DEGREE GROWTH OF MEROMORPHIC SURFACE MAPS

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Abstract

We study the degree growth of iterates of meromorphic self-maps of compact Kähler surfaces. Using cohomology classes on the Riemann-Zariski space, we show that the degrees grow similarly to those of mappings that are algebraically stable on some bimeromorphic model.

0. Introduction

Let X be a compact Kähler surface, and let $F : X \dashrightarrow X$ be a dominant meromorphic mapping. Fix a Kähler class ω on X, normalized by $(\omega^2)_X = 1$, and define the *degree* of F with respect to ω to be the positive real number

$$\deg_\omega(F) := (F^*\omega \cdot \omega)_X = (\omega \cdot F_*\omega)_X,$$

where $(\cdot)_X$ denotes the intersection form on $H^{1,1}_{\mathbf{R}}(X)$. When $X = \mathbf{P}^2$ and ω is the class of a line, this coincides with the usual algebraic degree of F. One can show that $\deg_\omega(F^{n+m}) \leq 2\deg_\omega(F^n)\deg_\omega(F^m)$ for all m, n. Hence the limit

$$\lambda_1 := \lim_{n\to\infty} \deg_\omega(F^n)^{1/n}$$

exists. We refer to it as the *asymptotic degree* of F. It follows from standard arguments (see Proposition 3.1) that λ_1 does not depend on the choice of ω, that λ_1 is invariant under bimeromorphic conjugacy, and that $\lambda_1^2 \geq \lambda_2$, where λ_2 is the topological degree of F.

MAIN THEOREM
Assume that $\lambda_1^2 > \lambda_2$. Then there exists a constant $b = b(\omega) > 0$ such that

$$\deg_\omega(F^n) = b\lambda_1^n + O(\lambda_2^{n/2}) \quad \textit{as } n \to \infty.$$

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The dependence of b on ω can be made explicit (see Remark 3.7). For the polynomial map $F(x, y) = (x^d, x^d y^d)$ on $\mathbf{C}^2$ (with ω the standard Fubini-Study form), one has $\lambda_2 = \lambda_1^2 = d^2$, $\deg_\omega(F^n) = nd^n$; hence the assertion in the main theorem may fail when $\lambda_1^2 = \lambda_2$.

Degree growth is an important component in the understanding of the complexity and dynamical behavior of a self-map and has been studied in a large number of works in both mathematics and physics literature. It is connected to topological entropy (see, e.g., [Fr], [G1], [G2], [DS]), and controlling it is necessary in order to construct interesting invariant measures and currents (see, e.g., [BF], [FS], [RS], [S]). Even in simple families of mappings, degree growth exhibits a rich behavior (see, e.g., the articles by Bedford and Kim [BK1], [BK2], which also contain references to the physics literature).

In [FS], Fornaess and Sibony connected the degree growth of rational self-maps to the interplay between contracted hypersurfaces and indeterminacy points. In particular, they proved that $\deg(F^n)$ is multiplicative if and only if F is what is now often called (algebraically) *stable*. This analysis was extended to slightly more general maps in [N]. Bonifant and Fornaess [BF] showed that only countably many sequences $(\deg(F^n))_1^\infty$ can occur, but in general, the precise picture is unclear.

For bimeromorphic maps of surfaces, the situation is quite well understood since the work of Diller and Favre [DF]. Using the factorization into blowups and blowdowns, they proved that any such map can be made stable by a bimeromorphic change of coordinates. This reduces the study of degree growth to the spectral properties of the induced map on the Dolbeault cohomology $H^{1,1}$. In particular, it implies that λ_1 is an algebraic integer and that $\deg(F^n)$ satisfies an integral recursion formula and gives a stronger version of our main theorem when $\lambda_1^2 > 1\,(= \lambda_2)$.

In the case that we consider, namely, (noninvertible) meromorphic surface maps, there are counterexamples to stability when $\lambda_1^2 = \lambda_2 > 1$ (see [F]). It is an interesting (and probably difficult) question whether counterexamples also exist with $\lambda_1^2 > \lambda_2 > 1$.

Instead of looking for a particular birational model in which the action of F^n on $H^{1,1}$ can be controlled, we take a different tack and study the action of F on cohomology classes on all modifications $\pi : X_\pi \to X$ at the same time. This idea already appeared in the study of cubic surfaces in [M] and was recently used by Cantat as a key tool in his investigation of the group of birational transformation of surfaces (see [C2]). In the context of noninvertible maps, Hubbard and Papadopol [HP] used similar ideas, but their methods apply only to a quite restricted class of maps.

Here we show that F acts (functorially) by pullback F^* and pushforward F_* on the vector space $W := \varprojlim H^{1,1}_{\mathbf{R}}(X_\pi)$ and on its dense subspace $C := \varinjlim H^{1,1}_{\mathbf{R}}(X_\pi)$. Compactness properties of W imply the existence of eigenvectors having eigenvalue λ_1 and certain positivity properties.

Following [DF], we then study the spectral properties of F^* and F_* under the assumption that $\lambda_1^2 > \lambda_2$. The space W is too big for this purpose, and we introduce a subspace L^2 that is the completion of C with respect to the (indefinite) inner product induced by the cup product, which is of Minkowski type by the Hodge index theorem. The main theorem then follows from the spectral properties of F^* and its adjoint F_* on L^2.

Using a different method, polynomial mappings of $\mathbf{C}^2$ were studied in detail by Favre and Jonsson in [FJ4]: in that case, λ_1 is a quadratic integer. However, our main theorem for polynomial maps does not immediately follow from the analysis in [FJ4]; the methods of the two articles can be viewed as complementary.

The space W above can be thought of as the Dolbeault cohomology $H^{1,1}$ of the *Riemann-Zariski space* of X. While we do not need the structure of the latter space in this article, the general philosophy of considering all bimeromorphic models at the same time is very useful for handling asymptotic problems in geometry, analysis, and dynamics (see [BFJ], [C1], [M], [FJ1], [FJ2], [FJ3]). In the present setting, it allows us to bypass the intricacies of indeterminacy points: heuristically, a meromorphic map becomes holomorphic on the Riemann-Zariski space.

The article is organized in three sections. In the first, we recall some definitions and introduce cohomology classes on the Riemann-Zariski space. In the second, we study the actions of meromorphic mappings on these classes. Finally, Section 3 deals with the spectral properties of these actions under iteration, concluding with the proof of the main theorem.

Remark on the setting. We choose to state our main result in the context of a complex manifold because the study of degree growth is particularly important for applications to holomorphic dynamics. However, our methods are purely algebraic, so that our main result actually holds in the case when X is a projective surface over any algebraically closed field of any characteristic, and $\omega = c_1(L)$ for some ample line bundle. In this context, one has to replace $H^{1,1}_{\mathbf{R}}(X)$ with the real Néron-Severi vector space and work with the suitable notion of pseudoeffective and nef classes, as defined in [L, Sections 1.4, 2.2].

1. Classes on the Riemann-Zariski space

Let X be a complex compact Kähler surface (for background, see [BHPV]), and write $H^{1,1}_{\mathbf{R}}(X) := H^{1,1}(X) \cap H^2(X, \mathbf{R})$.

1.1. The Riemann-Zariski space

By a *blowup* of X we mean a bimeromorphic morphism $\pi : X_\pi \to X$, where X_π is a smooth surface. Up to isomorphism, π is then a finite composition of point blowups. If π and π' are two blowups of X, we say that π' *dominates* π and write $\pi' \geq \pi$ if there exists a bimeromorphic morphism $\mu : X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \mu$. The

Riemann-Zariski space of X is the projective limit

$$\mathfrak{X} := \varprojlim_{\pi} X_\pi .$$

While suggestive, the space $\mathfrak{X}$ is, strictly speaking, not needed for our analysis, and we refer to [ZS, Chapter 6, Section 17], [V, Section 7] for details on its structure.

1.2. Weil and Cartier classes

When one blowup $\pi' = \pi \circ \mu$ dominates another one π, we have two induced linear maps, $\mu_* : H^{1,1}_{\mathbf{R}}(X_{\pi'}) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ and $\mu^* : H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(X_{\pi'})$, which satisfy the projection formula $\mu_*\mu^* = \mathrm{id}$. This allows us to define the following spaces.

Definition 1.1

The space of Weil classes on $\mathfrak{X}$ is the projective limit

$$W(\mathfrak{X}) := \varprojlim_{\pi} H^{1,1}_{\mathbf{R}}(X_\pi)$$

with respect to the pushforward arrows. The space of Cartier classes on $\mathfrak{X}$ is the inductive limit

$$C(\mathfrak{X}) := \varinjlim_{\pi} H^{1,1}_{\mathbf{R}}(X_\pi)$$

with respect to the pullback arrows.

The space $W(\mathfrak{X})$ is endowed with its projective limit topology, that is, the coarsest topology for which the projection maps $W(\mathfrak{X}) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ are continuous. There is also an inductive limit topology on $C(\mathfrak{X})$, but we do not use it.

Concretely, a Weil class $\alpha \in W(\mathfrak{X})$ is given by its *incarnations* $\alpha_\pi \in H^{1,1}_{\mathbf{R}}(X_\pi)$, compatible by pushforward; that is, $\mu_* a_{\pi'} = \alpha_\pi$ whenever $\pi' = \pi \circ \mu$. The topology on $W(\mathfrak{X})$ is characterized as follows: a sequence (or net[†]) $\alpha_j \in W(\mathfrak{X})$ converges to $\alpha \in W(\mathfrak{X})$ if and only if $\alpha_{j,\pi} \to \alpha_\pi$ in $H^{1,1}_{\mathbf{R}}(X_\pi)$ for each blowup π.

The projection formula recalled above shows that there is an injection $C(\mathfrak{X}) \subset W(\mathfrak{X})$, so that a Cartier class is, in particular, a Weil class. In fact, if $\alpha \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is a class in some blowup X_π of X, then α defines a Cartier class, also denoted α, whose incarnation $\alpha_{\pi'}$ in any blowup $\pi' = \pi \circ \mu$ dominating π is given by $\alpha_{\pi'} = \mu^*\alpha$. We say that α is *determined* in X_π. (It is then also determined in $X_{\pi'}$ for any blowup dominating π.) Each Cartier class is obtained that way. The space $C(\mathfrak{X})$ is dense in $W(\mathfrak{X})$: if α is a given Weil class, the net α_π of Cartier classes determined by the incarnations of α on all models X_π tautologically converges to α in $W(\mathfrak{X})$.

[†]A net is a family indexed by a directed set (see [Fo]).

Remark 1.2

The spaces of Weil classes and Cartier classes are denoted $Z_\bullet(X)$ and $Z^\bullet(X)$ by Manin [M]. He views these classes as living on the *bubble space* $\varinjlim X_\pi$ rather than the Riemann-Zariski space $\varprojlim X_\pi$.

1.3. Exceptional divisors

This section can be skipped on a first reading, the main technical issue being Proposition 1.6, which is used for the proof of Theorem 3.2.

The spaces $C(\mathfrak{X})$ and $W(\mathfrak{X})$ are clearly bimeromorphic invariants of X. Once the model X is fixed, an alternative and somewhat more explicit description of these spaces can be given in terms of exceptional divisors.

Definition 1.3

The set $\mathscr{D}$ of exceptional primes over X is defined as the set of all exceptional prime divisors of all blowups $X_\pi \to X$ modulo the following equivalence relation: two divisors E and E' on X_π and $X_{\pi'}$ are equivalent if the induced meromorphic map $X_\pi \dashrightarrow X_{\pi'}$ sends E onto E'.

When X is a projective surface, $\mathscr{D}$ is the set of divisorial valuations on the function field $\mathbf{C}(X)$ whose center on X is a point.

If $E \in \mathscr{D}$ is an exceptional prime and X_π is any model of X, one can consider the *center* of E on X_π, denoted by $c_\pi(E)$. It is a subvariety defined as follows. Choose a blowup $\pi' \geq \pi$ so that E appears as a curve on $X_{\pi'}$. Then $c_\pi(E)$ is defined as the image of $E \subset X_{\pi'}$ by the map $X_{\pi'} \to X_\pi$. It does not depend on the choice of π' and is either a point or an irreducible curve. In this 2-dimensional setting, there is a unique minimal blowup π_E such that $c_\pi(E)$ is a curve if and only if $\pi \geq \pi_E$. (In particular, $c_{\pi_E}(E)$ is a curve.)

Using these facts, one can construct an explicit basis for the vector space $C(\mathfrak{X})$ as follows (cf. [M, Proposition 35.6]). Let $\alpha_E \in C(\mathfrak{X})$ be the Cartier class determined by the class of E on X_{π_E}. Write $\mathbf{R}^{(\mathscr{D})}$ for the direct sum $\bigoplus_{\mathscr{D}} \mathbf{R}$ or, equivalently, for the space of real-valued functions on $\mathscr{D}$ with finite support.

PROPOSITION 1.4

The set $\{\alpha_E \mid E \in \mathscr{D}\}$ is a basis for the vector space of Cartier classes $\alpha \in C(\mathfrak{X})$ which are exceptional over X, that is, whose incarnations on X vanish. In other words, the map $H^{1,1}_{\mathbf{R}}(X) \oplus \mathbf{R}^{(\mathscr{D})} \to C(\mathfrak{X})$, sending $\alpha \in H^{1,1}_{\mathbf{R}}(X)$ to the Cartier class it determines, and $E \in \mathscr{D}$ to α_E is an isomorphism.

We now describe $W(\mathfrak{X})$ in terms of exceptional primes. If $\alpha \in W(\mathfrak{X})$ is a given Weil class, let $\alpha_X \in H^{1,1}_{\mathbf{R}}(X)$ be its incarnation on X. For each π, the Cartier class

$\alpha_\pi - \alpha_X$ is determined on X_π by a unique $\mathbf{R}$-divisor Z_π exceptional over X. If E is a π-exceptional prime, we set $\operatorname{ord}_E(\alpha) := \operatorname{ord}_E(Z_\pi)$ so that $Z_\pi = \sum_E \operatorname{ord}_E(Z_\pi)E$. It is easily seen to depend only on the class of E in $\mathscr{D}$. Let $\mathbf{R}^{\mathscr{D}}$ denote the (product) space of all real-valued functions on $\mathscr{D}$. We obtain a map $W(\mathfrak{X}) \to H^{1,1}_{\mathbf{R}}(X) \times \mathbf{R}^{\mathscr{D}}$, which is easily seen to be a bijection, and even naturally a homeomorphism, as the following straightforward lemma shows.

LEMMA 1.5
A net $\alpha_j \in W(\mathfrak{X})$ converges to $\alpha \in W(\mathfrak{X})$ if and only if $\alpha_{j,X}$ converges to α_X in $H^{1,1}_{\mathbf{R}}(X)$ and $\operatorname{ord}_E(\alpha_j) \to \operatorname{ord}_E(\alpha)$ for each exceptional prime $E \in \mathscr{D}$.

A result of Zariski (cf. [Ko, Theorem 3.17], [FJ1, Proposition 1.12]) states that the process of successively blowing up the center of a given exceptional prime $E \in \mathscr{D}$ starting from any given model must stop after finitely many steps with the center becoming a curve. In other words, if $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ is an infinite sequence of blowups such that the center of each blowup $X_n \leftarrow X_{n+1}$ meets $c_{X_n}(E)$, then X_n must dominate X_{π_E} for n large enough. Using this result, we record the following fact, which is used later on in the article.

PROPOSITION 1.6
Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ be an infinite sequence of blowups, and for each n, suppose that $\alpha_n \in C(\mathfrak{X})$ is a Cartier class that is determined in X_{n+1} and whose incarnation on X_n is zero. Then $\alpha_n \to 0$ in $W(\mathfrak{X})$ as $n \to \infty$.

Proof
In view of Proposition 1.5, we have to show that for every given exceptional prime $E \in \mathscr{D}$, $\operatorname{ord}_E(\alpha_n)$ converges to zero as $n \to \infty$. In fact, we claim that $\operatorname{ord}_E(\alpha_n) = 0$ for $n \geq n(E)$ large enough. Indeed, according to Zariski's result, there are two possibilities: either there exists N such that $c_{X_N}(E)$ is a curve, or there exists N such that the center of the blowup $X_{n+1} \to X_n$ does not meet $c_{X_n}(E)$ for all $n \geq N$. In the first case, it is clear that $\operatorname{ord}_E(\alpha_n) = 0$ for $n \geq N$ since α_n is exceptional over X_N. In the second case, the center of E on X_n does not meet the exceptional divisor of $X_n \to X_{n-1}$ for $n > N$, which supports the exceptional class α_n; thus $\operatorname{ord}_E(\alpha_n) = 0$ for $n > N$ as well. □

1.4. Intersections and L^2*-classes*
For each π, the intersection pairing $H^{1,1}_{\mathbf{R}}(X_\pi) \times H^{1,1}_{\mathbf{R}}(X_\pi) \to \mathbf{R}$ is denoted by $(\alpha \cdot \beta)_{X_\pi}$. It is nondegenerate and satisfies the projection formula: $(\mu_*\alpha \cdot \beta)_{X_\pi} = (\alpha \cdot \mu^*\beta)_{X_{\pi'}}$ if $\pi' = \pi \circ \mu$. It thus induces a pairing $W(\mathfrak{X}) \times C(\mathfrak{X}) \to \mathbf{R}$ which is denoted simply by $(\alpha \cdot \beta)$.

PROPOSITION 1.7
The intersection pairing induces a topological isomorphism between $W(\mathfrak{X})$ and $C(\mathfrak{X})^*$ endowed with its weak-$*$ topology.

Proof
A linear form L on $C(\mathfrak{X}) = \varinjlim_\pi H^{1,1}_{\mathbf{R}}(X_\pi)$ is the same thing as a collection of linear forms L_π on $H^{1,1}_{\mathbf{R}}(X_\pi)$, compatible by restriction. Now, such a collection is by definition an element of the projective limit $\varprojlim_\pi H^{1,1}_{\mathbf{R}}(X_\pi)^*$, which is identified to $W(\mathfrak{X})$ via the intersection pairing. This shows that the intersection pairing identifies $W(\mathfrak{X})$ with the dual of $C(\mathfrak{X})$ endowed with its weak-$*$ topology. $\square$

The intersection pairing defined above restricts to a nondegenerate quadratic form on $C(\mathfrak{X})$, denoted by $\alpha \mapsto (\alpha^2)$. However, it does *not* extend to a continuous quadratic form on $W(\mathfrak{X})$. For instance, if $z_1, z_2, \dots$ is a sequence of distinct points on X and π_n denotes the blowup of X at $z_1, \dots, z_n$, with exceptional divisor $F_n = E_1 + \cdots + E_n$, we have $(F_n^2) = -n$, but $\{F_n\} \in C(\mathfrak{X})$ converges in $W(\mathfrak{X})$. We thus introduce the maximal space to which the intersection form extends.

Definition 1.8
The space of L^2-classes $\mathrm{L}^2(\mathfrak{X})$ is defined as the completion of $C(\mathfrak{X})$ with respect to the intersection form.

The usual setting in which to perform a completion is that of a definite quadratic form on a vector space, which is not the case of the intersection form on $C(\mathfrak{X})$. However, the Hodge index theorem implies that it is of Minkowski type, and it is easy to show that the completion exists in that setting.

Let us be more precise. If $\omega \in C(\mathfrak{X})$ is a given class with $(\omega^2) > 0$, the intersection form is negative definite on its orthogonal complement $\omega^\perp := \{\alpha \in C(\mathfrak{X}) \mid (\alpha \cdot \omega) = 0\}$ as a consequence of the Hodge index theorem applied to each $H^{1,1}_{\mathbf{R}}(X_\pi)$. We have an orthogonal decomposition $C(\mathfrak{X}) = \mathbf{R}\omega \oplus \omega^\perp$, and we then let $\mathrm{L}^2(\mathfrak{X}) := \mathbf{R}\omega \oplus \overline{\omega^\perp}$, where $\overline{\omega^\perp}$ is the completion in the usual sense of $\omega^\perp$ endowed with the negative definite quadratic form (α^2). Note that $t\omega \oplus \alpha \mapsto t^2 - (\alpha^2)$ is then a norm on $\mathrm{L}^2(\mathfrak{X})$ which makes it a Hilbert space, but this norm depends on the choice of ω. However, the topological vector space $\mathrm{L}^2(\mathfrak{X})$ does not depend on the choice of ω.

In fact, the completion can be characterized by the following universal property: if (Y, q) is a complete topological vector space with a continuous nondegenerate quadratic form of Minkowski type, any isometry $T : C(\mathfrak{X}) \to Y$ continuously extends to $\mathrm{L}^2(\mathfrak{X}) \to Y$.

The intersection form on $L^2(\mathfrak{X})$ is also of Minkowski type, so that it satisfies the Hodge index theorem: if a nonzero class $\alpha \in L^2(\mathfrak{X})$ satisfies $(\alpha^2) > 0$, then the intersection form is negative definite on $\alpha^\perp \subset L^2(\mathfrak{X})$.

Remark 1.9
The direct sum decomposition $C(\mathfrak{X}) = H^{1,1}_{\mathbf{R}}(X) \oplus \mathbf{R}^{(\mathscr{D})}$ of Proposition 1.4 is orthogonal with respect to the intersection form. Furthermore, the intersection form is negative definite on $\mathbf{R}^{(\mathscr{D})}$, and $\{\alpha_E \mid E \in \mathscr{D}\}$ forms an orthonormal basis for $-(\alpha^2)$. Indeed, the center of $E \in \mathscr{D}$ on the minimal model X_{π_E} on which it appears is necessarily the last exceptional divisor to have been created in any factorization of π_E into a sequence of point blowups; thus it is a (-1)-curve.

Using this, one sees that $L^2(\mathfrak{X})$ is isomorphic to the direct sum $H^{1,1}_{\mathbf{R}}(X) \oplus \ell^2(\mathscr{D}) \subset W(\mathfrak{X})$, where $\ell^2(\mathscr{D})$ denotes the set of real-valued, square-summable functions $E \mapsto a_E$ on $\mathscr{D}$.

The different spaces that we have introduced so far are related as follows.

PROPOSITION 1.10
There is a natural continuous injection $L^2(\mathfrak{X}) \to W(\mathfrak{X})$*, and the topology on* $L^2(\mathfrak{X})$ *induced by the topology of* $W(\mathfrak{X})$ *coincides with its weak topology as a Hilbert space.*

If $\alpha \in W(\mathfrak{X})$ *is a given Weil class, then the intersection number* (α_π^2) *is a decreasing function of* π*, and* $\alpha \in L^2(\mathfrak{X})$ *if and only if* (α_π^2) *is bounded from below, in which case,* $(\alpha^2) = \lim_\pi (\alpha_\pi^2)$.

Proof
The injection $L^2(\mathfrak{X}) \to W(\mathfrak{X})$ is dual to the dense injection $C(\mathfrak{X}) \subset L^2(\mathfrak{X})$. By Proposition 1.7, a net $\alpha_k \in L^2(\mathfrak{X})$ converges to $\alpha \in L^2(\mathfrak{X})$ in the topology induced by $W(\mathfrak{X})$ if and only if $(\alpha_k \cdot \beta) \to (\alpha \cdot \beta)$ for each $\beta \in C(\mathfrak{X})$. Since $C(\mathfrak{X})$ is dense in $L^2(\mathfrak{X})$, this implies that $\alpha_k \to \alpha$ weakly in $L^2(\mathfrak{X})$.

For the last part, one can proceed using the abstract definition of $L^2(\mathfrak{X})$ as a completion, but it is more transparent to use the explicit representation of Remark 1.9. For any π, we have $\alpha_\pi = \alpha_X + \sum_{E \in \mathscr{D}_\pi} (\alpha \cdot \alpha_E)\alpha_E$, where $\mathscr{D}_\pi \subset \mathscr{D}$ is the set of exceptional primes of π. Then $(\alpha_\pi^2) = (\alpha_X^2) - \sum_{E \in \mathscr{D}_\pi} (\alpha \cdot \alpha_E)^2$, which is decreasing in π. It is then clear that $\alpha \in L^2(\mathfrak{X})$ if and only if (α_π^2) is uniformly bounded from below and $(\alpha^2) = \lim(\alpha_\pi^2)$. $\square$

1.5. Positivity
Recall that a class in $H^{1,1}_{\mathbf{R}}(X)$ is pseudoeffective (psef) if it is the class of a closed positive $(1,1)$-current on X. It is numerically effective (nef) if it is the limit of Kähler

classes. Any nef class is psef. The cone in $H^{1,1}_{\mathbf{R}}(X)$ consisting of psef classes is strict: if α and $-\alpha$ are both psef, then $\alpha = 0$.

If $\pi' = \pi \circ \mu$ is a blowup dominating some other blowup π, then $\alpha \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is psef (nef) if and only if $\mu^*\alpha \in H^{1,1}_{\mathbf{R}}(X_{\pi'})$ is psef (nef). On the other hand, if $\alpha' \in H^{1,1}_{\mathbf{R}}(X_{\pi'})$ is psef (nef), then so is $\mu_*\alpha' \in H^{1,1}_{\mathbf{R}}(X_\pi)$. (For the nef part of the last assertion, it is important that we work in dimension two.)

Definition 1.11
A Weil class $\alpha \in W(\mathfrak{X})$ is psef (nef) if its incarnation $\alpha_\pi \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is psef (nef) for any blowup $\pi : X_\pi \to X$.

We denote by $\mathrm{Nef}(\mathfrak{X}) \subset \mathrm{Psef}(\mathfrak{X}) \subset W(\mathfrak{X})$ the convex cones of nef and psef classes. The remarks above imply that a Cartier class $\alpha \in C(\mathfrak{X})$ is psef (nef) if and only if $\alpha_\pi \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is psef (nef) for one (or any) X_π in which α is determined. We write $\alpha \geq \beta$ as a shorthand for $\alpha - \beta \in W(\mathfrak{X})$ being psef.

PROPOSITION 1.12
The nef cone $\mathrm{Nef}(\mathfrak{X})$ *and the psef cone* $\mathrm{Psef}(\mathfrak{X})$ *are strict, closed, convex cones in* $W(\mathfrak{X})$ *with compact bases.*

Proof
The nef (resp., psef) cone is the projective limit of the nef (resp., psef) cones of each $H^{1,1}_{\mathbf{R}}(X_\pi)$. These are strict, closed, convex cones with compact bases, so the result follows from the Tychonoff theorem. □

Nef classes satisfy the following monotonicity property.

PROPOSITION 1.13
If $\alpha \in W(\mathfrak{X})$ *is a nef Weil class, then* $\alpha \leq \alpha_\pi$ *for each* π. *In particular,* $\alpha_\pi \neq 0$ *for each* π *unless* $\alpha = 0$.

Proof
By induction on the number of blowups, it suffices to prove that $\alpha_{\pi'} \leq \mu^*\alpha_\pi$ when $\pi' = \pi \circ \mu$ and μ is the blowup of a point in X_π. But then $\mu^*\alpha_\pi = \alpha_{\pi'} + cE$, where E is the class of the exceptional divisor and $c = (\alpha_{\pi'} \cdot E) \geq 0$. To get the second point, note that $\alpha_\pi = 0$ for some π implies that $\alpha \leq 0$. On the other hand, $\alpha \geq 0$ as α is nef. Since $\mathrm{Psef}(\mathfrak{X})$ is a strict cone, we infer that $\alpha = 0$. □

PROPOSITION 1.14
The nef cone $\mathrm{Nef}(\mathfrak{X})$ *is contained in* $\mathrm{L}^2(\mathfrak{X})$. *If* $\alpha_i \geq \beta_i$, $i = 1, 2$, *are nef classes, then we have* $(\alpha_1 \cdot \alpha_2) \geq (\beta_1 \cdot \beta_2) \geq 0$.

Proof
If $\alpha \in W(\mathfrak{X})$ is nef, each incarnation α_π is nef, and thus $(\alpha_\pi^2) \geq 0$, so that $\alpha \in \mathrm{L}^2(\mathfrak{X})$ by Proposition 1.10 with $(\alpha^2) = \inf_\pi(\alpha_\pi^2) \geq 0$. To get the second point, note that $(\alpha_1 \cdot \alpha_2) \geq (\alpha_1 \cdot \beta_2)$ since $\alpha_2 - \beta_2$ is psef and α_1 is nef and, similarly, $(\alpha_1 \cdot \beta_2) \geq (\beta_1 \cdot \beta_2)$. □

These two propositions together show that if $\omega \in C(\mathfrak{X})$ is a Cartier class determined by a Kähler class down on X, then $(\alpha \cdot \omega) > 0$ for any nonzero nef class $\alpha \in W(\mathfrak{X})$.

PROPOSITION 1.15
We have $2\,(\alpha \cdot \beta)\,\alpha \geq (\alpha^2)\,\beta$ for any nef Weil classes $\alpha, \beta \in W(\mathfrak{X})$. In particular, if $\omega \in C(\mathfrak{X})$ is determined by a Kähler class on X normalized by $(\omega^2) = 1$, we have, for any nonzero nef Weil class α,

$$\frac{(\alpha^2)}{2(\alpha \cdot \omega)}\,\omega \leq \alpha \leq 2(\alpha \cdot \omega)\,\omega. \tag{1.1}$$

Proof
The second assertion is a special case of the first one. To prove the first one, we may assume that $(\alpha \cdot \beta) > 0$, or else α and β are proportional by the Hodge index theorem, and the result is clear. It is a known fact (see the remark after [B, Theorem 4.1]) that if $\gamma \in C(\mathfrak{X})$ is a Cartier class with $(\gamma^2) \geq 0$, then either γ or $-\gamma$ is psef. In view of Proposition 1.10, the same result is true for any $\gamma \in \mathrm{L}^2(\mathfrak{X})$. Apply this to $\gamma = \alpha - t\beta$, where $t = ((\alpha \cdot \alpha)/2(\alpha \cdot \beta))$. As $(\gamma \cdot \gamma) \geq 0$ and $(\gamma \cdot \alpha) \geq 0$, γ must be psef. □

1.6. The canonical class
The canonical class $K_{\mathfrak{X}}$ is the Weil class whose incarnation in any blowup X_π is the canonical class K_{X_π}. It is not Cartier and does not even belong to $\mathrm{L}^2(\mathfrak{X})$. However, $K_{X_{\pi'}} \geq K_{X_\pi}$ whenever $\pi' \geq \pi$, and $K_{\mathfrak{X}}$ is the smallest Weil class dominating all the K_{X_π}. This allows us to intersect $K_{\mathfrak{X}}$ with any nef Weil class α in a slightly ad hoc way: we set $(\alpha \cdot K_{\mathfrak{X}}) := \sup_\pi(\alpha_\pi \cdot K_{X_\pi})_{X_\pi} \in \mathbf{R} \cup \{+\infty\}$.

2. Functorial behavior

Throughout this section, let $F : X \dashrightarrow Y$ be a dominant meromorphic map between compact Kähler surfaces. Following [M, Section 34.7], we introduce the action of F on Weil and Cartier classes. We then describe the continuity properties of these actions on the Hilbert space $\mathrm{L}^2(\mathfrak{X})$.

For each blowup Y_ϖ of Y, there exists a blowup X_π of X such that the induced map $X_\pi \to Y_\varpi$ is holomorphic. The associated pushforward $H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(Y_\varpi)$ and pullback $H^{1,1}_{\mathbf{R}}(Y_\varpi) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ are compatible with the projective and injective systems defined by pushforwards and pullbacks that define Weil and Cartier classes, respectively, so we can consider the induced morphisms on the respective projective and inductive limits.

Definition 2.1
Given $F : X \dashrightarrow Y$ as above, we denote by $F_* : W(\mathfrak{X}) \to W(\mathfrak{Y})$ the induced pushforward operator and by $F^* : C(\mathfrak{Y}) \to C(\mathfrak{X})$ the induced pullback operator.

Concretely, if $\alpha \in W(\mathfrak{X})$ is a Weil class, the incarnation of $F_*\alpha \in W(\mathfrak{Y})$ on a given blowup Y_ϖ is the pushforward of $\alpha_\pi \in H^{1,1}_{\mathbf{R}}(X_\pi)$ by the induced map $X_\pi \to Y_\varpi$ for any π such that the latter map is holomorphic. Similarly, if $\beta \in C(\mathfrak{Y})$ is a Cartier class determined on a blowup Y_ϖ, its pullback $F^*\beta \in C(\mathfrak{X})$ is the Cartier class determined on X_π by the pullback of $\beta_\varpi \in H^{1,1}_{\mathbf{R}}(Y_\varpi)$ by the induced map $X_\pi \to Y_\varpi$, whenever the latter is holomorphic.

These constructions are functorial, that is, $(F \circ G)_* = F_* \circ G_*$ and $(F \circ G)^* = G^* \circ F^*$, and are compatible with the duality between C and W since this is true for each holomorphic map $X_\pi \to Y_\varpi$. In other words, for any $\alpha \in W(\mathfrak{X})$ and $\beta \in C(\mathfrak{Y})$, we have $(F_*\alpha \cdot \beta) = (\alpha \cdot F^*\beta)$.

We also see that F_* preserves nef and psef Weil classes and that F^* preserves nef and psef Cartier classes. Indeed, the pullback and pushforward by a surjective holomorphic map both preserve nef and psef $(1, 1)$-classes in dimension two.

Remark 2.2
If $\pi : X_\pi \to X$ and $\varpi : Y_\varpi \to Y$ are arbitrary blowups, then the pullback operator $H^{1,1}_{\mathbf{R}}(Y_\varpi) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ usually associated to the meromorphic map $X_\pi \dashrightarrow Y_\varpi$ is given by the restriction of $F^* : C(\mathfrak{Y}) \to C(\mathfrak{X})$ to $H^{1,1}_{\mathbf{R}}(Y_\varpi)$, followed by the projection of $C(\mathfrak{X})$ onto $H^{1,1}_{\mathbf{R}}(X_\pi)$. Similarly, the pushforward operator $H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(Y_\varpi)$ usually associated to $X_\pi \dashrightarrow Y_\varpi$ is given by the restriction of $F_* : W(\mathfrak{X}) \to W(\mathfrak{Y})$ to $H^{1,1}_{\mathbf{R}}(X_\pi)$, followed by the projection of $W(\mathfrak{Y})$ onto $H^{1,1}_{\mathbf{R}}(Y_\varpi)$.

The intersection forms on $C(\mathfrak{X})$ and $C(\mathfrak{Y})$ are related by F^* as follows: $(F^*\beta^2) = e(F)(\beta^2)$, where $e(F) > 0$ is the topological degree of F. In view of the universal property of completions mentioned in Section 1.4, we get the following.

PROPOSITION 2.3
The pullback $F^* : C(\mathfrak{Y}) \to C(\mathfrak{X})$ extends to a continuous operator $F^* : \mathrm{L}^2(\mathfrak{Y}) \to \mathrm{L}^2(\mathfrak{X})$, so that $((F^*\beta)^2) = e(F)(\beta^2)$ for each $\beta \in \mathrm{L}^2(\mathfrak{Y})$. By duality, the pushforward $F_* : W(\mathfrak{X}) \to W(\mathfrak{Y})$ induces a continuous operator $F_* : \mathrm{L}^2(\mathfrak{X}) \to \mathrm{L}^2(\mathfrak{Y})$, so that $(F_*\alpha \cdot \beta) = (\alpha \cdot F^*\beta)$ for any $\alpha, \beta \in \mathrm{L}^2(\mathfrak{X})$.

Next, we show that the pullback $F^* : C(\mathfrak{Y}) \to C(\mathfrak{X})$ continuously extends to Weil classes and—dually—that the pushforward $F_* : W(\mathfrak{X}) \to W(\mathfrak{Y})$ preserves Cartier classes.

In doing so, we repeatedly use a consequence of the result of Zariski mentioned in Section 1. Namely, given $F : X \dashrightarrow Y$ and a blowup $\pi : X_\pi \to X$, there exists a blowup Y_ϖ of Y such that the induced meromorphic map $X_\pi \dashrightarrow Y_\varpi$ does not contract any curve to a point.

LEMMA 2.4
Suppose that $\pi : X_\pi \to X$ and $\varpi : Y_\varpi \to Y$ are two blowups such that the induced meromorphic map $X_\pi \dashrightarrow Y_\varpi$ does not contract any curve to a point. Then for each Cartier class $\beta \in C(\mathfrak{Y})$, the incarnations of $F^*\beta$ and $F^*\beta_\varpi$ on X_π coincide.

Proof
Any Cartier class is a difference of nef Cartier classes, so we may assume that β is nef and determined in some blowup ϖ' dominating ϖ. Pick π' dominating π so that the induced map $X_{\pi'} \to Y_{\varpi'}$ is holomorphic. Set $\alpha := F^*(\beta_\varpi - \beta)$. Then $\alpha \in C(\mathfrak{X})$ is psef and determined in $X_{\pi'}$. We must show that $\alpha_\pi = 0$. If $\alpha_\pi \neq 0$, then $\alpha \geq \lambda C$, where $\lambda > 0$ and C is the class of an irreducible curve on X_π. Now, C is not contracted by $X_\pi \dashrightarrow Y_\varpi$, so the incarnation of $F_*\alpha$ on Y_ϖ is nonzero. But this is a contradiction since this incarnation equals $e(F)(\beta_\varpi - \beta)_\varpi = 0$. □

COROLLARY 2.5
The pullback operator $F^* : C(\mathfrak{Y}) \to C(\mathfrak{X})$ continuously extends to $F^* : W(\mathfrak{Y}) \to W(\mathfrak{X})$ and preserves nef and psef Weil classes.

More precisely, if X_π is a given blowup of X and Y_ϖ is a blowup of Y such that the induced meromorphic map $X_\pi \dashrightarrow Y_\varpi$ does not contract curves, then for any Weil class $\gamma \in W(\mathfrak{Y})$, one has $(F^*\gamma)_\pi = (F^*\gamma_\varpi)_\pi$.

COROLLARY 2.6
The pushforward operator $F_* : W(\mathfrak{X}) \to W(\mathfrak{Y})$ preserves Cartier classes. More precisely, if $\alpha \in C(\mathfrak{X})$ is a Cartier class determined on some X_π, then $F_*\alpha$ is Cartier, determined on Y_ϖ as soon as the induced meromorphic map $X_\pi \dashrightarrow Y_\varpi$ does not contract curves.

Proof
For any $\beta \in C(\mathfrak{Y})$, the incarnations of $F^*\beta$ and $F^*\beta_\varpi$ on X_π coincide by Corollary 2.5. Hence

$$(F_*\alpha \cdot \beta) = (\alpha \cdot F^*\beta) = (\alpha \cdot F^*\beta_\varpi) = (F_*\alpha \cdot \beta_\varpi) = \big((F_*\alpha)_\varpi \cdot \beta\big).$$

As this holds for any Cartier class $\beta \in C(\mathfrak{Y})$, we must have $F_*\alpha = (F_*\alpha)_\varpi$ by Proposition 1.7. □

3. Dynamics

Now, consider a dominant meromorphic self-map $F : X \dashrightarrow X$ of a compact Kähler surface X. Write $\lambda_2 = e(F)$ for the topological degree of F. If $\omega \in \mathrm{Nef}(\mathfrak{X})$ is a nef Weil class such that $(\omega^2) > 0$, we define the degree of F with respect to ω as

$$\deg_\omega(F) := (F^*\omega \cdot \omega) = (\omega \cdot F_*\omega).$$

This coincides with the usual notion of degree when $X = \mathbf{P}^2$ and ω is the Cartier class determined by a line on $\mathbf{P}^2$.

PROPOSITION 3.1
The limit

$$\lambda_1 := \lambda_1(F) := \lim_{n\to\infty} \deg_\omega(F^n)^{1/n} \tag{3.1}$$

exists and does not depend on the choice of the nef class $\omega \in \mathrm{Nef}(\mathfrak{X})$ *with* $(\omega^2) > 0$. *Moreover,* λ_1 *is invariant under bimeromorphic conjugacy and* $\lambda_1^2 \geq \lambda_2$.

The result above is well known, but we include the proof for completeness. We call λ_1 the *asymptotic degree* of F. It is also known as the *first dynamical degree* and can be computed (see [DF]) as $\lambda_1 = \lim_{n\to\infty} \rho_n^{1/n}$, where ρ_n is the spectral radius of F^n acting on $H^{1,1}_{\mathbf{R}}(X)$ by pullback or pushforward (cf. Remark 2.2).

Proof of Proposition 3.1
Upon scaling ω, we can assume that $(\omega^2) = 1$. By (1.1), we then have $G^*\omega \leq 2(G^*\omega \cdot \omega)\,\omega$ for any dominant mapping $G : X \dashrightarrow X$. Applying this with $G = F^m$ yields

$$\deg_\omega(F^{n+m}) = (F^{n*}F^{m*}\omega \cdot \omega) \leq 2(F^{n*}\omega \cdot \omega)(F^{m*}\omega \cdot \omega) = 2\deg_\omega(F^n)\deg_\omega(F^m).$$

This implies (see, e.g., [KH, Proposition 9.6.4]) that the limit in (3.1) exists. Let us temporarily denote it by $\lambda_1(\omega)$. If $\omega' \in C(\mathfrak{X})$ is another nef class with $(\omega'^2) > 0$, then it follows from (1.1) that $\omega' \leq C\omega$ for some $C > 0$. By Proposition 1.14, this gives

$$\deg_{\omega'}(F^n) = (F^{n*}\omega' \cdot \omega') \leq C^2(F^{n*}\omega \cdot \omega) = C^2 \deg_\omega(F^n).$$

Taking nth roots and letting $n \to \infty$ shows that $\lambda_1(\omega') \leq \lambda_1(\omega)$, and thus $\lambda_1(\omega') = \lambda_1(\omega)$ by symmetry, so that λ_1 is indeed independent of ω. It is then invariant by bimeromorphic conjugacy since $\mathfrak{X}$ and all the spaces attached to it are.

Finally, Proposition 1.14 yields $F^{n*}\omega \leq 2(F^{*n}\omega \cdot \omega)\,\omega$, which implies that

$$e(F)^n = e(F^n) = (F^{n*}\omega^2) \leq 4(F^{n*}\omega \cdot \omega)^2 = 4\deg_\omega(F^n)^2,$$

and letting $n \to \infty$ yields $\lambda_2 = e(F) \leq \lambda_1^2$. $\square$

3.1. Existence of eigenclasses

To begin, we do not assume that $\lambda_1^2 > \lambda_2$.

THEOREM 3.2

Let $F : X \dashrightarrow X$ be any dominant meromorphic self-map of a smooth Kähler surface X with asymptotic degree λ_1. Then we can find nonzero nef Weil classes θ_* and θ^* with $F_*\theta_* = \lambda_1\theta_*$ and $F^*\theta^* = \lambda_1\theta^*$.

Note that by Proposition 1.14, both classes θ_*, θ^* belong to $\mathrm{L}^2(\mathfrak{X})$.

Proof

We use the pushforward and pullback operators

$$S_\pi : H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(X_\pi) \qquad \text{and} \qquad T_\pi : H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(X_\pi),$$

usually associated to the meromorphic map $X_\pi \dashrightarrow X_\pi$ induced by F for a given blowup $\pi : X_\pi \to X$. Thus S_π (resp., T_π) is the restriction to $H^{1,1}_{\mathbf{R}}(X_\pi)$ of $F_* : C(\mathfrak{X}) \to C(\mathfrak{X})$ (resp., $F^* : C(\mathfrak{X}) \to C(\mathfrak{X})$) followed by the projection $C(\mathfrak{X}) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ (cf. Remark 2.2). These operators are typically denoted F_* and F^* in the literature, but here that notation conflicts with the corresponding operators on $C(\mathfrak{X})$ or $W(\mathfrak{X})$.

The spectral radius $\rho_\pi > 0$ of T_π can be computed as follows: if $\theta \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is any nef class with $(\theta^2) > 0$, then $(T_\pi^n\theta \cdot \theta)^{1/n} \to \rho_\pi$ as $n \to \infty$.

LEMMA 3.3

We have $\lambda_1 \leq \rho_{\pi'} \leq \rho_\pi$ for all $\pi' \geq \pi$.

Proof

Let $\theta \in C(\mathfrak{X})$ be a given nef class determined on $X_{\pi'}$ with $(\theta^2) > 0$, so that $\theta \leq \theta_\pi$ by Proposition 1.13. Then $T_{\pi'}\theta$ is the incarnation on $X_{\pi'}$ of the nef class $F^*\theta$ on $X_{\pi'}$, and $T_\pi\theta_\pi$ is the incarnation on X_π of the nef class $F^*\theta_\pi \geq F^*\theta$; thus $F^*\theta \leq T_{\pi'}\theta \leq T_\pi\theta_\pi$ holds by Proposition 1.13. By induction, we get $F^{n*}\theta \leq T_{\pi'}^n\theta \leq T_\pi^n\theta_\pi$ for all n; hence $(F^{n*}\theta \cdot \theta)^{1/n} \leq (T_{\pi'}^n\theta \cdot \theta)^{1/n} \leq (T_\pi^n\theta_\pi \cdot \theta_\pi)^{1/n}$ by Proposition 1.14, and $\lambda_1 \leq \rho_{\pi'} \leq \rho_\pi$ follows by letting $n \to \infty$. □

Now, the set of nef classes in $H^{1,1}_{\mathbf{R}}(X_\pi)$ is a closed convex cone with compact basis invariant by T_π; thus a Perron-Frobenius-type argument (see [DF, Lemma 1.12]) establishes the existence of a nonzero nef class $\theta(\pi) \in H^{1,1}_{\mathbf{R}}(X_\pi)$ with $T_\pi\theta(\pi) = \rho_\pi\theta(\pi)$.

If we identify $\theta(\pi)$ with the nef Cartier class that it determines, this says that the nef Cartier classes $F^*\theta(\pi)$ and $\rho_\pi\theta(\pi)$ have the same incarnation on X_π. We have thus obtained approximate eigenclasses, and now the plan is to get the desired class θ^* as a limit of classes of the form $\theta(\pi)$. We then explain how to modify the argument to construct θ_*.

We normalize $\theta(\pi)$ by $(\theta(\pi)\cdot\omega) = 1$ for a fixed class $\omega \in C(\mathfrak{X})$ determined by a Kähler class on X with $(\omega^2) = 1$, so that the $\theta(\pi)$ all lie in a compact subset of the nef cone $\mathrm{Nef}(\mathfrak{X})$ by Proposition 1.12.

Let $X = X_0 \leftarrow X_1 \leftarrow \cdots$ be an infinite sequence of blowups so that the lift of F as a map from X_{n+1} to X_n is holomorphic for $n \geq 0$.

For each n, let ρ_n denote the spectral radius of T_n on $H^{1,1}_{\mathbf{R}}(X_n)$ as above, and pick a nonzero nef Cartier class $\theta_n \in C(\mathfrak{X})$ determined on X_n and such that $T_n\theta_n = \rho_n\theta_n$. Then $F^*\theta_n$ is a Cartier class determined in X_{n+1}, and by definition, $T_n\theta_n$ is the incarnation of this class in X_n. Therefore $F^*\theta_n$ and $\rho_n\theta_n$ coincide on X_n. By Proposition 1.6, it follows that $F^*\theta_n - \rho_n\theta_n$ converges to zero in $W(\mathfrak{X})$ as $n \to \infty$.

We have seen above that ρ_n is a decreasing sequence. Let $\rho_\infty := \lim \rho_n$, so that $\rho_\infty \geq \lambda_1$ by Lemma 3.3. Since the θ_n lie in a compact subset of $\mathrm{Nef}(\mathfrak{X})$, we can find a cluster point θ^* for the sequence θ_n, which is also a nef Weil class with $(\theta^*\cdot\omega) = 1$. Since $F^*\theta_n - \rho_n\theta_n$ converges to zero in $W(\mathfrak{X})$, it follows that $F^*\theta^* = \rho_\infty\theta^*$.

To complete the proof, we show that $\rho_\infty = \lambda_1$. In fact, if $\alpha \in W(\mathfrak{X})$ is any nonzero nef eigenclass of F^* with $F^*\alpha = t\,\alpha$ for some $t \geq 0$, then $t \leq \lambda_1$. Indeed, we have $\alpha \leq C\omega$ for some $C > 0$ by Proposition 1.15, and it follows that $(F^{n*}\omega\cdot\omega) \geq C^{-1}(F^{n*}\alpha\cdot\omega) = C^{-1}t^n(\alpha\cdot\omega)$. Taking nth roots and letting $n \to \infty$ yields $\lambda_1 \geq t$.

In order to construct θ_*, we modify the above argument as follows. Let $S_\pi : H^{1,1}_{\mathbf{R}}(X_\pi) \to H^{1,1}_{\mathbf{R}}(X_\pi)$ be the pushforward operator defined above. As F^* and F_* are adjoint to each other with respect to the intersection pairing, it follows that S_π and T_π are adjoint with respect to Poincaré duality on $H^{1,1}_{\mathbf{R}}(X_\pi)$, so that they have the same spectral radius ρ_π. By a Perron-Frobenius-type argument, there exists a nonzero nef class $\vartheta(\pi) \in H^{1,1}_{\mathbf{R}}(X_\pi)$ such that $S_\pi\vartheta(\pi) = \rho_\pi\vartheta(\pi)$.

Now, pick $X = X_0 \leftarrow X_1 \leftarrow \cdots$ to be an infinite sequence of blowups such that the lifts of F from X_n to X_{n+1} do not contract any curves. For each n, we get a nef class $\vartheta_n \in C(\mathfrak{X})$ determined on X_n normalized by $(\vartheta_n\cdot\omega) = 1$. By Corollary 2.6, the class $F_*\vartheta_n$ is determined in X_{n+1}, so $F_*\vartheta_n$ and $\rho_n\vartheta_n$ coincide in X_n. Proposition 1.6 then shows that $F_*\vartheta_n - \rho_n\vartheta_n$ converges to zero in $W(\mathfrak{X})$ as $n \to \infty$; hence $\theta_* \in \mathrm{Nef}(\mathfrak{X})$ can be taken to be any cluster value of ϑ_n. □

Remark 3.4

When K_X is not psef (i.e., if X is rational or ruled) we may also achieve $(\theta_*\cdot K_{\mathfrak{X}}) \leq 0$. To see this, first note that $F^*K_{\mathfrak{X}} \leq K_{\mathfrak{X}}$ as classes in $W(\mathfrak{X})$ since $K_{X_{\pi'}} - F^*K_{X_\pi}$

is represented by the effective zero divisor of the Jacobian determinant of the map $X_{\pi'} \to X_\pi$ induced by F, assuming that this is holomorphic. Now, for each blowup X_π, let C_π be the set of nef classes $\alpha \in H^{1,1}_{\mathbf{R}}(X_\pi)$ such that $(\alpha \cdot K_{\mathfrak{X}}) \le 0$. Then C_π is a closed convex cone with compact basis and is not reduced to zero since $K_{\mathfrak{X}}$ is not psef. It is, furthermore, invariant by S_π. Indeed, if $\alpha \in H^{1,1}_{\mathbf{R}}(X_\pi)$ is a nef class, we have

$$(S_\pi \alpha \cdot K_{\mathfrak{X}}) = (F_* \alpha \cdot K_{X_\pi}) \le (F_* \alpha \cdot K_{\mathfrak{X}}) = (\alpha \cdot F^* K_{\mathfrak{X}}) \le (\alpha \cdot K_{\mathfrak{X}}).$$

We can thus assume that the nonzero eigenclasses ϑ_n in the proof of Theorem 3.2 belong to C_n, and we get $(\theta_* \cdot K_{\mathfrak{X}}) \le 0$.

The same argument does *not* work for θ^* since $F_* K_{\mathfrak{X}} \le K_{\mathfrak{X}}$ does not hold in general.

3.2. Spectral properties

Theorem 3.2 asserts the existence of eigenclasses for F_* and F^* with eigenvalue λ_1. We now further analyze the spectral properties under the assumption that $\lambda_1^2 > \lambda_2$.

THEOREM 3.5
Assume that $\lambda_1^2 > \lambda_2$. Then the nonzero nef Weil classes $\theta_*, \theta^* \in \mathrm{L}^2(\mathfrak{X})$ such that $F^*\theta^* = \lambda_1 \theta^*$ and $F_*\theta_* = \lambda_1 \theta_*$ are unique up to scaling. We have $(\theta_* \cdot \theta^*) > 0$ and $(\theta^{*2}) = 0$. We rescale them so that $(\theta_* \cdot \theta^*) = 1$. Let $\mathscr{H} \subset \mathrm{L}^2(\mathfrak{X})$ be the orthogonal complement of θ^* and θ_*, so that we have the decomposition $\mathrm{L}^2(\mathfrak{X}) = \mathbf{R}\theta^* \oplus \mathbf{R}\theta_* \oplus \mathscr{H}$. The intersection form is negative definite on $\mathscr{H}$, and $\|\alpha\|^2 := -(\alpha^2)$ defines a Hilbert norm on $\mathscr{H}$. The actions of F^* and F_* with respect to this decomposition are as follows.

(i) *The subspace $\mathscr{H}$ is F^*-invariant, and*

$$\begin{cases} F^{n*}\theta^* = \lambda_1^n \theta^*, \\ F^{n*}\theta_* = \left(\dfrac{\lambda_2}{\lambda_1}\right)^n \theta_* + (\theta_*^2)\, \lambda_1^n \left(1 - \left(\dfrac{\lambda_2}{\lambda_1^2}\right)^n\right) \theta^* + h_n \\ \quad \text{with } h_n \in \mathscr{H},\ \|h_n\| = O(\lambda_2^{n/2}), \\ \|F^{n*} h\| = \lambda_2^{n/2} \|h\| \quad \text{for all } h \in \mathscr{H}. \end{cases}$$

(ii) *The subspace $\mathscr{H}$ is not F_*-invariant in general, but*

$$\begin{cases} F_*^n \theta_* = \lambda_1^n \theta_*, \\ F_*^n \theta^* = \left(\dfrac{\lambda_2}{\lambda_1}\right)^n \theta^*, \\ \|F_*^n h\| \le C \lambda_2^{n/2} \|h\| \quad \text{for some } C > 0 \text{ and all } h \in \mathscr{H}. \end{cases}$$

COROLLARY 3.6
For any Weil class $\alpha \in \mathrm{L}^2(\mathfrak{X})$*, we have*

$$\frac{1}{\lambda_1^n} F^{n*}\alpha = (\alpha \cdot \theta_*)\theta^* + O\Big(\Big(\frac{\lambda_2}{\lambda_1^2}\Big)^{n/2}\Big)$$

and

$$\frac{1}{\lambda_1^n} F_*^n\alpha = (\alpha \cdot \theta^*)\theta_* + O\Big(\Big(\frac{\lambda_2}{\lambda_1^2}\Big)^{n/2}\Big).$$

Proof
The decomposition of α in $\mathrm{L}^2(\mathfrak{X}) = \mathbf{R}\theta^* \oplus \mathbf{R}\theta_* \oplus \mathscr{H}$ is given by

$$\alpha = \big((\alpha \cdot \theta_*) - (\alpha \cdot \theta^*)(\theta_*^2)\big)\theta^* + (\alpha \cdot \theta^*)\theta_* + \alpha_0, \tag{3.2}$$

where $\alpha_0 \in \mathscr{H}$. The result follows from (3.2) using Theorem 3.5(i), (ii). □

Proof of the main theorem
Applying Corollary 3.6 to $\alpha = \omega$ (which is nef and hence in $\mathrm{L}^2(\mathfrak{X})$) gives

$$\deg_\omega(F^n) = (F^{n*}\omega \cdot \omega) = (\omega \cdot \theta^*)(\omega \cdot \theta_*)\lambda_1^n + O(\lambda_2^{n/2}).$$

This completes the proof with $b := (\omega \cdot \theta^*)(\omega \cdot \theta_*)$. □

Proof of Theorem 3.5
Using Theorem 3.2, we may find nonzero nef Weil classes θ_*, θ^* such that $F_*\theta_* = \lambda_1\theta_*$ and $F^*\theta^* = \lambda_1\theta^*$. Fix two such classes for the duration of the proof. In the end, we see that they are unique up to scaling.

The proof amounts to a series of simple arguments using general facts for transformations of a complete vector space endowed with a Minkowski form. We provide the details for the benefit of the reader.

First, note that $\lambda_1 F_*\theta^* = F_*F^*\theta^* = \lambda_2\theta^*$, so that $F_*\theta^* = (\lambda_2/\lambda_1)\theta^*$. Since $F_*\theta_* = \lambda_1\theta_*$ and $\lambda_1^2 > \lambda_2$, it follows that θ^* and θ_* cannot be proportional.

Applying the relation $(F^*\alpha^2) = \lambda_2(\alpha^2)$ to $\alpha = \theta^*$ yields $\lambda_1^2(\theta^{*2}) = \lambda_2(\theta^{*2})$, and thus $(\theta^{*2}) = 0$ since $\lambda_1^2 > \lambda_2$. By the Hodge index theorem, θ_* and θ^* would thus have to be proportional if they were orthogonal. We infer that $(\theta^* \cdot \theta_*) > 0$, and we rescale θ^* so that $(\theta^* \cdot \theta_*) = 1$.

Let us first prove the properties in (i) for the pullback. As both θ_* and θ^* are eigenvectors for F_*, the space $\mathscr{H}$ is invariant under F^*. Using (3.2) and the invariance properties of θ_* and θ^*, we get

$$F^*\theta_* = \frac{\lambda_2}{\lambda_1}\theta_* + \lambda_1\Big(1 - \frac{\lambda_2}{\lambda_1^2}\Big)(\theta_*^2)\theta^* + h_1, \tag{3.3}$$

where $h_1 \in \mathscr{H}$. Inductively, (3.3) gives

$$F^{n*}\theta_* = \left(\frac{\lambda_2}{\lambda_1}\right)^n \theta_* + \lambda_1^n \left(1 - \left(\frac{\lambda_2}{\lambda_1^2}\right)^n\right)(\theta_*^2)\theta^* + h_n, \tag{3.4}$$

where $h_{n+1} = F^* h_n + (\lambda_2/\lambda_1)^n h_1 \in \mathscr{H}$. Using the fact that $\|F^* h\|^2 = \lambda_2 \|h\|^2$ on $\mathscr{H}$, we get $\|h_{n+1}\| \leq \lambda_2^{1/2}\|h_n\| + (\lambda_2/\lambda_1)^n \|h_1\|$, which is easily seen to imply that $\|h_n\| = O(\lambda_2^{n/2})$ since $\sum_k (\lambda_2^{1/2}/\lambda_1)^k < +\infty$. This concludes the proof of (i).

Let us now turn to the pushforward operator. The first two equations are clear. As θ_* may not be an eigenvector for F^*, $\mathscr{H}$ need not be invariant by F_*, but since $F_* h$ is orthogonal to θ^* for any $h \in \mathscr{H}$, we can write $F_*^n h = a_n \theta^* + g_n$ with $a_n = (F^{n*}\theta_* \cdot h)$ and $g_n \in \mathscr{H}$. We have seen that $F^{n*}\theta_* = h_n$ modulo θ^*, θ_* with $\|h_n\| = O(\lambda_2^{n/2})$; thus $|a_n| = |(h_n \cdot h)| \leq C\lambda_2^{n/2}\|h\|$. On the other hand, we have $(g_n^2) = (F^{n*} g_n \cdot h)$; thus $\|g_n\|^2 \leq \lambda_2^{n/2}\|g_n\|\|h\|$, and this shows that $\|F_*^n h\| \leq C\lambda_2^{n/2}\|h\|$. □

Remark 3.7
It follows from the proof of the main theorem that there exist nef classes $\alpha_*, \alpha^* \in H^{1,1}_{\mathbf{R}}(X)$ such that for any Kähler classes ω, ω' on X, we have

$$\frac{\deg_\omega(F^n)}{\deg_{\omega'}(F^n)} = \frac{(\alpha^* \cdot \omega)_X \, (\alpha_* \cdot \omega)_X}{(\alpha^* \cdot \omega')_X \, (\alpha_* \cdot \omega')_X} + O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{n/2}\right).$$

Indeed, we can take α^* and α_* as the incarnations in X of θ^* and θ_*, respectively.

Remark 3.8
When F is bimeromorphic, we have $\theta_*(F) = \theta^*(F^{-1})$; hence $(\theta_*^2) = 0$. However, in general, we may have $(\theta_*^2) > 0$. For example, let F be any polynomial map of $\mathbf{C}^2$ whose extension to $\mathbf{P}^2$ is not holomorphic but does not contract any curve. If ω is the class of a line on $\mathbf{P}^2$, then $\deg_\omega(F) > \sqrt{\lambda_2} > 1$. On the other hand, $F_*\omega = \deg_\omega(F)\omega$ by Corollary 2.6, so $\lambda_1 = \deg_\omega(F)$, $\theta_* = \omega$ and $(\theta_*^2) = 1$.

Remark 3.9
The case when θ_* (or θ^*) is Cartier is very special. For example, when F is bimeromorphic, it follows from [DF, Theorem 0.4] that θ_* (or, equivalently, θ^*) is Cartier if and only if F is *biholomorphic* in some birational model. In the general noninvertible case, similar rigidity results are expected (see [C1] for work in this direction).

Note also that F being algebraically stable in some birational model does not imply that the eigenclasses are Cartier. We do not know whether having a Cartier eigenclass implies algebraic stability in some model, but having a Cartier eigenclass has many of the same consequences as stability: λ_1 is an algebraic integer, and the sequence of degrees $(\deg_\omega F^n)_1^\infty$ satisfies a linear recurrence relation.

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