma \in N^1(X)$ is arbitrary, then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle \alpha^{n-1}\rangle \cdot \gamma.$$

The right-hand side of the equation above involves the *positive intersection product* $\langle \alpha^{n-1}\rangle \in N^1(X)^*$ of the big class α, first introduced in the analytic context in [BDPP]. We shall return to its algebraic definition later in this introduction, when discussing our method of proof.

We then proceed to show that the derivative of the volume in the direction of a class determined by a prime divisor can also be interpreted as a restricted volume, as introduced and studied in [ELMNP3]. Recall that if V is a subvariety of X, the *restricted volume* on V of a line bundle L on X measures the growth of sections in $H^0(V, kL|_V)$ that extend to X. It is defined as

$$\mathrm{vol}_{X|V}(L) := \limsup_{k\to\infty} \frac{d!}{k^d}\, h^0(X|V, kL)$$

where $d := \dim V$ and $h^0(X|V, kL)$ denotes the rank of the restriction map

$$H^0(X, kL) \to H^0(V, kL|_V).$$

Restricted volumes have recently played a crucial role in the proof of the boundedness of pluricanonical systems of varieties of general type in [Ta] and implicitly in [HM]. Here we relate the positive intersection products with restricted volumes, and show the following.

Theorem B. *If D is a prime divisor on the smooth projective variety X and L is a big line bundle on X, then the restricted volume of L on D satisfies*

$$\mathrm{vol}_{X|D}(L) = \langle L^{n-1} \rangle \cdot D.$$

This statement in particular implies that the restricted volume only depends on the numerical class of both L and D in $N^1(X)$. When V is an irreducible subvariety, [ELMNP3] and [Ta] have independently shown that the restricted volume $\mathrm{vol}_{X|V}(L)$ of a big line bundle L can be expressed as the asymptotic intersection number of the moving parts of $|kL|$ with the strict transforms of V on appropriate birational models X_k of X, when L satisfies an additional positivity assumption along V. The main (and difficult) result of [ELMNP3] says that $\mathrm{vol}_{X|V}(L) = 0$ otherwise. Our contribution in Theorem B is to show that the asymptotic intersection number along a *prime divisor* $V = D$ in fact coincides with the intersection number $\langle L^{n-1} \rangle \cdot D$ when the additional positivity condition is satisfied, using our differentiability result, and that both sides are 0 otherwise, relying on the orthogonality of Zariski decompositions instead of the main result of [ELMNP3].

Theorems A and B yield the following corollary, which was kindly communicated to us by R. Lazarsfeld and M. Mustaţă and which inspired our differentiability result.

Corollary C. *If D is a prime divisor on the smooth projective variety X and L is a big line bundle, then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(L + tD) = n\,\mathrm{vol}_{X|D}(L).$$

We also give an application of our differentiability theorem and characterize the equality case in the Khovanskii–Teissier inequalities for big and nef classes, a problem considered by Teissier in [Te2, p.96] and [Te3, p.139]. Recall that one version of the Khovanskii–Teissier inequalities [Te1] for a pair of nef classes $\alpha, \beta \in N^1(X)$ asserts that the sequence $k \mapsto \log(\alpha^k \cdot \beta^{n-k})$ is concave. Here we prove the following theorem.

Theorem D. *If $\alpha, \beta \in N^1(X)$ are big and nef classes, then the following are equivalent:*

(i) *the concave sequence $k \mapsto \log(\alpha^k \cdot \beta^{n-k})$ is affine;*
(ii) $(\alpha^{n-1} \cdot \beta) = (\alpha^n)^{1-\frac{1}{n}} (\beta^n)^{\frac{1}{n}}$*;*
(iii) *α and β are proportional.*

An equivalent formulation is:

Corollary E. *The function $\alpha \mapsto (\alpha^n)^{\frac{1}{n}}$ is strictly concave on the big and nef cone of $N^1(X)$.*

Note that when α and β are ample, these results are direct consequences of the Hodge–Riemann relations; see for instance [Vo, Theorem 6.32]. However, our approach is different and purely algebraic. In fact, the two statements above and their proofs are inspired by their counterparts in convex geometry [Sch]. When X is a projective toric variety, a big and nef class $\alpha \in N^1(X)$ corresponds to a convex polytope in $\mathbf{R}^n$ (unique up to translation) with Euclidean volume $(\alpha^n)/n! > 0$. The Khovanskii–Teissier inequalities are known in this setting as the Alexandrov–Fenchel inequalities for mixed volumes, a far-reaching generalization of the classical isoperimetric inequality. A different, "effective", strengthening of the isoperimetric inequality was given by Bonnesen [Bon] (in dimension two) and Diskant [D] (in any dimension). It immediately implies the analog of Theorem D in convex geometry; in particular, equality holds in the isoperimetric inequality iff the convex body is a ball. The proof by Diskant is based on the differentiability of the volume function of (inner parallel) convex bodies, a fact established by Alexandrov. Following the same strategy, and using Theorem A, we prove the following version of the Diskant inequality in our setting, thus providing a solution to [Te2, Problem B].

Theorem F. *If $\alpha, \beta \in N^1(X)$ are big and nef classes and s is the largest real number such that $\alpha - s\beta$ is pseudo-effective, then*

$$(\alpha^{n-1} \cdot \beta)^{\frac{n}{n-1}} - (\alpha^n)(\beta^n)^{\frac{1}{n-1}} \geq \left((\alpha^{n-1} \cdot \beta)^{\frac{1}{n-1}} - s(\beta^n)^{\frac{1}{n-1}}\right)^n.$$

Theorem D immediately follows from Theorem F.

Let us now present our method of proof of Theorem A. It is a fundamental fact that the volume of an arbitrary big line bundle L admits an intersection-theoretic interpretation, generalizing the equality $\mathrm{vol}(L) = (L^n)$ when L is ample. Indeed, a remarkable theorem of Fujita [Fuj] (see also [DEL]) states that $\mathrm{vol}(L)$ is the supremum of all intersection numbers (A^n), where A is an ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor on a modification $\pi : X_\pi \to X$ such that $\pi^* L = A + E$.

In view of this result, it is natural to put *all* birational models of X on equal footing, and study numerical classes defined on all birational models at the same time. Being purely algebraic, this technique extends to any projective variety over any algebraically closed field of characteristic zero.

More precisely, we introduce the *Riemann–Zariski space* $\mathfrak{X}$ of X as the projective limit of all birational models of X. A (codimension p numerical) class in $\mathfrak{X}$ is then a collection of (codimension p numerical) classes in each birational model of X that are compatible under push-forward. The set of all these classes is an infinite dimensional vector space that we denote by $N^p(\mathfrak{X})$. It naturally contains the space $CN^p(\mathfrak{X})$, the union of the spaces $N^p(X_\pi)$ of numerical classes of codimension p cycles of all birational models X_π of X. One can extend to $CN^1(\mathfrak{X})$ the usual notions of pseudo-effective, big and nef classes. Note that related objects have already been introduced in the context of Mori's minimal model program by Shokurov [Sh]. The notion of *b-divisors*, crucial to his approach, can be interpreted as divisors on $\mathfrak{X}$. Chow groups on the Riemann–Zariski space also appeared in the work of Aluffi [A] and numerical classes in [BFJ1], [C], and [M] in the case of surfaces; see also [FJ] and [BFJ2] for a local study.

We then introduce the notion of the *positive intersection product*

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$$

of big classes $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$, defined as the least upper bound of the products $\beta_1 \cdot \ldots \cdot \beta_p$ of nef classes β_i on modifications $\pi : X_\pi \to X$ such that $\beta_i \le \pi^* \alpha_i$. Although this product is non-linear, it is homogeneous and increasing in each variable, and continuous on the big cone. In this terminology, Fujita's theorem is equivalent to the relation

$$\mathrm{vol}(\alpha) = \langle \alpha^n \rangle$$

for any big class $\alpha \in N^1(X)$. Using the easy but fundamental inequality

$$\mathrm{vol}(A - B) \ge (A^n) - n(A^{n-1} \cdot B)$$

for any two nef Cartier classes $A, B \in CN^1(\mathfrak{X})$, we deduce a sub-linear control for the volume function $\mathrm{vol}(\alpha + t\gamma) \ge \mathrm{vol}(\alpha) + tn\langle \alpha^{n-1} \rangle \cdot \gamma + O(t^2)$ for any two Cartier classes $\alpha, \gamma \in CN^1(\mathfrak{X})$ such that α is big, from which we easily infer Theorem A. As an immediate consequence, we get the following orthogonality relation

$$\langle \alpha^n \rangle = \langle \alpha^{n-1} \rangle \cdot \alpha$$

for any psef class $\alpha \in CN^1(\mathfrak{X})$, which was the crucial point in the dual characterization of pseudo-effectivity of [BDPP].

In a similar vein, the proof of Theorem B is based on a suitable generalization of Fujita's theorem for restricted volumes, obtained independently in [ELMNP3] and [Ta]. If D is a prime divisor on a smooth projective variety X, we say that a line bundle L is *D-big* if there exists a decomposition $L = A + E$, where A is an ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor on X whose support does not contain D. The generalization of Fujita's theorem expresses the restricted volume of a D-big line bundle as the supremum of all intersection numbers $A^{n-1} \cdot D_\pi$, where $\pi : X_\pi \to X$ is a modification, D_π denotes the strict transform of D, A is an ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor on X_π whose support does not contain D_π and such that $\pi^*L = A + E$. In Section 4, we again interpret this result in the framework of the Riemann–Zariski space and define the restricted positive intersection product

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle|_{\mathfrak{D}}$$

of D-big classes α_i on X as a numerical class of the Riemann–Zariski space $\mathfrak{D}$ associated to D, in such a way that the restricted positive intersection product $\langle L^{n-1} \rangle|_{\mathfrak{D}}$ coincides with the asymptotic intersection number $||L^{n-1} \cdot D||$ of L and D as introduced in [ELMNP3] if L is D-big. The generalized Fujita theorem mentioned above then reads

$$\mathrm{vol}_{X|D}(L) = \langle L^{n-1} \rangle|_{\mathfrak{D}}$$

when L is D-big, and Theorem B asserts that the restricted positive intersection product $\langle L^{n-1} \rangle|_{\mathfrak{D}}$ coincides with the intersection number $\langle L^{n-1} \rangle \cdot D$.

The former is computed as a limit as $k \to \infty$ of intersection numbers of the form $A_k^{n-1} \cdot D_k$ with A_k an ample $\mathbf{Q}$-divisor $\leq \pi_k^* L$ on a blow-up $\pi_k : X_k \to X$ and D_k the strict transform of D on X_k, and the latter is then the limit of $A_k^{n-1} \cdot \pi_k^* D$. We easily deduce from this relation that $\langle L^{n-1} \rangle|_{\mathfrak{D}} \leq \langle L^{n-1} \rangle \cdot D$. The reverse inequality is obtained by showing that

$$\limsup_{t \to 0+} \frac{1}{t} \left(\mathrm{vol}(L) - \mathrm{vol}(L - tD) \right) \leq n \, \mathrm{vol}_{X|D}(L),$$

building on the basic relation $h^0(X, L) = h^0(X, L - D) + h^0(X|D, L)$. This yields Theorem B in the D-big case. When L is big but not D-big, we prove that the inequality $\langle L^{n-1} \rangle \cdot D \geq \mathrm{vol}_{X|D}(L)$ still holds, and that $\langle L^{n-1} \rangle \cdot D = 0$, as a consequence of the orthogonality relation mentioned above. We get in particular that $\mathrm{vol}_{X|D}(L) = 0$ when L is not D-big, thus reproving by a different method a special case of the main result of [ELMNP3].

Let us briefly indicate the structure of our article. We introduce numerical classes in the Riemann–Zariski space in Section 1, and study their positivity properties. The positive intersection product is then defined in Section 2. We turn to the volume function on the set of psef classes in the Riemann–Zariski space in Section 3. In particular, this section contains the proof of the Differentiability Theorem, and its applications to the characterization of the equality case in the Khovanskii–Teissier inequalities: Theorems D and F and Corollary E. We also include in Section 3.4 an informal discussion making a link between positive intersection products and Zariski-type decompositions for psef classes, intended to shed further light on our construction. The paper concludes with a presentation of the restricted volume from the Riemann–Zariski point of view in Section 4, allowing us to prove Theorem B and Corollary C.

1. Classes on the Riemann–Zariski space

1.1. The Riemann–Zariski space. Let X be a projective variety. Since we shall deal with classes living on arbitrary modifications of X, it is convenient to introduce the following terminology. By a *blow-up* of X, we mean a birational morphism $\pi : X_\pi \to X$, where X_π is a *smooth* projective variety. If π and π' are two blow-ups of X, we say that π' *dominates* π, and write $\pi' \geq \pi$, if there exists a birational morphism (necessarily unique) $\mu : X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \mu$. This endows the set of blow-ups of X with a partial order relation. By the Hironaka desingularization theorem, this ordered set is non-empty and forms a directed family. The *Riemann–Zariski space* of X is the projective limit

$$\mathfrak{X} := \varprojlim_{\pi} X_\pi.$$

We refer to [ZS] and [Va] for a thorough discussion of the structure of this space, which is introduced here merely in order to make later definitions more suggestive.

1.2. Weil and Cartier classes. For any smooth projective variety Y of dimension n, and any integer $0 \leq p \leq n$, let $N^p(Y)$ be the real vector space of numerical equivalence classes of codimension p cycles. It is a finite dimensional space. Any birational morphism $\mu : Y' \to Y$ induces in a contravariant way a pull-back morphism

$$\mu^* : N^p(Y) \to N^p(Y')\ ;$$

and in a covariant way a push-forward morphism

$$\mu_* : N^p(Y') \to N^p(Y).$$

There is an intersection pairing $N^p(Y) \times N^{n-p}(Y) \to \mathbf{R}$, which is preserved under pull-back by birational morphisms, and for which push-forward and pull-back are adjoint to each other. We refer to [Ful, Chapter 19] for further details on the space N^p and its link with the cohomology groups $H^{p,p}$ in the case of a complex projective variety.

We now consider an arbitrary projective variety X. The inverse family of blow-ups $\pi : X_\pi \to X$ induces two families of arrows between the spaces $N^p(X_\pi)$: one is the inverse family with push-forward morphisms as arrows $\mu_* : N^p(X_{\pi'}) \to N^p(X_\pi)$, whenever $\mu : X_{\pi'} \to X_\pi$, and the other is the direct family with pull-back morphisms as arrows $\mu^* : N^p(X_\pi) \to N^p(X_{\pi'})$.

Definition 1.1.

- The space of p-codimensional Weil classes on the Riemann–Zariski space $\mathfrak{X}$ is defined as the projective limit

$$N^p(\mathfrak{X}) := \varprojlim_{\pi} N^p(X_\pi)$$

with arrows defined by push-forward. It is endowed with its projective limit topology, which will be called the *weak topology*.

- The space of p-codimensional Cartier classes on $\mathfrak{X}$ is defined as the inductive limit

$$CN^p(\mathfrak{X}) := \varinjlim_{\pi} N^p(X_\pi)$$

with arrows induced by pull-back. It is endowed with its inductive limit topology, which will be called the *strong topology*.

The terminology stems from the fact that Weil divisors are meant to be pushed forward, whereas Cartier divisors are meant to be pulled back. Note that these spaces are infinite dimensional as soon as $\dim X \geq 2$ and $p \neq 0, n$.

A Weil class $\alpha \in N^p(\mathfrak{X})$ is by definition described by its *incarnations* $\alpha_\pi \in N^p(X_\pi)$ on each smooth birational model X_π of X, compatible with each other by push-forward. Convergence of a sequence (or a net) $\alpha_n \to \alpha$ in the weak topology means $\alpha_{n,\pi} \to \alpha_\pi$ in $N^p(X_\pi)$ for each π.

On the other hand, the relation $\mu_*\mu^*\alpha = \alpha$ when μ is a birational morphism shows that there is an injection

$$CN^p(\mathfrak{X}) \to N^p(\mathfrak{X})$$

i.e. a Cartier class is in particular a Weil class. Concretely, a Weil class α is Cartier iff there exists π such that its incarnations $\alpha_{\pi'}$ on higher blow-ups $X_{\pi'}$ are obtained by pulling back α_π. We will call such a π a *determination* of α. Conversely, there are natural injective maps

$$N^p(X_\pi) \hookrightarrow CN^p(\mathfrak{X})$$

which extend a given class $\beta \in N^p(X_\pi)$ to a Cartier class by pulling it back, and of course this Cartier class is by construction determined by π. In the sequel, we shall always identify a class $\beta \in N^p(X_\pi)$ with its image in $CN^p(\mathfrak{X})$.

Note that the topology induced on $CN^p(\mathfrak{X})$ by the weak topology of $N^p(\mathfrak{X})$ is coarser than the strong topology. By definition, a map on $CN^p(\mathfrak{X})$ is continuous in the strong topology iff its restriction to each of the finite dimensional subspaces $N^p(X_\pi)$ is continuous.

Finally we note that

Lemma 1.2. *The natural continuous injective map*

$$CN^p(\mathfrak{X}) \to N^p(\mathfrak{X})$$

has dense image (in the weak topology).

Proof. If $\alpha \in N^p(\mathfrak{X})$ is a given Weil class, then we can consider the Cartier classes α_π determined by the incarnation of α on X_π, and it is obvious that the net α_π converges to α in the weak topology as $\pi \to \infty$, since α_π and α have by definition the same incarnation on X_π. □

1.3. Divisors. We shall refer to the space $CN^1(\mathfrak{X}) := \varinjlim_\pi N^1(X_\pi)$ as the *Néron–Severi space* of $\mathfrak{X}$. Its elements are related to the so-called b-Cartier divisors on X, in the sense of Shokurov [Sh] and [I]. These are by definition elements of the inductive limit

$$\mathrm{CDiv}(\mathfrak{X}) := \varinjlim_{\pi} \mathrm{Div}(X_\pi)$$

of the spaces of (Cartier) $\mathbf{R}$-divisors on each X_π. According to our present point of view, we will change the terminology and call an element of $\mathrm{CDiv}(\mathfrak{X})$ a *Cartier divisor on* $\mathfrak{X}$.

It is then immediate to see that the Néron–Severi space $CN^1(\mathfrak{X})$ is the quotient of $\mathrm{CDiv}(\mathfrak{X})$ modulo numerical equivalence. In particular, a Cartier divisor D on X itself, maybe induces a Cartier class in $CN^1(\mathfrak{X})$.

1.4. Positive classes. When Y is a smooth projective variety, we define the *psef cone* (a short-hand for pseudo-effective cone) of $N^p(Y)$ as the closure of the convex cone generated by effective codimension p cycles.

Proposition 1.3. *For any smooth projective variety Y, $N^p(Y)$ is a finite dimensional real vector space in which the psef cone is convex, closed, strict (i.e. $\pm\alpha$ psef implies $\alpha = 0$) and has a compact basis (i.e. the set of classes $\beta \in N^p(Y)$ with $\alpha - \beta$ psef is compact for every psef class α).*

Proof. The fact that $N^p(Y)$ is finite dimensional is proved in [Ful, Example 19.1.4]. By definition, the psef cone is convex and closed. Any strict closed convex cone in a finite dimensional space has a compact basis, so it remains to prove that the psef cone is strict. If $\alpha \in N^p(Y)$ is psef, then clearly $\alpha \cdot h_1 \cdot \ldots \cdot h_{n-p} \geq 0$ for all ample classes h_i on Y. Therefore if $\pm\alpha$ are psef, we get that α is zero against any complete intersection class, i.e. $\alpha \cdot \beta_1 \cdot \ldots \cdot \beta_{n-p} = 0$ for all classes $\beta_i \in N^1(Y)$ (since any such class can be written as a difference of ample classes). Now if $f : Z \to Y$ is a surjective smooth morphism, $\pm f^*\alpha$ is also psef, and thus also zero against any complete intersection class on Z. Applying this to $f : Z = \mathbf{P}(E) \to Y$ for a vector bundle E on Y yields that α is zero against the $(n-p)$th Segre class of any vector bundle E on Y, which is the push-forward of the appropriate power of the tautological line bundle of $\mathbf{P}(E)$. Since such Segre classes generate $N^{n-p}(Y)$, this finally shows that α is zero in $N^p(Y)$. □

In the sequel, we shall write $\alpha \geq 0$ when $\alpha \in N^p(Y)$ is a psef class. If $\mu : Y' \to Y$ is a birational morphism, the push forward $\mu_*\alpha \in N^p(Y)$ of a psef class $\alpha \in N^p(Y')$ is also psef, thus we can introduce the following definition.

Definition 1.4. A Weil class $\alpha \in N^p(\mathfrak{X})$ is *psef* iff all its incarnations $\alpha_\pi \in N^p(X_\pi)$ are psef.

The set of all psef classes is a strict convex cone in $N^p(\mathfrak{X})$ that is closed in the weak topology. We will write $\alpha \geq 0$ if $\alpha \in N^p(\mathfrak{X})$ is psef.

Proposition 1.5. *The psef cone has compact basis. In other words, if $\alpha \in N^p(\mathfrak{X})$ is a given psef class, the set of Weil classes $\beta \in N^p(\mathfrak{X})$ such that $0 \leq \beta \leq \alpha$ is compact (in the weak topology).*

Proof. The set in question is the projective limit of the corresponding sets $K_\pi := \{\gamma \in N^p(X_\pi) \mid 0 \leq \beta_\pi \leq \alpha_\pi\}$, and each of these sets is compact as recalled above, so the result follows from the Tychonoff theorem. □

We now consider positive Cartier classes on $\mathfrak{X}$. If Y is a projective variety, recall [L, §1.4.C] that the *nef cone* of $N^1(Y)$ is the closure of the cone of classes determined by ample divisors. A nef class is thus psef. The interior of the psef cone of $N^1(Y)$ is called the *big cone*. If $\mu : Y' \to Y$ is a birational morphism, then a class $\alpha \in N^1(Y)$ is nef (resp. psef, big) iff $\mu^*\alpha$ is. This property enables us to extend the definitions to the Riemann–Zariski space.

Definition 1.6. A Cartier class $\alpha \in CN^1(\mathfrak{X})$ is said to be nef (resp. psef, big) iff its incarnation α_π is nef (resp. psef, big) for some determination π of α.

The set of psef classes of $CN^1(\mathfrak{X})$ will be called the *psef cone* of $CN^1(\mathfrak{X})$. It is obviously a closed convex cone, and in fact it coincides with the inverse image of the psef cone of $N^1(\mathfrak{X})$ under the continuous injection $CN^1(\mathfrak{X}) \to N^1(\mathfrak{X})$. In other words, a Cartier class $\alpha \in CN^1(\mathfrak{X})$ is psef as a Cartier class iff it is psef as a Weil class, thus the terminology chosen is not ambiguous.

Proposition 1.7. *The psef cone of $CN^1(\mathfrak{X})$ is the closure of the big cone, i.e. the set of big classes. The big cone is the interior of the psef cone.*

Proof. The analogous statement is true in each space $N^1(X_\pi)$, so the result follows easily from the behavior of psef and big classes under pull-back recalled above and the definition of the strong topology. □

The set of nef classes of $CN^1(\mathfrak{X})$ will be called the *nef cone* of $CN^1(\mathfrak{X})$. It is a closed convex cone.

Remark 1.8. A *b*-divisor is *b*-big (resp. *b*-nef) in the terminology of Shokurov iff its class in $CN^1(\mathfrak{X})$ is big (resp. nef).

Remark 1.9. There is no "ample cone" on $\mathfrak{X}$, i.e. the interior of the nef cone of $CN^1(\mathfrak{X})$ is empty. Indeed, if α is a nef class determined by π, then for any strictly higher blow-up $\pi' \geq \pi$ its incarnation $\alpha_{\pi'} \in N^1(X_{\pi'})$ lies on the boundary of the nef cone.

Remark 1.10. One can also define the nef cone in $N^1(\mathfrak{X})$ as the closure in the weak topology of the nef cone of $CN^1(\mathfrak{X})$; see [BFJ2] for details in a similar situation.

1.5. Toric varieties. Let X be an n-dimensional projective toric variety; see [O] for background. If we restrict to toric blow-ups $\pi : X_\pi \to X$ in the definitions above, we obtain a toric Riemann–Zariski space $\mathfrak{X}_{\rm tor}$ and a toric Néron–Severi space $CN^1(\mathfrak{X}_{\rm tor}) \subset CN^1(\mathfrak{X})$. One easily checks that $CN^1(\mathfrak{X}_{\rm tor})$ is canonically isomorphic to the space of functions $g : N_{\mathbf{R}} \to \mathbf{R}$ that are piecewise linear with respect to some rational fan decomposition of $N_{\mathbf{R}}$, modulo linear forms. Here as usual $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, where $N = \mathbf{Z}^n$ is the lattice of 1-parameter subgroups of the torus $(\mathbf{C}^*)^n$.

To a piecewise linear function g as above is associated its Newton polytope $\mathrm{Nw}(g)$ in the dual space $M_{\mathbf{R}} = N^*_{\mathbf{R}}$. It is defined as the set of linear forms $m \in M_{\mathbf{R}}$ such that $m \le g$ everywhere on $N_{\mathbf{R}}$. This polytope only depends on the cohomology class $\alpha \in CN^1(\mathfrak{X}_{\rm tor})$, up to translation, thus we will denote it by $\mathrm{Nw}(\alpha) = \mathrm{Nw}(g)$. It is a standard fact in toric geometry that:

(i) α is psef iff $\mathrm{Nw}(\alpha)$ is non-empty;
(ii) α is big iff $\mathrm{Nw}(\alpha)$ has non-empty interior;
(iii) α is nef iff g is convex.

Note that when α is nef, g is equal to the maximum of all linear forms lying in $Nw(\alpha)$. Hence, a toric nef class can be recovered from its Newton polytope, and we can identify toric nef classes and polytopes in $M_{\mathbf{R}} = \mathbf{R}^n$ with rational maximal faces, up to translation.

2. Positive intersection product

2.1. Intersections of Cartier classes. On a smooth projective variety Y, the intersection product of p classes $\alpha_1, \dots, \alpha_p \in N^1(Y)$ defines an element $\alpha_1 \cdot \ldots \cdot \alpha_p \in N^p(Y)$. Recall that for any birational morphism $\mu : Y' \to Y$, we have $\mu^*\alpha_1 \cdot \ldots \cdot \mu^*\alpha_p = \mu^*(\alpha_1 \cdot \ldots \cdot \alpha_p)$; see [Ful, Chapter 19].

We now define the intersection product of p Cartier classes $\alpha_1, \dots, \alpha_p \in CN^1(\mathfrak{X})$ having a common determination on X_π as the Cartier class in $CN^p(\mathfrak{X})$ determined by $\alpha_{1,\pi} \cdot \ldots \cdot \alpha_{p,\pi}$. Since the intersection product is preserved by pull-back, this is independent on the choice of determination. It is, moreover, continuous for the strong topology on $CN^1(\mathfrak{X})$.

Remark 2.1. By the projection formula we have $f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta$ for any birational morphism f between smooth projective varieties. Thus, it is possible to extend the intersection product to a bilinear map $CN^p(\mathfrak{X}) \times \mathrm{N}^q(\mathfrak{X}) \to \mathrm{N}^{p+q}(\mathfrak{X})$. In the case $p = 1$ and $q = n - 1$, we get a pairing

$CN^1(\mathfrak{X}) \times \mathrm{N}^{n-1}(\mathfrak{X}) \to \mathbf{R}$, and under this pairing the psef cone of $\mathrm{N}^{n-1}(\mathfrak{X})$ is the dual of the nef cone in $CN^1(\mathfrak{X})$.

When the α_i are nef classes, it is easy to see that $\alpha_1 \cdot \ldots \cdot \alpha_p \in CN^p(\mathfrak{X}) \subset N^p(\mathfrak{X})$ is psef. More generally, we have:

Lemma 2.2. *If $\alpha_i \in CN^1(\mathfrak{X})$, $1 \le i \le p$ are Cartier classes with α_1 psef and α_i nef for $i \ge 2$, then their intersection product $\alpha_1 \cdot \ldots \cdot \alpha_p \in N^p(\mathfrak{X})$ is psef.*

Proof. By continuity, it is enough to check this when for some common determination π, the class α_1 is represented on X_π by an effective divisor and the α_i are represented by very ample divisors for $i \ge 2$. But then the result is clear. □

The lemma implies the following monotonicity property which will be crucial in what follows.

Proposition 2.3. *Let α_i and α_i', $1 \le i \le p$, be nef classes in $CN^1(\mathfrak{X})$, and suppose that $\alpha_i \ge \alpha_i'$ for $i = 1, \ldots, p$. Then we have*

$$\alpha_1 \cdot \ldots \cdot \alpha_p \ge \alpha_1' \cdot \ldots \cdot \alpha_p'$$

in $N^p(\mathfrak{X})$.

Proof. By successively replacing each α_i by α_i' and using the symmetry of the intersection product, it is enough to consider the case where $\alpha_i = \alpha_i'$ for $i \ge 2$. But then $(\alpha_1 - \alpha_1') \cdot \alpha_2 \cdot \ldots \cdot \alpha_p \ge 0$ since $\alpha_1 - \alpha_1'$ is psef by assumption. □

As a consequence, we get the following useful uniformity result.

Corollary 2.4. *Let $\alpha_1, \ldots, \alpha_n \in CN^1(\mathfrak{X})$ be arbitrary Cartier classes, and suppose that for some $0 \le p \le n$, α_i is nef for $i \le p$, and that we are given a nef class $\omega \in CN^1(\mathfrak{X})$ such that $\omega \pm \alpha_i$ is nef for each $i > p$. Then we have*

$$|(\alpha_1 \cdot \ldots \cdot \alpha_n)| \le C_n(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \omega^{n-p})$$

for some constant C_n only depending on n.

Proof. Write α_i as the difference of two nef classes $\beta_i := \alpha_i + \omega$ and ω for $i > p$. Expanding out the product of the α_i, we see that it is enough to bound terms of the form $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \beta_{i_1} \cdot \ldots \cdot \beta_{i_k} \cdot \omega^{n-p-k})$. But we also have $\beta_i \le 2\omega$ by assumption, thus the result follows from Proposition 2.3. □

2.2. Positive intersections of big classes. The aim of this section is to justify

Definition 2.5. If $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$ are big classes, their positive intersection product

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$$

is defined as the least upper bound of the set of classes

$$(\alpha_1 - D_1) \cdot \ldots \cdot (\alpha_p - D_p) \in N^p(\mathfrak{X})$$

where D_i is an effective Cartier $\mathbf{Q}$-divisor on $\mathfrak{X}$ such that $\alpha_i - D_i$ is nef.

We shall see in Proposition 2.13 that one can in fact replace the D_i by arbitrary psef Cartier classes.

To justify the definition above, we rely on two lemmas. Recall that a partially ordered set is *directed* if any two elements can be dominated by a third. Dually, it is *filtered* if any two elements dominate a third.

Lemma 2.6. *Let $\alpha \in CN^1(\mathfrak{X})$ be a big Cartier class. Then the set $\mathcal{D}(\alpha)$ of effective Cartier $\mathbf{Q}$-divisors D on $\mathfrak{X}$ such that $\alpha - D$ is nef is non-empty and filtered.*

Lemma 2.7. *Let V be a Hausdorff topological vector space and K a strict closed convex cone, with associated partial order relation $\leq$. Then any non-empty directed subset $S \subset V$ that is contained in a compact subset of V admits a least upper bound with respect to $\leq$.*

The existence of the least upper bound in Definition 2.5 is then obtained by applying Lemma 2.7 to $V = N^p(\mathfrak{X})$, $K =$ the psef cone and

$$S = \{(\alpha_1 - D_1) \cdot \ldots \cdot (\alpha_p - D_p) \text{ with } D_i \in \mathcal{D}(\alpha_i)\}.$$

The fact that S is directed is a consequence of Lemma 2.6 and Proposition 2.3, and S is contained in the compact set (see Proposition 1.5) of classes $\leq \omega^p$ if ω is any Cartier class determined in a common determination of the α_i, and sufficiently big so that $\omega \geq \alpha_i$ for all i.

Proof of Lemma 2.6. Since α is big, one can find an effective $\mathbf{Q}$-divisor D on a determination X_π of α such that $\alpha_\pi - D$ is ample, and this proves that $\mathcal{D}(\alpha)$ is non-empty. In order to prove that it is filtered, we will show that given two effective $\mathbf{Q}$-divisors D_1, D_2 such that $\alpha - D_i$ are nef, the Weil $\mathbf{Q}$-divisor $D := \min(D_1, D_2)$ on $\mathfrak{X}$, defined coefficient-wise, is in fact Cartier and that $\alpha - D$ is nef. First, we can assume that α and the D_i are determined on X. By homogeneity, we can also assume that the D_i have integer coefficients. We then have the following characterization of D. If we introduce the ideal sheaf $\mathcal{I} := \mathcal{O}_X(-D_1) + \mathcal{O}_X(-D_2)$, then it is easy to see that the incarnation D_π of $D = \min(D_1, D_2)$ on a blow-up X_π is the divisorial part of the ideal $\mathcal{I}\mathcal{O}_{X_\pi}$. This shows that D is Cartier and determined in X_π if $\mathcal{I}\mathcal{O}_{X_\pi}$ is a principal ideal sheaf, which is the case as soon as π dominates the normalized blow-up of $\mathcal{I}$. Since nefness is a closed condition, in order to show that $\alpha - D$ is nef, we can add an arbitrarily small ample class to α and reduce by homogeneity to the case where $\alpha = c_1(L)$ for some line bundle L on X_π such that $L - D_i$ are globally generated on X. It then follows that $\mathcal{O}_X(L) \otimes \mathcal{I}$ is also globally generated on X, and thus that $\pi^*L - D$ is globally generated on X_π. In particular, it is nef, and this concludes the proof of the lemma. □

Proof of Lemma 2.7. The assumption that S is directed means that it is a net with respect to the order relation $\leq$. Obviously, any accumulation point of this net is a least upper bound for S; in particular, there can be at most one such accumulation point. But since this net lives in a compact set, it admits an accumulation point, and therefore converges towards the least upper bound of S. □

Remark 2.8. We have shown above that $\min(D_1, D_2)$ is a Cartier divisor on $\mathfrak{X}$ for any two effective Cartier $\mathbf{Q}$-divisors D_i on $\mathfrak{X}$. The relation $\min(D_1, D_2) = \min(D_1 + E, D_2 + E) - E$ then shows that the min (and thus also the max) of arbitrary Cartier $\mathbf{Q}$-divisors on $\mathfrak{X}$ is Cartier. All this fails, however, for arbitrary Cartier $\mathbf{R}$-divisors, as can be seen already in the toric setting.

2.3. Continuity properties. We now proceed by describing some general properties of the positive intersection product.

Proposition 2.9. *The positive intersection product*

$$(\alpha_1, \ldots, \alpha_p) \mapsto \langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$$

is symmetric, homogeneous of degree 1, *and super-additive in each variable. Moreover, it varies continuously on the p-fold product of the big cone of* $CN^1(\mathfrak{X})$.

Note that in general the positive intersection product is not linear in each variable, as can be already seen for the map $\alpha \mapsto \langle \alpha \rangle$ on surfaces, using its interpretation in terms of the Zariski decomposition; see Section 3.4 below.

Proof. Only the continuity statement is not clear. Let $\alpha_i \in CN^1(\mathfrak{X})$, $1 \leq i \leq p$, be big classes and $\varepsilon > 0$. Since α_i lies in the interior of the psef cone, we have $\varepsilon\alpha_i \geq \pm\gamma_i$ for every small enough Cartier class $\gamma_i \in CN^1(\mathfrak{X})$. It follows that

$$(1 - \varepsilon)\ \alpha_i \leq \alpha_i + \gamma_i \leq (1 + \varepsilon)\ \alpha_i$$

and thus

$$(1 - \varepsilon)^p\ \langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \leq \langle (\alpha_1 + \gamma_1) \cdot \ldots \cdot (\alpha_p + \gamma_p) \rangle \leq (1 + \varepsilon)^p\ \langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle$$

for γ_i small enough, which concludes the proof. □

We now extend the definition of the positive intersection product as follows.

Definition 2.10. If $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$ are psef classes, their positive intersection product

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$$

is defined as the limit

$$\lim_{\varepsilon \to 0+} \langle (\alpha_1 + \varepsilon\omega) \cdot \ldots \cdot (\alpha_p + \varepsilon\omega) \rangle$$

where $\omega \in CN^1(\mathfrak{X})$ is *any* big Cartier class.

This definition makes sense, because the positive intersection products on the right-hand side decrease as ε decreases to 0. In particular, they lie in a compact subset of $N^p(\mathfrak{X})$, so it is easy to see that the limit in question exists. Furthermore, if ω' is another big class, then there exists $C > 0$ such that $C^{-1}\omega \le \omega' \le C\omega$, and this shows that the limit is independent of the choice of the big class ω.

Remark 2.11. The extension of the positive intersection product is not continuous up to the boundary of the psef cone in general; see Example 3.8. It is however upper semi-continuous in the appropriate sense.

An important property of the positive intersection product is that it coincides with the usual intersection product on nef classes.

Proposition 2.12. *If $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$ are nef classes, then*

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle = \alpha_1 \cdot \ldots \cdot \alpha_p.$$

Proof. When the α_i are big and nef, the divisor $D = 0$ is allowed in Definition 2.5, thus Proposition 2.3 immediately yields the desired equality. The general case follows, since ω can be chosen to be big and nef in Definition 2.10, in which case the big classes $\alpha_i + \varepsilon\omega$ are also nef. □

In view of this result and monotonicity, we get the following characterization of the positive intersection product of big classes.

Proposition 2.13. *If $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$ are big Cartier classes, their positive intersection product $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$ is the least upper bound of the set of all intersection products*

$$\beta_1 \cdot \ldots \cdot \beta_p \in N^p(\mathfrak{X})$$

with $\beta_i \in CN^1(\mathfrak{X})$ a nef class such that $\beta_i \le \alpha_i$.

Remark 2.14. We do not know if this result still holds for arbitrary psef classes in general. It does hold when the α_i admit a Zariski decomposition; see Section 3.4.

2.4. Concavity properties. The intersection products of nef classes satisfy several remarkable inequalities, whose proofs are based on the Hodge index theorem. We refer to [L, §1.6] for their statements. Thanks to Proposition 2.13, we can transfer them to the positive intersection product on big classes, and by approximation to psef classes. This yields the following two results.

Theorem 2.15 (Khovanskii–Teissier inequalities)**.** *If $\alpha_1, \cdots, \alpha_n \in CN^1(\mathfrak{X})$ are psef classes, then we have*

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_n \rangle \geq \langle \alpha_1^n \rangle^{\frac{1}{n}} \cdots \langle \alpha_n^n \rangle^{\frac{1}{n}}.$$

More generally, we have

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_n \rangle \geq \langle \alpha_1^p \cdot \alpha_{p+1} \cdot \ldots \cdot \alpha_n \rangle^{\frac{1}{p}} \cdots \langle \alpha_p^p \cdot \alpha_{p+1} \cdot \ldots \cdot \alpha_n \rangle^{\frac{1}{p}}$$

for every $1 \leq p \leq n$.

Note that this implies in particular that the sequence $k \mapsto \log\langle \alpha^k \cdot \beta^{n-k} \rangle$ is concave for any two psef classes $\alpha, \beta \in CN^1(\mathfrak{X})$.

Theorem 2.16. *The function $\alpha \mapsto \langle \alpha^n \rangle^{1/n}$ is concave and homogeneous on the psef cone. More generally, given psef classes $\alpha_{p+1}, \cdots, \alpha_n$, the function*

$$\alpha \mapsto \langle \alpha^p \cdot \alpha_{p+1} \cdot \ldots \cdot \alpha_n \rangle^{1/p}$$

is homogeneous and concave on the psef cone of $CN^1(\mathfrak{X})$.

3. The volume function

3.1. Continuity properties. Recall that the volume of a line bundle L on X is defined by

$$\mathrm{vol}(L) = \limsup_{k \to \infty} \frac{n!}{k^n}\ h^0(X, kL).$$

As explained in the introduction, Fujita's theorem [Fuj] can be stated as follows.

Theorem 3.1 (Fujita's theorem)**.** *If L is any big line bundle on X, then*

$$\mathrm{vol}(L) = \langle L^n \rangle.$$

Since the function $\alpha \mapsto \langle \alpha^n \rangle$ is homogeneous of degree n and continuous on the big cone, it follows that it coincides with the (necessarily unique) continuous and n-homogeneous extension of the volume function to the big cone as constructed (using different arguments) in [L, §2.2.C]. Now positive intersection products are not continuous up to the boundary of the psef cone in general (see Example 3.8) and it is thus remarkable that the following holds:

Theorem 3.2. *The function $\alpha \mapsto \langle \alpha^n \rangle$ is strongly continuous on the psef cone of $CN^1(\mathfrak{X})$ and vanishes on its boundary (and only there).*

This result follows from [L, Cor. 2.2.45], and is also proved for arbitrary $(1,1)$-classes using analytic techniques in [Bou1]. Note that since $\alpha \mapsto \langle \alpha^n \rangle$ is non-negative and upper semi-continuous up to the boundary of the psef cone, its continuity follows in fact from its vanishing on the boundary. All in all we get that the function $\mathrm{vol} : CN^1(\mathfrak{X}) \to \mathbf{R}$ defined by

$$\mathrm{vol}(\alpha) := \langle \alpha^n \rangle$$

when α is psef, and

$$\mathrm{vol}(\alpha) := 0$$

when α is not psef, is continuous and coincides with the volume function defined in [L, §2.2.C].

3.2. Proof of Theorem A. Let us first recall the following fundamental Morse-type inequality:

Proposition 3.3. *For any two nef Cartier classes $A, B \in CN^1(\mathfrak{X})$, we have*

$$\mathrm{vol}(A - B) \geq (A^n) - n(A^{n-1} \cdot B).$$

We refer to [L, Proof of Theorem 2.2.15] for an elementary algebraic proof using the interpretation of the volume in terms of growth of sections. As an immediate consequence, we get the following sub-linear control of the volume near a nef class.

Corollary 3.4. *Let $\beta \in CN^1(\mathfrak{X})$ be a nef class, and $\gamma \in CN^1(\mathfrak{X})$ an arbitrary Cartier class. If $\omega \in CN^1(\mathfrak{X})$ is a given nef and big class such that $\beta \leq \omega$ and $\omega \pm \gamma$ is nef, then we have*

$$\mathrm{vol}(\beta + t\gamma) \geq (\beta^n) + nt(\beta^{n-1} \cdot \gamma) - Ct^2$$

for every $0 \leq t \leq 1$ and some constant $C > 0$ only depending on (ω^n).

Proof. We claim that $(\beta + t\gamma)^n = (\beta^n) + nt(\beta^{n-1} \cdot \gamma) + O(t^2)$ for $0 \leq t \leq 1$, with O only depending on (ω^n). Indeed, O is controlled by $(\beta^k \cdot \gamma^{n-k})$, $k = 0, \ldots, n-2$, and thus by $(\beta^k \cdot \omega^{n-k})$ thanks to Corollary 2.4, which in turn is bounded by (ω^n) according to Proposition 2.3 since $\beta \leq \omega$. Now we write $\beta + t\gamma$ as the difference of the two nef classes $A := \beta + t(\gamma + \omega)$ and $B := t\omega$. Then we also have

$$(\beta + t\gamma)^n = (A - B)^n = (A^n) - n(A^{n-1} \cdot B) + O(t^2)$$

with O only depending on (ω^n). Indeed, this O is controlled by $(A^k \cdot \omega^{n-k})$, $k = 0, \ldots, n-2$, and we have $A \leq 3\omega$, so we again get a control in terms of (ω^n) only. All in all we have

$$(A^n) - n(A^{n-1} \cdot B) = (\beta^n) + nt(\beta^{n-1} \cdot \gamma) + O(t^2)$$

and the result now follows from an application of Proposition 3.3 to $\beta + t\gamma = A - B$. $\square$

We are now in a position to prove our main result, Theorem A. We consider a big Cartier class $\alpha \in CN^1(\mathfrak{X})$ and an arbitrary Cartier class $\gamma \in CN^1(\mathfrak{X})$, and fix a sufficiently nef and big class $\omega \in CN^1(\mathfrak{X})$ such that $\alpha \le \omega$ and $\omega \pm \gamma$ is nef. If $\beta \le \alpha$ is a nef Cartier class, then *a fortiori* $\beta \le \omega$, and we deduce from Corollary 3.4 that

$$\mathrm{vol}(\alpha + t\gamma) \ge \mathrm{vol}(\beta + t\gamma) \ge (\beta^n) + nt(\beta^{n-1} \cdot \gamma) - Ct^2$$

for every $0 \le t \le 1$ and some constant $C > 0$ only depending on (ω^n). Taking the supremum over all such nef classes $\beta \le \alpha$ yields

$$\mathrm{vol}(\alpha + t\gamma) \ge \mathrm{vol}(\alpha) + nt\langle \alpha^{n-1} \rangle \cdot \gamma - Ct^2,$$

for some $C > 0$ only depending on (ω^n). This holds for every $0 \le t \le 1$, and in fact also for every $-1 \le t \le 1$, merely by replacing γ by $-\gamma$. Exchanging the roles of $\alpha + t\gamma \le 2\omega$ and $\alpha = (\alpha + t\gamma) + t(-\gamma)$, this yields

$$\mathrm{vol}(\alpha) \ge \mathrm{vol}(\alpha + t\gamma) - nt\langle (\alpha + t\gamma)^{n-1} \rangle \cdot \gamma - Ct^2,$$

for some possibly larger $C > 0$ also depending only on (ω^n). The combination of these two inequalities immediately shows that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma$$

as desired, because $\langle (\alpha + t\gamma)^{n-1} \rangle$ converges to $\langle \alpha^{n-1} \rangle$ by Proposition 2.9, α being big. We have thus shown that the volume admits a directional derivative in any direction, and that this derivative is given by the linear form $n \langle \alpha^{n-1} \rangle \in N^1(X)^*$. On the big cone, this form varies continuously, so vol is of class $\mathcal{C}^1$.

Remark 3.5. In fact, we have proved that the volume function admits a directional derivative in any direction in the infinite dimensional space $CN^1(\mathfrak{X})$, which is induced by a continuous linear form on $CN^1(\mathfrak{X})$. In particular, the restriction of vol to any finite dimensional subspace is $\mathcal{C}^1$.

As a consequence of Theorem A, we get the following orthogonality property, which was the key point in the characterization of pseudo-effectivity in [BDPP].

Corollary 3.6. *For any psef class $\alpha \in CN^1(\mathfrak{X})$, we have*

$$\langle \alpha^n \rangle = \langle \alpha^{n-1} \rangle \cdot \alpha.$$

Proof. It is enough to show this when α is big. Applying Theorem A to $\gamma := \alpha$ yields that $n\alpha \cdot \langle \alpha^{n-1} \rangle$ coincides with the derivative at $t = 0$ of $\langle (\alpha + t\alpha)^n \rangle = (1 + t)^n \langle \alpha^n \rangle$, which is of course nothing but $n\langle \alpha^n \rangle$. $\square$

3.3. The Diskant inequality. In this section, we prove the Diskant inequality (Theorem F), and its consequences Theorem D and Corollary E on the characterization of the equality case in the Khovanskii–Teissier inequalities.

Before starting the proof, we formalize one ingredient in the statement of Diskant inequality, and define the *slope* of a big class with respect to another. Recall that since the big cone is the interior of the psef cone by Proposition 1.7, if α and β are big classes, there exists $t > 0$ such that $t\beta \leq \alpha$.

Definition 3.7. The *slope* of β with respect to α is defined as

$$s = s(\alpha, \beta) = \sup\{t > 0 \mid \alpha \geq t\beta\}.$$

Since the psef cone is closed, we have $\alpha \geq s\beta$, and $\alpha - t\beta$ is big for $t < s$. Note that $\alpha = \beta$ iff $s(\alpha, \beta) = s(\beta, \alpha) = 1$.

Proof of Theorem D. Since $k \mapsto \log(\alpha^k \cdot \beta^{n-k})$ is concave, (i) and (ii) are equivalent. Moreover, (iii) trivially implies (i), so it only remains to prove that (i) implies (iii). By homogeneity we may assume that $(\alpha^n) = (\beta^n) = 1$, which implies $s(\alpha, \beta) \leq 1$ by Proposition 2.3. From (i) we get $(\alpha^{n-1} \cdot \beta) = 1$ so Diskant's inequality gives $s(\alpha, \beta) = 1$. By symmetry we get $s(\beta, \alpha) = 1$. Hence $\alpha = \beta$. □

Proof of Corollary E. Pick α, β big and nef with $((\alpha + \beta)^n)^{\frac{1}{n}} = (\alpha^n)^{\frac{1}{n}} + (\beta^n)^{\frac{1}{n}}$. Then

$$\begin{aligned} ((\alpha + \beta)^n) &= \sum_{k=0}^{n} \binom{n}{k} (\alpha^k \cdot \beta^{n-k}) \\ &\leq \sum_{k=0}^{n} \binom{n}{k} (\alpha^n)^{\frac{k}{n}} (\beta^n)^{\frac{n-k}{n}} = ((\alpha^n)^{\frac{1}{n}} + (\beta^n)^{\frac{1}{n}})^n, \end{aligned}$$

so the function $k \mapsto \log(\alpha^k \cdot \beta^{n-k})$ must be affine. By Theorem D, this implies that α and β are proportional. □

Proof of Theorem F. Set $\alpha_t := \alpha - t\beta$ for $t \geq 0$. By the definition of the slope $s = s(\alpha, \beta)$, α_t is big iff $t < s$. By Theorem A, $f(t) := \mathrm{vol}(\alpha_t)$ is differentiable for $t < s$, with $f'(t) = -n\langle \alpha_t^{n-1} \rangle \cdot \beta$. We also have $f(0) = (\alpha^n)$ and $f(t) \to 0$ as $t \to s_-$ by continuity so it follows that

$$(\alpha^n) = n \int_{t=0}^{s} \langle \alpha_t^{n-1} \rangle \cdot \beta \, dt.$$

If $\gamma_t \in CN^1(\mathfrak{X})$ is a nef class with $\gamma_t \leq \alpha_t = \alpha - t\beta$, then $\gamma_t + t\beta \leq \alpha$ implies

$$(\gamma_t^{n-1} \cdot \beta)^{\frac{1}{n-1}} + t(\beta^n)^{\frac{1}{n-1}} \leq (\alpha^{n-1} \cdot \beta)^{\frac{1}{n-1}}$$

by Theorem 2.16. Taking the supremum over all such nef classes γ_t yields

$$(\langle \alpha_t^{n-1} \rangle \cdot \beta)^{\frac{1}{n-1}} + t(\beta^n)^{\frac{1}{n-1}} \leq (\alpha^{n-1} \cdot \beta)^{\frac{1}{n-1}},$$

by Proposition 2.13. We thus obtain

$$(\alpha^n) \le n \int_{t=0}^{s} \left((\alpha^{n-1} \cdot \beta)^{1/n-1} - t(\beta^n)^{1/n-1} \right)^{n-1} dt$$

and the result follows since

$$\frac{d}{dt} \left((\alpha^{n-1} \cdot \beta)^{\frac{1}{n-1}} - t(\beta^n)^{\frac{1}{n-1}} \right)^n$$
$$= n(\beta^n)^{\frac{1}{n-1}} \left((\alpha^{n-1} \cdot \beta)^{\frac{1}{n-1}} - t(\beta^n)^{\frac{1}{n-1}} \right)^{n-1}.$$

□

3.4. Positive intersection and Zariski decomposition. Although the discussion to follow is strictly speaking not necessary for the understanding of the rest of the article, we would like to emphasize at this point that there is a very close relationship between positive intersection products and Zariski-type decompositions of psef classes. A survey of the different definitions of Zariski decompositions that have been proposed in higher dimension (and which all coincide for big classes) is given in [P], and a very complete account on the existence problem can be found in [N].

For a psef class $\alpha \in CN^1(\mathfrak{X})$, set $P(\alpha) := \langle \alpha \rangle \in N^1(\mathfrak{X})$, so that $P(\alpha) \le \alpha$. By definition, any nef Cartier class $\beta \in CN^1(\mathfrak{X})$ such that $\beta \le \alpha$ already satisfies $\beta \le P(\alpha)$. One easily deduces that the incarnation of the Weil class $P(\alpha)$ on each X_π coincides with the positive part of α_π in its divisorial Zariski decomposition, as first introduced by Nakayama (see [N], where it is called σ-decomposition), and independently for arbitrary $(1,1)$-classes via analytic tools by the first author in [Bou2]. This means that the collection of all the divisorial Zariski decompositions in all models gives rise to a decomposition $\alpha = P(\alpha) + N(\alpha)$ in $N^1(\mathfrak{X})$, where $N(\alpha)$ is (the class of) an effective Weil divisor on $\mathfrak{X}$.

We will say that α *admits a Zariski decomposition* if $P(\alpha)$ is a Cartier class on $\mathfrak{X}$. This is equivalent to requiring that the positive part of α_π in its divisorial Zariski decomposition on X_π is nef for some π, i.e. that $P(\alpha)$ is a nef Cartier class on $\mathfrak{X}$, a situation called "generalized Fujita decomposition" in [P]. When psef classes $\alpha_1, \dots, \alpha_p$ admit a Zariski decomposition, it is clear that

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle = P(\alpha_1) \cdot \ldots \cdot P(\alpha_p).$$

The Fujita theorem and the orthogonality relation (Corollary 3.6) can thus be formulated respectively as

$$\mathrm{vol}(\alpha) = P(\alpha)^n \text{ and } P(\alpha)^{n-1} \cdot N(\alpha) = 0$$

when α admits a Zariski decomposition (the terminology "orthogonality relation" in fact stems from this second equality).

In dimension $n = 2$, it follows from classical results that Zariski decompositions always exist. However, in higher dimensions, psef (or even big) Cartier classes do not admit a Zariski decomposition in general. A counter-example is constructed in [N], in which α is the big Cartier class induced by the tautological line bundle on $X := \mathbf{P}_S(L_1 \oplus L_2 \oplus L_3)$, the L_i being appropriate line bundles on an abelian surface S.

In the general case where classes do not necessarily admit Zariski decompositions, one can interpret the construction of positive intersection products as making sense of the intersection $P_1 \cdot \ldots \cdot P_p$ of the Weil classes $P_i = P(\alpha_i)$, even though it is definitely not possible to make sense of the intersection of arbitrary Weil classes on $\mathfrak{X}$.

Using the existence and characterization of Zariski decompositions on surfaces recalled above, we can give the following counter-example to continuity of positive intersection products up to the boundary of the psef cone.

Example 3.8. Let X be any projective surface with infinitely many exceptional curves, i.e. irreducible curves C_k such that $(C_k^2) < 0$ (for instance the blow-up of $\mathbf{P}^2$ at 9 points). If ω is a given ample class on X and $t_k := (\omega \cdot C_k)^{-1} > 0$, then $\beta_k := t_k C_k$ is bounded in $N^1(X)$, thus we can assume that it converges to a non-zero class $\beta \in N^1(X)$. Since the C_k are distinct, the limit β is nef, and thus $P(\beta) = \beta$. But as C_k is contractible, we get $P(\beta_k) = 0$, which shows that $\alpha \mapsto P(\alpha)$ is not continuous at β. If α is any ample class on X, we have $\langle \alpha \cdot \beta_k \rangle = \alpha \cdot P(\beta_k) = 0$, whereas $\langle \alpha \cdot \beta \rangle = \alpha \cdot \beta \neq 0$, and this shows indeed that the positive intersection product is not continuous at (α, β). Note that β lies on the boundary of the psef cone.

Beyond surfaces, toric varieties constitute an important class of varieties on which Zariski decompositions always exist. Indeed, if X is a projective toric variety, then the nef part of a toric psef class $\alpha \in CN^1(\mathfrak{X}_{\mathrm{tor}})$ is just the toric nef class $P(\alpha) \in CN^1(\mathfrak{X}_{\mathrm{tor}})$ associated to the Newton polytope $\mathrm{Nw}(\alpha)$ of α (cf. Section 1.5). In other words, if g is the homogeneous function on $\mathbf{R}^n$ corresponding to α, which is piecewise linear with respect to some rational fan decomposition Σ of $\mathbf{R}^n$, then its nef part corresponds to its convex minorant, i.e. the largest homogeneous and convex function $h \leq g$. This function h is also piecewise linear with respect to some refinement Σ' of the fan Σ. This means that if α is determined on some toric blow-up X_π of X, the Cartier class $P(\alpha)$ is determined on some higher toric blow-up $\pi' \geq \pi$, which cannot be taken to be π in general (as was the case for surfaces). On the other hand, for each fixed π the map $\alpha \mapsto P(\alpha)$ is piecewise linear on the psef cone of $N^1(X_\pi)$ with respect to the Gel′fand–Kapranov–Zelevinskij decomposition of [OP]. This implies in particular that the volume $\mathrm{vol}(\alpha) = P(\alpha)^n$ is piecewise polynomial on the psef cone of $N^1(X_\pi)$, as explained in [ELMNP2].

We conclude this section by describing the relationship between positive products and the Brunn–Minkowski theory. If L is a big line bundle on some toric blow-up X_π, then $h^0(X, L)$ is equal to the number of integral points in the Newton polytope $\mathrm{Nw}(L) \subset \mathbf{R}^n$. Since $\mathrm{Nw}(kL) = k\mathrm{Nw}(L)$, it follows that $\frac{\mathrm{vol}(L)}{n!} = \lim_{k\to\infty} h^0(X, kL)/k^n$ equals the Euclidean volume of $\mathrm{Nw}(L)$. This then extends by linearity and continuity to show that if $\alpha_i \in CN^1(\mathfrak{X}_{\mathrm{tor}})$, $i = 1, \dots, n$ are psef toric classes, the positive intersection product $\langle \alpha_1 \cdot \ldots \cdot \alpha_n \rangle$ $= P(\alpha_1) \cdot \ldots \cdot P(\alpha_n)$ is nothing but the mixed volume of their Newton polytopes $\mathrm{Nw}(\alpha_i)$ as defined in the Brunn–Minkowski theory (up to the factor $n!$); see [Sch].

4. Restricted volumes

4.1. Restriction of positive intersection products to divisors. From now on, we assume that X is a projective *normal* variety, and let D be a prime divisor on X. In this situation, we want to define a restriction map from classes on the Riemann–Zariski space $\mathfrak{X}$ of X to classes on the Riemann–Zariski space $\mathfrak{D}$ of D. For Cartier classes, this is easily done as follows. Since X is normal, any blow-up of X is an isomorphism over the generic point of D, and can be dominated by a blow-up $\pi : X_\pi \to X$ such that the strict transform D_π of D on X_π is smooth, by embedded resolution of singularities. The corresponding system of restriction maps $N^p(X_\pi) \to N^p(D_\pi)$ is compatible under pull-back, and thus defines a continuous restriction map on Cartier classes

$$CN^p(\mathfrak{X}) \to CN^p(\mathfrak{D})$$
$$\alpha \mapsto \alpha|_{\mathfrak{D}}.$$

However, it is not possible to extend this map to a continuous linear map on Weil classes $N^p(\mathfrak{X}) \to N^p(\mathfrak{D})$ $\alpha \mapsto \alpha|_{\mathfrak{D}}$ in general. Indeed, such a continuous extension is necessarily unique by density of Cartier classes, and writing a given Weil class $\alpha \in N^p(\mathfrak{X})$ as a limit of Cartier classes $\alpha = \lim_\pi \alpha_\pi$ as in Lemma 1.2 shows that $\alpha \mapsto \alpha|_{\mathfrak{D}}$ should satisfy

$$(\alpha|_{\mathfrak{D}})_\pi = \alpha_\pi|_{D_\pi}$$

for every blow-up π of X. But this relation already fails for Cartier classes, as can be seen, for instance, if X is $\mathbf{P}^2$, D is a line and α is the Cartier class on $\mathfrak{X}$ determined by the strict transform of D on the blow-up X_π of X at a point of D.

The goal of this section will be to show that it is possible to define the restriction to $\mathfrak{D}$ of positive intersection products $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle \in N^p(\mathfrak{X})$ of psef classes $\alpha_i \in CN^1(\mathfrak{X})$ under a suitable positivity assumption with respect to D. The main point is that the positive intersection product of big classes $\alpha_i \in CN^1(\mathfrak{X})$ is by definition a monotone limit of Cartier classes of the form $\beta_1 \cdot \ldots \cdot \beta_p$ with $\beta_i \le \alpha_i$ a nef Cartier class. We can therefore try to define the restriction of $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle$ to $\mathfrak{D}$ as the monotone limit of the Cartier classes $(\beta_1 \cdot \ldots \cdot \beta_p)|_{\mathfrak{D}}$. The trouble is that monotonicity is meant with respect to pseudo-effectivity, which is in general destroyed upon restricting to $\mathfrak{D}$. We thus introduce the following definitions.

Definition 4.1. If D is a prime divisor on a normal projective variety Y, we say that a class $\alpha \in N^1(Y)$ is *D-psef* and write $\alpha \ge_D 0$ iff it belongs to the closed convex cone generated by effective divisors whose support does not contain D. A class α is *D-big* if it lies in the interior of the D-psef cone.

It is clear that if α is D-psef, then $\alpha|_D$ is psef. Note also that a D-psef class is in particular psef, and thus that a D-big class is in particular big.

Remark 4.2. If $\alpha \in N^1(Y)$ is psef, then:

- α is D-psef iff D is not contained in the restricted base locus (or non-nef locus) of α (see [ELMNP2] and [Bou2]);
- α is D-big iff D is not contained in the augmented base locus (or non-ample locus) of α (see [ELMNP2] and [Bou2]).

If $\mu : Y' \to Y$ is a birational morphism, and if D' denotes the strict transform of D, then $\mu^*\alpha$ is D'-psef iff $\alpha \in N^1(Y)$ is D-psef. Considering again a normal projective variety X and a prime divisor D on X, it follows that the following definition makes sense:

Definition 4.3. A class $\alpha \in CN^1(\mathfrak{X})$ is $\mathfrak{D}$-psef (resp. $\mathfrak{D}$-big) iff there exists a determination π of α such that $\alpha_\pi \in N^1(X_\pi)$ is D_π-psef (resp. D_π-big).

We will write $\alpha \ge_{\mathfrak{D}} 0$ if $\alpha \in CN^1(\mathfrak{X})$ is $\mathfrak{D}$-psef. Note that a class in $CN^1(\mathfrak{X})$ is $\mathfrak{D}$-big iff it belongs to the interior of the $\mathfrak{D}$-psef cone in the strong topology.

Proceeding as in Section 2, one shows that the following definition makes sense.

Definition 4.4. If $\alpha_1, \ldots, \alpha_p \in CN^1(\mathfrak{X})$ are $\mathfrak{D}$-big classes, one defines their restricted positive intersection product on $\mathfrak{D}$,

$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle|_{\mathfrak{D}} \in N^p(\mathfrak{D})$$

as the least upper bound of the set of classes

$$(\beta_1 \cdot \ldots \cdot \beta_p)|_{\mathfrak{D}} \in N^p(\mathfrak{D})$$

where $\beta_i \in CN^1(\mathfrak{X})$ is a nef class such that $\beta_i \le_{\mathfrak{D}} \alpha_i$.

Remark 4.5. The $\mathfrak{D}$-big classes $\alpha_i \in CN^1(\mathfrak{X})$ restrict to big classes $\alpha_i|_{\mathfrak{D}} \in CN^1(\mathfrak{D})$, so we can also consider their positive intersection on $\mathfrak{D}$, to wit

$\langle\alpha_1|_{\mathfrak{D}}\cdot\ldots\cdot\alpha_p|_{\mathfrak{D}}\rangle\in N^p(\mathfrak{D})$. It is the least upper bound of the set of classes $\gamma_1\cdot\ldots\cdot\gamma_p\in N^p(\mathfrak{D})$ where $\gamma_i\in CN^1(\mathfrak{D})$ is nef and such that $\gamma_i\le\alpha_i|_{\mathfrak{D}}$. The point of Definition 4.4 above is that we only consider nef classes $\gamma_i\le\alpha_i|_{\mathfrak{D}}$ that are *restrictions* to $\mathfrak{D}$ of nef classes $\beta_i\le_{\mathfrak{D}}\alpha_i$ on $\mathfrak{X}$. In particular, it follows that

$$\langle\alpha_1\cdot\ldots\cdot\alpha_p\rangle|_{\mathfrak{D}}\le\langle\alpha_1|_{\mathfrak{D}}\cdot\ldots\cdot\alpha_p|_{\mathfrak{D}}\rangle$$

in $N^p(\mathfrak{D})$, but equality does not hold in general.

Since the restricted positive intersection product is homogeneous and increasing in each variable (with respect to $\ge_{\mathfrak{D}}$), continuity holds as in Proposition 2.9.

Proposition 4.6. *The restricted positive intersection product* $\langle\alpha_1\cdot\ldots\cdot\alpha_p\rangle|_{\mathfrak{D}}\in N^p(\mathfrak{D})$ *depends continuously on the* $\mathfrak{D}$*-big classes* $\alpha_i\in CN^1(\mathfrak{X})$.

Again the definition can be extended to $\mathfrak{D}$-psef classes $\alpha_i\in CN^1(\mathfrak{X})$ by setting

$$\langle\alpha_1\cdot\ldots\cdot\alpha_p\rangle|_{\mathfrak{D}}:=\lim_{\varepsilon\to0+}\langle(\alpha_1+\varepsilon\omega)\cdot\ldots\cdot(\alpha_p+\varepsilon\omega)\rangle|_{\mathfrak{D}}$$

for $\omega\in CN^1(\mathfrak{X})$ a $\mathfrak{D}$-big class. Indeed, the limit in question does not depend on the choice of ω. It depends upper semi-continuously on the $\mathfrak{D}$-psef classes α_i, but continuity does not hold up to the boundary in general.

Proposition 4.7. *If* $\alpha_1,\ldots,\alpha_p\in CN^1(\mathfrak{X})$ *are nef, then*

$$\langle\alpha_1\cdot\ldots\cdot\alpha_p\rangle|_{\mathfrak{D}}=(\alpha_1\cdot\ldots\cdot\alpha_p)|_{\mathfrak{D}}.$$

We also note that the concavity properties of Section 2.4 also hold in this setting, again because the Khovanskii–Teissier inequalities for nef classes hold on $\mathfrak{D}$.

We will use the following easy inequality:

Proposition 4.8. *Let D be a prime divisor on X and assume that D is also Cartier. If* $\alpha_1,\ldots,\alpha_{n-1}\in CN^1(\mathfrak{X})$ *are* $\mathfrak{D}$*-psef classes, then we have*

$$\langle\alpha_1\cdot\ldots\cdot\alpha_{n-1}\rangle|_{\mathfrak{D}}\le\langle\alpha_1\cdot\ldots\cdot\alpha_{n-1}\rangle\cdot D.$$

Proof. It is enough to consider the case of $\mathfrak{D}$-big classes α_i. Pick $\beta_i\in CN^1(\mathfrak{X})$ a nef class with $\beta_i\le_{\mathfrak{D}}\alpha_i$. If π is a determination of β_i, then

$$\begin{aligned}(\beta_1\cdot\ldots\cdot\beta_{n-1})|_{\mathfrak{D}}&=\beta_{1,\pi}\cdot\ldots\cdot\beta_{n-1,\pi}\cdot D_\pi\\&\le\beta_{1,\pi}\cdot\ldots\cdot\beta_{n-1,\pi}\cdot\pi^*D\le\langle\alpha_1\cdot\ldots\cdot\alpha_{n-1}\rangle\cdot D,\end{aligned}$$

where D_π denotes as before the strict transform of D on X_π. The desired inequality follows by taking the supremum over all such nef classes β_i. □

We can now give the following characterization of $\mathfrak{D}$-big classes.

Theorem 4.9. *If* $\alpha\in CN^1(\mathfrak{X})$ *is big and* $\mathfrak{D}$*-psef, then* α *is* $\mathfrak{D}$*-big iff* $\langle\alpha^{n-1}\rangle|_{\mathfrak{D}}>0$.

Proof. If α is $\mathfrak{D}$-big, then $\alpha \geq_{\mathfrak{D}} \omega$ is, for some nef class $\omega \in CN^1(\mathfrak{X})$ which is determined by an ample class on X. We thus have $\langle \alpha^{n-1} \rangle|_{\mathfrak{D}} \geq (\omega|_D)^{n-1} > 0$. Conversely, assume that α is big and $\mathfrak{D}$-psef but not $\mathfrak{D}$-big, and let us show that $\langle \alpha^{n-1} \rangle|_{\mathfrak{D}} = 0$. Upon replacing X by some higher birational model, we can also assume that α is determined on X, and that X is smooth, so that D is in particular Cartier. In view of the preceding proposition, the result follows from Lemma 4.10. $\square$

Lemma 4.10. *If D is Cartier on X and α is a big class determined on X which is not D-big, then $\langle \alpha^{n-1} \rangle \cdot D = 0$.*

Proof. If ω is an ample class on X, then $\alpha - \varepsilon\omega$ is big but not D-psef for every small enough $\varepsilon > 0$. By continuity, it is thus enough to show that $\langle \alpha^{n-1} \rangle \cdot D = 0$ for any big class α on X which is not D-psef. But in that case, D is contained in the non-nef locus of α, i.e. $\alpha \geq P + tD$ for some $t > 0$ if we denote by P the incarnation on X of $\langle \alpha \rangle$ (which coincides with the positive part in the divisorial Zariski decomposition of α, cf. [Bou2]). It follows that $\langle \alpha^{n-1} \rangle \cdot \alpha \geq \langle P^{n-1} \rangle \cdot P + \langle \alpha^{n-1} \rangle \cdot tD$. But we also have $\langle P^{n-1} \rangle \cdot P = \langle P^n \rangle = \langle \alpha^n \rangle = \langle \alpha^{n-1} \rangle \cdot \alpha$ by the orthogonality relation (Corollary 3.6) applied to P and α, and thus we get $\langle \alpha^{n-1} \rangle \cdot D = 0$ as claimed. $\square$

As noticed after Theorem 3.2, this implies the following.

Corollary 4.11. *The function $\alpha \mapsto \langle \alpha^{n-1} \rangle|_{\mathfrak{D}}$ is continuous on the cone of big and $\mathfrak{D}$-psef classes of $CN^1(\mathfrak{X})$.*

Remark 4.12. The theorem fails when α is not big. For instance, if X is a smooth surface, D is ample and $\alpha \in CN^1(\mathfrak{X})$ is nef but not big, then $\langle \alpha \rangle|_{\mathfrak{D}} = \alpha \cdot D > 0$ as soon as α is non-zero, but α is not $\mathfrak{D}$-big since it is not even big.

Remark 4.13. The lemma fails if α is not determined on X. For instance, if X is a ruled surface, D is a ruling and $\pi : X_\pi \to X$ is the blow-up of X at a point $p \in D$, with exceptional divisor E, then the strict transform D_π of D can be blown-down by $\mu : X_\pi \to X'$. If $\alpha_\pi := \mu^*\omega$ for some ample class ω on X', then clearly the Cartier class $\alpha \in CN^1(\mathfrak{X})$ determined by α_π is nef and big, but not $\mathfrak{D}$-big. However, we have

$$\alpha \cdot D = \alpha_\pi \cdot (D_\pi + E) = \omega \cdot E' > 0$$

with E' the image of E on X'.

4.2. Restricted volumes. Again, let X be a normal projective variety and let D be a prime divisor. Given a line bundle L on X, we will denote by $h^0(X|D, L)$ the rank of the restriction map

$$H^0(X, L) \to H^0(D, L|_D).$$

Recall from the introduction that the restricted volume of L on D is then defined as

$$\operatorname{vol}_{X|D}(L) := \limsup_{k\to\infty} \frac{(n-1)!}{k^{n-1}}\ h^0(X|D, kL).$$

It is thus the growth coefficient of the number of sections of $\mathcal{O}_D(kL)$ on D that extend to X. As stated in the introduction, we have the following theorem.

Theorem 4.14 (Generalized Fujita's Theorem, [ELMNP3] and [Ta])**.** *If L is a D-big line bundle, then*

$$\operatorname{vol}_{X|D}(L) = \langle L^{n-1}\rangle|_{\mathfrak{D}}. \tag{4.1}$$

This result is in fact established for a *smooth* projective variety X, but it immediately extends to the case when X is merely normal in view of the relation $\operatorname{vol}_{X|D}(L) = \operatorname{vol}_{X'|D'}(\mu^* L)$ if $\mu : X' \to X$ is a blow-up and D' is the strict transform of D (because μ has connected fibers, X being normal). The theorem shows in particular that $\operatorname{vol}_{X|D}(L)$ only depend on $c_1(L) \in N^1(X)$, and it follows that $\alpha \mapsto \langle \alpha^{n-1}\rangle|_{\mathfrak{D}}$ is the (necessarily unique) extension of the restricted volume to an $(n-1)$-homogeneous and continuous function on the open cone of D-big classes.

Using Theorem 4.9, we now extend this to arbitrary big line bundles.

Theorem 4.15. *If L is a big and D-psef line bundle, then* (4.1) *holds. In particular, if L is a big line bundle, then L is D-big iff* $\operatorname{vol}_{X|D}(L) > 0$.

As noted before, L is D-big iff D is not contained in the augmented base locus of L. Hence, we recover with a different proof, a special case of the main result of [ELMNP3]. (The general case deals with irreducible components of the augmented base locus of any codimension.)

Remark 4.16. If a line bundle L is not D-psef, then clearly $\operatorname{vol}_{X|D}(L) = 0$, so the theorem shows that $\operatorname{vol}_{X|D}(L)$ only depends on the numerical class of the big line bundle L. It is worthwhile to note that this fails for non-big line bundles in general. For instance, if D is (Cartier and) ample and L is not big, we have $h^0(X, kL - D) = 0$ for each k, and thus $h^0(X|D, kL) = h^0(X, kL)$. Now if $X = C_1 \times C_2$ is a product of two smooth curves and L is the sum of the pull-back of an ample bundle L_1 on C_1 and the pull-back of a numerically trivial bundle L_2 on C_2, then $\operatorname{vol}_{X|D}(L) = \limsup_{k\to\infty} \frac{1}{k} h^0(X, kL) = 0$ when L_2 is not torsion in $\operatorname{Pic}^0(C_2)$, whereas $\operatorname{vol}_{X|D}(L) = \limsup_{k\to\infty} \frac{1}{k} h^0(X, kL) = \deg L_1 > 0$ when L_2 is torsion. Note that L is nef, so that we have $\langle L\rangle|_{\mathfrak{D}} = L\cdot D = (D\cdot F_1)\deg L_1$, where F_1 denotes the fiber of the projection $X \to C_1$.

Proof of Theorem 4.15. Let L be an arbitrary D-psef line bundle. We claim that

$$\langle L^{n-1}\rangle|_{\mathfrak{D}} \geq \operatorname{vol}_{X|D}(L). \tag{4.2}$$

Indeed, if A is an ample divisor on X, then clearly $\mathrm{vol}_{X|D}(L+\varepsilon A) \geq \mathrm{vol}_{X|D}(L)$ for every rational $\varepsilon > 0$. Since $L + \varepsilon A$ is D-big, the left-hand side coincides with $\langle (L+\varepsilon A)^{n-1}\rangle|_{\mathfrak{D}}$ by Theorem 4.14. Thus (4.2) follows by letting $\varepsilon \to 0$, by definition of $\langle L^{n-1}\rangle|_{\mathfrak{D}}$ for the $\mathfrak{D}$-psef class determined by L.

Now if L is big and D-psef but not D-big, then $\langle L^{n-1}\rangle|_{\mathfrak{D}} = 0$ by Theorem 4.9, and thus (4.2) gives $\mathrm{vol}_{X|D}(L) = 0$, which completes the proof. □

4.3. Proof of Theorem B. Let L be a big line bundle on X. We also assume that the prime divisor D is Cartier on X, which of course holds if X is smooth. We have to show that $\mathrm{vol}_{X|D}(L) = \langle L^{n-1}\rangle \cdot D$. If L is not D-big, then the left-hand side is zero by Theorem 4.15, and the right-hand side is zero by Lemma 4.10. We can thus assume that L is D-big. In that case, we have $\langle L^{n-1}\rangle \cdot D \geq \langle L^{n-1}\rangle|_{\mathfrak{D}} = \mathrm{vol}_{X|D}(L)$ by Proposition 4.8 and the generalized Fujita theorem. In order to prove the converse inequality, we rely on the following two simple remarks. First, for every line bundle M the kernel of the restriction map $H^0(X, M) \to H^0(D, M|_D)$ is precisely $H^0(X, M - D)$, hence

$$h^0(X, M) - h^0(X, M - D) = h^0(X|D, M). \tag{4.3}$$

Second, for any effective divisor B whose support does not contain D, multiplication by the canonical section σ of $H^0(X, B)$ yields injections $H^0(X, M - B) \hookrightarrow H^0(X, M)$ and $H^0(D, M - B) \hookrightarrow H^0(D, M)$. This implies the second basic relation:

$$h^0(X|D, M - B) \leq h^0(X|D, M). \tag{4.4}$$

Now fix an integer k, apply (4.3) to $M = kL - jD$ for $j = 0, \dots, k$, and sum all these relations. This gives

$$h^0(X, kL) - h^0(X, k(L - D)) = \sum_{j=0}^{k-1} h^0(X|D, kL - jD).$$

Fix a sufficiently ample divisor A not containing D in its support such that $A - D$ is linearly equivalent to an effective divisor B not containing D in its support. By repeated use of (4.4), we get $h^0(X|D, kL - jD) \leq h^0(X|D, kL - jD + jA) = h^0(X|D, kL + jB) \leq h^0(X|D, kL + kB)$ for each $j \leq k$. Hence

$$h^0(X, kL) - h^0(X, k(L - D)) \leq k\, h^0(X|D, k(L + B)).$$

Dividing by k^n, and taking the limit when $k \to \infty$, we infer

$$\mathrm{vol}(L) - \mathrm{vol}(L - D) \leq n\, \mathrm{vol}_{X|D}(L + B).$$

Replacing L by kL, and expressing the (restricted) volumes as (restricted) positive intersection products, we finally conclude that

$$\frac{1}{k}\left(\langle L^n\rangle - \langle (L-\frac{1}{k}D)^n\rangle\right) \le n\langle (L+\frac{1}{k}B)^{n-1}\rangle|_{\mathfrak{D}}.$$

When $k\to\infty$, the left-hand side tends to $n\,\langle L^{n-1}\rangle\cdot D$ by Theorem A, whereas the right-hand side converges to $n\,\langle L^{n-1}\rangle|_{\mathfrak{D}} = n\,\mathrm{vol}_{X|D}(L)$ by continuity (Proposition 4.6). This proves the required inequality $\langle L^{n-1}\rangle\cdot D \le \mathrm{vol}_{X|D}(L)$ and concludes the proof.

Example 4.17 (Surfaces)**.** In dimension $n=2$, a big class $\alpha\in CN^1(X)$ is not D-big iff $P(\alpha)\cdot D=0$, where $P(\alpha)$ is the nef part of its Zariski decomposition. In particular, α is always D-big when D is ample (or even nef by the Hodge index theorem). Now the continuous extension of $L\mapsto \mathrm{vol}_{X|D}(L)$ to the big cone of $CN^1(X)$, to wit $\alpha\mapsto (P(\alpha)\cdot D)$, is not $\mathcal{C}^1$ in general on the open cone of D-big classes, because $\alpha\mapsto P(\alpha)$ is not $\mathcal{C}^1$ in general, as exemplified by the blow-up of $\mathbf{P}^2$ at a point. This means that the analogue of Theorem A fails for restricted volumes.

Acknowledgment

It is a great pleasure to thank the authors of [ELMNP3] for sharing their work with us. Their ideas have obviously had a great influence on the present paper. In particular, the statement of Corollary C was suggested to us by R. Lazarsfeld and M. Mustaţă, and they informed us after completion of the present paper that they had independently obtained a proof of it. We also thank B. Teissier for explaining the history of the problems considered here; S. Takayama for interesting discussions and for sharing his preprint; and F. Wittke for spotting a mistake in a previous version of the paper.

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