

A REFINEMENT OF IZUMI'S THEOREM

SÉBASTIEN BOUCKSOM, CHARLES FAVRE, AND MATTIAS JONSSON

ABSTRACT. We improve Izumi's inequality, which states that any divisorial valuation v centered at a closed point 0 on an algebraic variety Y is controlled by the order of vanishing at 0 . More precisely, as v ranges through valuations that are monomial with respect to coordinates in a fixed birational model X dominating Y , we show that for any regular function f on Y at 0 , the function $v \mapsto v(f)/\text{ord}_0(f)$ is uniformly Lipschitz continuous as a function of the weight defining v . As a consequence, the volume of v is also a Lipschitz continuous function. Our proof uses toroidal techniques as well as positivity properties of the images of suitable nef divisors under birational morphisms.

CONTENTS

Introduction	1
1. Background	6
2. Some convex analysis	8
3. Proof of Theorem B	9
4. Consequences of Theorem B	15
References	22

INTRODUCTION

Let Y be a normal variety over an algebraically closed field k and let $0 \in Y$ be a closed point. Write \mathfrak{m}_0 for the maximal ideal of the local ring $\mathcal{O}_{Y,0}$ at 0 and define $\text{ord}_0 : \mathcal{O}_{Y,0} \rightarrow \mathbf{Z}_{\geq 0} \cup \{\infty\}$, the order of vanishing at 0 , by

$$\text{ord}_0(f) := \max\{j \geq 0 \mid f \in \mathfrak{m}_0^j\},$$

This is a valuation if 0 is a smooth point, but not in general. See §4.1 for more details.

Recall that a valuation v of the function field $k(Y)$ is *divisorial* if there exists a projective birational morphism $X \rightarrow Y$, with X normal, and an irreducible Cartier divisor E on X such that v is proportional to ord_E , the order of vanishing along E . We say that v is *centered at 0* if E lies above 0 , or, equivalently, $v \geq 0$ on $\mathcal{O}_{Y,0}$ and $v > 0$ on \mathfrak{m}_0 .

Izumi's Theorem¹ says that any divisorial valuation centered at 0 is comparable to the order of vanishing at 0 :

Date: September 20, 2012.

Second author supported by the ANR-project BERKO. Third author supported by the CNRS and the NSF.

¹In fact the original statement is slightly more general assuming only $(Y, 0)$ to be analytically irreducible.

Izumi's Theorem. *For any divisorial valuation v of $k(Y)$ centered at 0 there exists a constant $C = C(v) > 0$ such that*

$$C^{-1} \operatorname{ord}_0(f) \leq v(f) \leq C \operatorname{ord}_0(f). \quad (*)$$

Only the right-hand inequality in (*) is nontrivial. Indeed, if we set $c = v(\mathfrak{m}_0) = \min\{v(f) \mid f \in \mathfrak{m}_0\}$ then $c > 0$ and $v \geq c \operatorname{ord}_0$.

Several versions of Izumi's Theorem can be found in the literature. In the case when k is of characteristic zero and Y is smooth, it goes back at least to Tougeron, see [Tou72, p.178] (the same proof was used in the context of plurisubharmonic functions by the second author in [Fav99]). A proof based on multiplier ideals is given in [ELS03]. These approaches give an estimate on the optimal constant C in (*) in terms of log-discrepancies.

Izumi himself was mainly interested in the case of singular complex analytic spaces, see [Izu81, Izu85]. His argument has been generalized by Rees [Ree89], and alternative proofs given by Hübl and Swanson [HS01], and Beddani [Bed09]. Another approach, based on the notion of key polynomials, was recently developed by Moghaddam [Mog11], see [FJ04] in the two-dimensional case. For a connection between Izumi's Theorem and the Artin-Rees Lemma, see [Ron06].

Our objective is not to generalize the setting of Izumi's Theorem, but to make the statement more precise. Consider a projective birational morphism $\pi : X \rightarrow Y$ with X smooth. We assume that $Z := \pi^{-1}(0)$ is a divisor with simple normal crossing support such that any nonempty intersection between irreducible components of Z is irreducible. (We do not assume that Z is reduced and the exceptional set of π may be strictly larger than Z . If k has characteristic zero, the existence of such a morphism follows from Hironaka's Theorem.)

The *dual complex* $\Delta = \Delta(X, Z)$ is a simplicial complex encoding the intersections of the irreducible components of Z . We can view the elements of Δ as *quasimonomial valuations* on $\mathcal{O}_{Y,0}$ centered at 0 , see e.g. [JM10].² There is a natural (integral) affine structure on Δ . Pick a metric on Δ that is compatible with this structure.

Any function $f \in \mathcal{O}_{Y,0}$ defines a nonnegative function on Δ given by $v \mapsto v(f)$.

Theorem A. *There exists a constant $A > 0$ such that for any $f \in \mathcal{O}_{Y,0}$, the function $v \mapsto v(f)$ on Δ is concave on each face and Lipschitz continuous with Lipschitz constant at most $A \operatorname{ord}_0(f)$.*

The constant A depends on X and on the metric on $\Delta = \Delta(X, Z)$ but not on f .

It is not hard to see that Theorem A implies Izumi's Theorem in the case when the base field k has characteristic zero. Indeed, in view of Hironaka's theorem [Hir64], any divisorial valuation on $k(Y)$ centered at 0 is proportional to a point v in some dual complex Δ of some X as above; we can even choose v as a vertex. Further, Z , and hence Δ , is connected as a consequence of Zariski's Main Theorem. By Theorem A we have

$$\max_{v \in \Delta} v(f) \leq (1 + A \operatorname{diam}(\Delta)) \min_{v \in \Delta} v(f).$$

On the other hand, one can show (see §4.2) that $\min_{v \in \Delta} v(f)$ is comparable to $\operatorname{ord}_0(f)$; hence Izumi's Theorem follows.

²Quasimonomial valuations are also known as Abhyankar valuations.

One can also rephrase Theorem A in terms of Newton polyhedra. See §4.6 for details on what follows. Let E_i , $i \in I$ be the irreducible components of Z . Pick $J \subset I$ such that $E_J := \bigcap_{j \in J} E_j \neq \emptyset$, and let $z_j \in \mathcal{O}_{X, \xi_J}$, $j \in J$ be a system of coordinates at the generic point ξ_J of E_J such that $E_j = \{z_j = 0\}$ for $j \in J$. Using Cohen's Theorem, we can expand any $f \in \mathcal{O}_{Y,0} \subset \mathcal{O}_{X, \xi_J}$ as a formal power series in the z_j with coefficients in the residue field of ξ_J . Let $\text{Nw}(f, J) \subset \mathbf{R}_{\geq 0}^J$ be the Newton polyhedron of this expansion. Fix a norm on \mathbf{R}^J .

Theorem A'. *There exists a constant $A > 0$ such that for any $f \in \mathcal{O}_{Y,0}$ and any J with $E_J \neq \emptyset$, all extremal points of the Newton polyhedron $\text{Nw}(f, J)$ are of norm at most $A \text{ord}_0(f)$.*

Theorem A will be a consequence of a more general result that we now describe. Let X be a smooth, quasiprojective variety over k and $Z \subset X$ an effective divisor with proper and connected simple normal crossing support such that any nonempty intersection between irreducible components of Z is irreducible. We view the elements of the dual complex $\Delta = \Delta(X, Z)$ as rank 1 valuations on the function field of X normalized by $v(Z) = 1$.

Fix an effective divisor G on some open neighborhood of Z in X . We can define $v(G)$ for any $v \in \Delta$ using local defining equations of G . Thus G gives rise to a function $\chi = \chi_G$ on Δ , defined by $\chi(v) := v(G)$. Fix a line bundle $M \in \text{Pic}(X)$ that is ample on Z .

Theorem B. *There exist constants A and B such that for any G as above, the function $\chi = \chi_G$ on Δ is concave on each face and Lipschitz continuous with Lipschitz constant at most*

$$A \min_{\Delta} \chi + B \max_J |(G \cdot M^{n-|J|-1} \cdot E_J)|, \quad (**)$$

where the maximum is over subsets $J \subset I$ for which $E_J \neq \emptyset$. Here the constants A and B depend on X , M and the metric chosen on $\Delta = \Delta(X, Z)$, but not on G .

Theorem A follows from Theorem B by picking G as the divisor of $f \circ \pi$. Indeed, the second item in (**) vanishes, and one can show that $\min_{\Delta} \chi$ and $\text{ord}_0(f)$ are comparable.

Theorem B can also be applied to study polynomials at infinity. Fix an embedding $\mathbf{A}^m \subset \mathbf{P}^m$. Following the terminology introduced in [FJ11, Jon12] in a dynamical context, we say that an *admissible compactification* of \mathbf{A}^m is a smooth projective variety X over k together with a projective birational morphism $\pi : X \rightarrow \mathbf{P}^m$ that is an isomorphism over \mathbf{A}^m and such that if Z is the pullback of the hyperplane at infinity $\mathbf{P}^m \setminus \mathbf{A}^m$, then Z has simple normal crossing support and any nonempty intersection between irreducible components of Z is irreducible. By Zariski's Main Theorem, the support of Z is connected. We can view the elements of the dual complex $\Delta = \Delta(X, Z)$ as valuations on $k(X) = k(\mathbf{P}^m)$ as above. In particular, any polynomial $P \in k[\mathbf{A}^m]$ defines a function on Δ , given by $v \mapsto v(P)$. This function is easily seen to be concave and piecewise affine on the faces of Δ .

Corollary C. *There exists a constant $B > 0$ such that if $P \in k[\mathbf{A}^m]$ is a polynomial of degree $d \geq 1$, then the function $v \mapsto v(P)$ on Δ is Lipschitz continuous with Lipschitz constant at most Bd .*

This follows by taking the divisor G as the pullback to X of the hypersurface on \mathbf{P}^m defined as the zero locus of P . We have $v(P) = v(G) - d = \chi_G(v) - d$ for $v \in \Delta$, so the

Lipschitz constant of $v \mapsto v(P)$ is the same as that of χ_G . Now, $\min_{\Delta} \chi_G = 0$, so Corollary C follows from Theorem B.

Finally, we use Theorem A in order to study the variation of several natural numerical invariants associated to rank 1 valuations. As above, let Y be a normal variety of dimension m , defined over an algebraically closed field k , and let 0 be a closed point of Y . Pick any two \mathfrak{m}_0 -primary ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathcal{O}_{Y,0}$, and denote by $e(\mathfrak{a}_i)$ their Hilbert-Samuel multiplicities. It is a theorem due to Teissier and Risler [Tei72, §2] that the function $(r, s) \mapsto e(\mathfrak{a}_1^r \cdot \mathfrak{a}_2^s)$ is a homogeneous polynomial of degree m and that we can find nonnegative integers $e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]})$, $0 \leq i \leq m$, such that

$$e(\mathfrak{a}_1^r \cdot \mathfrak{a}_2^s) = \sum_{i=0}^m \binom{m}{i} e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]}) r^{m-i} s^i$$

for all $r, s \in \mathbf{Z}_{\geq 0}$.

Pick any rank 1 valuation v on $\mathcal{O}_{Y,0}$ centered at 0 . Then the sequence of valuation ideals $\mathfrak{a}(v, n) = \{f \in \mathcal{O}_{Y,0} \mid v(f) \geq n\}$ forms a graded sequence in the sense that $\mathfrak{a}(v, n) \cdot \mathfrak{a}(v, n') \subset \mathfrak{a}(v, n+n')$ for any n, n' . One can show (see §4.4 below) that for any integer $0 \leq i \leq m$, the following limit exists:

$$\alpha_i(v) := \lim_{n \rightarrow \infty} \frac{e(\mathfrak{a}(v, n)^{[i]}; \mathfrak{m}_0^{[m-i]})}{n^i}.$$

When $i = m$, it is a theorem due to [ELS03, LM09] (see also [Cut12]) show that the sequence $\frac{m!}{n^m} \dim_k(\mathcal{O}_{Y,0}/\mathfrak{a}(v, n))$ converges and that its limit is equal to $\alpha_m(v)$. This invariant is usually referred to as the *volume* of a valuation. This invariant is quite subtle, since it can be irrational even when the valuation is divisorial, see [CS93, example 6] or [Kur03].

Now let X, Z and $\Delta = \Delta(X, Z)$ be as in Theorem A.

Corollary D. *For any $0 \leq i \leq m$, the function $v \mapsto \alpha_i(v)$ is Lipschitz continuous on Δ .*

This result is new even in the case $i = m$. Note that Fulger [Fu12] has introduced a notion of local volume for divisors on X , and proved that this local volume is locally Lipschitz in the relative Néron-Severi space $N(X/Y)$, see Proposition 1.18 in *ibid*. It is unlikely that one can recover the Lipschitz continuity of α_m through his result since there is no canonical way to attach to a valuation $v \in \Delta(X, Z)$ a divisor in X that computes $\alpha_m(v)$.³ His result is, however, close in spirit to the continuity statement for the (global) volume function on the Néron-Severi space of a projective variety, see [Laz]. The latter statement has been strengthened in [BFJ09] to show that the global volume function is in fact differentiable on the Néron-Severi space. By analogy, one can ask whether or not Fulger's local volume, and the functions $v \mapsto \alpha_i(v)$ are differentiable.

In the case Y is smooth at 0 , we have

$$\alpha_1(v) = \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1}.$$

³When 0 is an isolated singularity, $Z(v) := \lim_{n \rightarrow \infty} \frac{1}{n} Z(\mathfrak{a}(v, n))$ is a nef b -divisor over 0 in the sense of [BdFF10], and one can show that $\alpha_m(v) = -Z(v)^m$. However it is unclear how to use this interpretation to prove the continuity of α_m .

In the general singular case, one can find a constant $C > 0$ such that

$$C^{-1} \alpha_1(v) \leq \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1} \leq C \alpha_1(v)$$

for all v , see Proposition 4.8. In particular, Corollary D gives a control on the variation of the optimal constant appearing in Izumi's theorem. More generally we obtain

Corollary E. *The function $(v, v') \mapsto \sup_{\mathfrak{m}_0} \frac{v}{v'}$ is Lipschitz continuous on $\Delta \times \Delta$.*

The constant $\sup_{\mathfrak{m}_0} \frac{v}{v'}$ is sometimes referred to as the linking number of two valuations, see [Huc70, Sam59].

In dimension 2 over a smooth point, then $\alpha_2(v) = \text{Vol}(v)$ is equal to $(\sup_{\mathfrak{m}_0} v / \text{ord}_0)^{-1}$ by [FJ04, Remark 3.33] for any valuation normalized by $v(\mathfrak{m}_0) = +1$. In this case, we thus have

$$\alpha_2(v) = \text{Vol}(v), \quad \alpha_1(v) = \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1}, \quad \text{and} \quad \alpha_2(v) = \frac{\alpha_1(v)}{v(\mathfrak{m}_0)}.$$

Observe that since the function $\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0}$ is affine on (each segment of) the dual graph Δ by [FJ04, §6], it follows that α_1 and α_2 are both differentiable functions on Δ .

Our approach to Theorem B follows [BFJ12a], where a similar result was proved in a slightly different context. The fact that χ is continuous, concave and piecewise affine on the faces on $\Delta = \Delta(X, Z)$ is a direct consequence of the way Δ is embedded into the set of valuations on the function field on X . After this observation the proof consists of two steps.

First we give an upper bound for χ on the vertices of Δ . Our argument for this uses elementary intersection theory and in fact is quite close to the original proof of Izumi's Theorem by Izumi himself.

Second, we prove the Lipschitz estimate. Because of the concavity, it suffices to bound certain directional derivatives of χ from above. To do this, we first define a suitable simplicial subdivision Δ' of Δ such that χ is affine on (a suitable subset of) the faces of Δ' . Using the toroidal techniques of [KKMS], we can associate to Δ' a projective birational morphism $X' \rightarrow X$, where X' is a normal, \mathbf{Q} -factorial variety. Roughly speaking, the directional derivatives on Δ translate into actual differences on Δ' , and these can be estimated more or less as in the first step.

One of our motivations behind this paper is to study pluripotential theory on Berkovich spaces [Ber90] over a field equipped with a trivial norm. The Lipschitz estimate in Theorem B implies the compactness of certain spaces of quasi-plurisubharmonic functions that appear in [BFJ08, BdFF10]. These applications to pluripotential theory will appear elsewhere; the corresponding results (including the Lipschitz estimate) for a discretely valued field can be found in [BFJ12a, BFJ12b]. For more on (pluri)potential theory in a non-Archimedean setting, see also [BR10, Thu05, FR10, FJ04, Jon12].

The paper is organized as follows. In §1 we recall some basic facts about valuations in general and quasimonomial valuations in particular. We also state a result that follows from [KKMS]. In §2 we recall some facts about Lipschitz constants for convex functions. The proof of our main result, Theorem B, is then given in §3 whereas its various consequences are established in §4.

1. BACKGROUND

Throughout the paper, k is an algebraically closed field. By a variety over k we mean a separated integral scheme of finite type over k . If Z is a subscheme of a scheme X , we denote by $|Z|$ its support.

1.1. Valuations. Let X be a normal, quasiprojective variety over k . By a *valuation on X* we mean a (rank 1) valuation $v : k(X) \rightarrow \mathbf{R}$ that is trivial on k and admits a *center* on X , that is, a point (not necessarily closed) $\xi \in X$ such that v is nonnegative on the local ring $\mathcal{O}_{X,\xi}$ and strictly positive on the maximal ideal of this ring. Since X is assumed separated, the center is unique if it exists. We write Val_X for the set of all valuations on X . For a closed point $0 \in X$, we shall also denote by $\text{Val}_{X,0}$ the subset of valuations $v \in \text{Val}_X$ such that $v(\mathfrak{m}_0) > 0$.

If G is a \mathbf{Q} -Cartier divisor on X and $v \in \text{Val}_X$, then we define $v(G) := \frac{1}{m}v(f_m)$, where $m \in \mathbf{Z}_{>0}$ is such that mG is a Cartier divisor and $f_m \in \mathcal{O}_{X,\xi}$ is a local equation for mG at the center ξ of v on X . If G is effective, then $v(G) \geq 0$ with strict inequality if and only if ξ is contained in the support of G .

Consider a proper birational map $\pi : X' \rightarrow X$ with X' normal. If $E \subset X'$ is a prime divisor, then ord_E , the order of vanishing along E defines an element of $\text{Val}_{X'} = \text{Val}_X$. Any valuation proportional to such a valuation will be called *divisorial*.

1.2. Dual complexes. Now assume (X, Z) is an *SNC pair*. By this we will mean that X is a smooth, quasiprojective variety over k and $Z \subset X$ is an effective divisor with projective, connected, simple normal crossing support such that any nonempty intersection of irreducible components of Z is connected. Thus we can write $Z = \sum_{i \in I} b_i E_i$, where E_i , $i \in I$ are the irreducible components of $|Z|$, $b_i \in \mathbf{Z}_{>0}$ and, for any $J \subset I$, the intersection $E_J := \bigcap_{j \in J} E_j$ is either empty or irreducible.

The *dual complex* $\Delta = \Delta(X, Z)$ is a simplicial complex defined in the usual way: to each $i \in I$ is associated a vertex e_i and to each $J \subset I$ with $E_J \neq \emptyset$ is associated a simplex σ_J containing all the e_j , $j \in J$.

Let $\text{Div}(X, Z) \simeq \bigoplus_{i \in I} \mathbf{Z}E_i$ be the free abelian group of divisors on X supported on $|Z|$. Set $\text{Div}(X, Z)_{\mathbf{R}} := \text{Div}(X, Z) \otimes_{\mathbf{Z}} \mathbf{R} \simeq \bigoplus_{i \in I} \mathbf{R}E_i$. We can embed Δ in the dual vector space $\text{Div}(X, Z)_{\mathbf{R}}^*$ as follows. A vertex e_i of Δ is identified with the element in $\text{Div}(X, Z)_{\mathbf{R}}^*$ satisfying $\langle e_i, E_i \rangle = b_i^{-1}$ and $\langle e_i, E_j \rangle = 0$ for $i \neq j$. A simplex σ_J of Δ is identified with the convex hull of $(e_j)_{j \in J}$ in $\text{Div}(X, Z)_{\mathbf{R}}^*$. In this way, Δ can be written

$$\Delta = \left\{ t = \sum_{i \in I} t_i e_i \mid t_i \geq 0, \sum_i b_i t_i = 1, \bigcap_{i|t_i > 0} E_i \neq \emptyset \right\} \subset \text{Div}(X, Z)_{\mathbf{R}}^*.$$

This embedding naturally equips Δ with an *integral affine structure*: the integral affine functions are the restrictions to Δ of the elements in $\text{Div}(X, Z)$.

1.3. Quasimonomial valuations. We can also embed the dual complex Δ into the valuation space Val_X . See [JM10, §3] for details on what follows.

Pick a point $t = \sum_{i \in I} t_i e_i \in \Delta \subset \text{Div}(X, Z)_{\mathbf{R}}^*$, let J be the set of indices $j \in I$ such that $t_j > 0$ and let ξ_J be the generic point of $E_J = \bigcap_{j \in J} E_j$. Pick local algebraic coordinates $z_j \in \mathcal{O}_{X,\xi_J}$, $j \in J$, such that $E_j = \{z_j = 0\}$. We then associate to t the valuation val_t , which is a monomial valuation in these coordinates with weight t_j on z_j , $j \in J$. More precisely, val_t

is defined as follows. Using Cohen's Theorem, we can write any $f \in \mathcal{O}_{X, \xi_J}$ in the complete ring $\widehat{\mathcal{O}_{X, \xi_J}}$ as a formal power series

$$f = \sum_{\alpha \in \mathbf{N}^J} f_\alpha z^\alpha. \quad (1.1)$$

where $f_\alpha \in \widehat{\mathcal{O}_{X, \xi_J}}$ and, for each α , either $f_\alpha = 0$ or $f_\alpha(\xi_J) \neq 0$. We then set

$$\text{val}_t(f) := \min\{\langle t, \alpha \rangle \mid f_\alpha \neq 0\}. \quad (1.2)$$

While the expansion (1.1) is not unique, one can show that (1.2) is well defined. Further, it suffices to take the minimum over finitely many α . If $t \in \Delta$, then the center of val_t on X is the generic point of E_J , where $J \subset I$ is defined by the property that v lies in the relative interior of σ_J .

Proposition 1.1. *Let (X, Z) be an SNC pair. Then, for any effective divisor G on X , the function $v \mapsto v(G)$ is continuous, concave and integral piecewise affine on $\Delta = \Delta(X, Z)$.*

Proof. The function $v \mapsto v(G)$ is continuous on Val_X , so its restriction to Δ is also continuous. Let $\sigma = \sigma_J$ be a face of Δ , determined by a subset $J \subset I$ such that $E_J \neq \emptyset$. Let $\xi = \xi_J$ be the generic point of E_J and $f \in \mathcal{O}_{X, \xi}$ a defining equation for G at ξ . It then follows from (1.2) that $t \mapsto \text{val}_t(f)$ is continuous, piecewise integral affine and convex on σ_J . \square

The valuation val_t is divisorial if and only if $t_j \in \mathbf{Q}$ for all j , see [JM10, Remark 3.9]. In particular, the set of $t \in \Delta$ for which val_t is divisorial is dense in Δ .

1.4. Subdivisions and blowups. A *subdivision* Δ' of $\Delta = \Delta(X, Z)$ is a compact rational polyhedral complex of $\text{Div}(X, Z)_{\mathbf{R}}^*$ refining Δ . A subdivision Δ' is *simplicial* if its faces are simplices. It is *projective* if there exists a convex, piecewise integral function h on Δ such that Δ' is the coarsest subdivision of Δ on each of whose faces h is affine. (Such a function h is called a support function for Δ' .)

Theorem 1.2. *Let Δ' be a simplicial projective subdivision of Δ . Then there exists a projective birational morphism $\rho : X' \rightarrow X$ with the following properties:*

- (i) ρ is an isomorphism on $X' \setminus |Z'|$, where $Z' := \rho^{-1}(Z)$;
- (ii) X' is normal, Z' has pure codimension 1 and every irreducible component of Z' is \mathbf{Q} -Cartier;
- (iii) the vertices $(e'_i)_{i \in I'}$ of Δ' are in bijection with the irreducible components $(E'_i)_{i \in I'}$ of Z' : for each $i \in I'$, the center on X' of e'_i is the generic point of E'_i ;
- (iv) If $J' \subset I'$, then $E'_{J'} := \bigcap_{j \in J'} E'_j$ is nonempty if and only if the corresponding vertices e'_j , $j \in J'$ of Δ' span a face $\sigma'_{J'}$ of Δ' ; in this case, $E'_{J'}$ is normal, irreducible, of codimension $|J'|$, and its generic point is the center of v on X' for all v in the relative interior of $\sigma'_{J'}$;
- (v) for each $i \in I'$, the function $\Delta \ni v \rightarrow v(E'_i) \in \mathbf{R}$ is affine on the simplices of Δ' .

Since Z is a divisor with simple normal crossing singularities on a smooth ambient space, the inclusion $X \setminus Z \subset X$ is a toroidal embedding in the sense of [KKMS, Chapter II]. This result is thus a consequence of the toroidal analysis in op. cit.

2. SOME CONVEX ANALYSIS

In this section we note some basic facts about convex functions. Let V be a finite dimensional real vector space and $\tau \subset V$ a compact convex set containing at least two points. Denote by $\mathcal{E}(\tau)$ the set of extremal points of τ .

Given a norm $\|\cdot\|$ on V , the Lipschitz constant of a continuous function $\varphi : \tau \rightarrow \mathbf{R}$ is defined as usual as

$$\text{Lip}_\tau(\varphi) := \sup_{v \neq v'} \frac{|\varphi(v) - \varphi(v')|}{\|v - v'\|} \in [0, +\infty]$$

and its $C^{0,1}$ -norm is then

$$\|\varphi\|_{C^{0,1}(\tau)} := \|\varphi\|_{C^0(\tau)} + \text{Lip}_\tau(\varphi),$$

where $\|\varphi\|_{C^0(\tau)} := \sup_\tau |\varphi|$. This quantity of course depends on the choice of $\|\cdot\|$, but since all norms on V are equivalent, choosing another norm only affects the estimates to follow by an overall multiplicative constant.

2.1. Directional derivatives. Now let $\varphi : \tau \rightarrow \mathbf{R}$ be convex and continuous. For $v, w \in \tau$ we define the directional derivative of φ at v towards w as

$$D_v \varphi(w) := \left. \frac{d}{dt} \right|_{t=0_+} \varphi((1-t)v + tw); \quad (2.1)$$

this limit exists by convexity of φ .

Lemma 2.1. *For any fixed $w \in \tau$, the function $v \mapsto D_v \varphi(w)$ is upper semicontinuous.*

Proof. Fix v and let $\varepsilon > 0$. Then there exists $0 < t < 1$ such that

$$D_v \varphi(w) \geq \frac{\varphi(tw + (1-t)v) - \varphi(v)}{t} - \varepsilon.$$

Since φ is continuous, we have

$$\frac{\varphi((1-t)v + tw) - \varphi(v)}{t} \geq \frac{\varphi((1-t)v' + tw) - \varphi(v')}{t} - \varepsilon$$

for any v' close to v . Now φ is convex hence

$$\frac{\varphi((1-t)v' + tw) - \varphi(v')}{t} \geq D_{v'} \varphi(w),$$

and we conclude that

$$D_v \varphi(w) \geq D_{v'} \varphi(w) - 2\varepsilon$$

for any v' close to v . This ends the proof. \square

Proposition 2.2. *There exists $C > 0$ such that every Lipschitz continuous convex function $\varphi : \tau \rightarrow \mathbf{R}$ satisfies*

$$C^{-1} \|\varphi\|_{C^{0,1}(\tau)} \leq \|\varphi\|_{C^0(\partial\tau)} + \sup_{e \in \mathcal{E}(\tau), v \in \text{int}(\tau)} |D_{\pi_e(v)} \varphi(e)| \leq C \|\varphi\|_{C^{0,1}(\tau)}.$$

Here $\pi_e(v) \in \partial\tau$ is the unique point in $\partial\tau$ such that $v \in [e, \pi_e(v)]$.

Observe that $\sup\{|D_{\pi_e(v)} \varphi(e)|, e \in \mathcal{E}(\tau), v \in \text{int}(\tau)\}$ equals $\sup\{|D_w \varphi(e)|, e \in \mathcal{E}(\tau), w \in \partial\tau, [w, e] \not\subset \partial\tau\}$.

For the proof, see [BFJ12a, Lemma A.2].

2.2. Newton polyhedra. Assume now that $\tau \subset V$ is a compact polytope whose affine span $\langle \tau \rangle$ is an affine hyperplane that does not contain the origin of V . Let $\varphi : \tau \rightarrow \mathbf{R}$ be a piecewise affine continuous convex function. It extends as a 1-homogeneous piecewise linear convex function on the polyhedral cone $\hat{\tau}$ over τ , whose *Newton polyhedron* $\text{Nw}(\varphi)$ is as usual defined as the convex subset of V^* consisting of all linear forms $m \in V^*$ such that $m \leq \varphi$ on $\hat{\tau}$ (or, equivalently, on τ). We endow V^* with the dual norm $\|m\| := \sup_{\|v\|=1} \langle m, v \rangle$.

Proposition 2.3. *There exists a constant $C > 0$, not depending on φ , such that*

$$C^{-1} \|\varphi\|_{C^{0,1}(\tau)} \leq \max_{m \in \mathcal{E}_\tau(\varphi)} \|m\| \leq C \|\varphi\|_{C^{0,1}(\tau)},$$

where $\mathcal{E}_\tau(\varphi) \subset V^*$ denotes the (finite) set of extremal points of the Newton polyhedron of φ .

Proof. Since τ is a non-empty compact subset disjoint from the linear hyperplane W parallel to $\langle \tau \rangle$, it is clear by homogeneity that there exists $C > 0$ such that

$$\|m\| \leq C \left(\|m|_W\| + \inf_{v \in \tau} |\langle m, v \rangle| \right) \quad (2.2)$$

for all $m \in M_{\mathbf{R}}$. On the other hand, elementary convex analysis tells us that

$$\varphi(v) = \max_{m \in \mathcal{E}_\tau(\varphi)} \langle m, v \rangle \quad (2.3)$$

for all $v \in \tau$, and that the set $\{v \in \tau \mid \varphi(v) = \langle m, v \rangle\}$ has non-empty interior in τ for each $m \in \mathcal{E}_\tau(\varphi)$. We thus see that the image of the gradient of φ on its differentiability locus is exactly the finite set $\{m|_W \mid m \in \mathcal{E}_\tau(\varphi)\} \subset W^*$, which implies that the Lipschitz constant of φ on τ satisfies

$$\text{Lip}_\tau(\varphi) = \max_{m \in \mathcal{E}_\tau(\varphi)} \|m|_W\|.$$

Since $\|\varphi\|_{C^0(\tau)} \leq \max_\tau \|v\| \max_{\mathcal{E}_\tau(\varphi)} \|m\|$, and $\|m|_W\| \leq \|m\|$, the left-hand inequality is now clear. Since for each $m \in \mathcal{E}_\tau(\varphi)$ there exists $v \in \tau$ such that $\varphi(v) = \langle m, v \rangle$, we have $\inf_{v \in \tau} |\langle m, v \rangle| \leq \|\varphi\|_{C^0(\tau)}$, and (2.2) yields the right-hand inequality. \square

3. PROOF OF THEOREM B

Write $Z = \sum_{i \in I} b_i E_i$. Fix a line bundle M on X that is ample on Z and set

$$\theta_G := \max_{J \subset I} |(G \cdot E_J \cdot M^{n-|J|-1})|.$$

Note that $\theta_G = 0$ if the line bundle $\mathcal{O}_X(G)|_Z$ is trivial.

Throughout the proof, $A \geq 1$ and $B \geq 0$ will denote various constants whose values may vary from line to line, but they do not depend on G .

We already know from Proposition 1.1 that the function $\chi = \chi_G$ is nonnegative, concave and integral piecewise affine on each simplex in Δ .

3.1. Bounding the values on vertices. We first prove the estimate

$$\chi(e_i) \leq A \min_{\Delta} \chi + B \theta_G \quad \text{for all } i \in I. \quad (3.1)$$

Since χ is concave on each simplex, its minimum on Δ must be attained at a vertex. Further, the 1-skeleton of Δ is connected since Z has connected support. We may therefore assume that $I = \{0, 1, \dots, m\}$, where $\chi(e_0) = \min \chi$, e_i is adjacent to e_{i+1} (i.e. $E_i \cap E_{i+1} \neq \emptyset$)

for $i = 0, \dots, l-1$ and $\chi(e_i) = \max_{i \in I} \chi$, where $1 \leq l \leq m$. It suffices to prove (3.1) for $1 \leq i \leq l$ and this we shall do by induction. Let us write

$$G = \sum_{j \in I} b_j \chi(e_j) E_j + \tilde{G},$$

where \tilde{G} is an effective divisor whose support does not contain any E_i . For each $i \in I$ we then have

$$\begin{aligned} \sum_{j \in I} b_j \chi(e_j) (E_i \cdot E_j \cdot M^{n-2}) &= (G \cdot E_i \cdot M^{n-2}) - (\tilde{G} \cdot E_i \cdot M^{n-2}) \\ &\leq (G \cdot E_i \cdot M^{n-2}) \\ &\leq \theta_G. \end{aligned} \tag{3.2}$$

Set

$$c_{ij} := b_j (E_i \cdot E_j \cdot M^{n-2})$$

for $i, j \in I$. Note that $c_{ij} \geq 0$ for $j \neq i$, with strict inequality if and only if e_i and e_j are adjacent. In particular, $c_{i,i+1} > 0$ for $0 \leq i < l$. Since $\chi \geq 0$ it follows from (3.2) that

$$\begin{aligned} \chi(e_{i+1}) &\leq \frac{|c_{ii}|}{c_{i,i+1}} \chi(e_i) + \frac{\theta_G}{c_{i,i+1}} \\ &\leq A_0 \chi(e_i) + B_0 \theta_G, \end{aligned}$$

for $0 \leq i < l$, where the constants A_0 and B_0 do not depend on i or G . We may assume that $A_0 \geq 1$. A simple induction now gives (3.1) for $0 \leq i < l$, with $A = A_0^l$ and $B = B_0(1 + A_0 + \dots + A_0^{l-1})$.

3.2. Bounding Lipschitz constants. Let τ be a face of Δ . Our aim is to prove by induction on $\dim \tau$ that

$$\|\chi|_\tau\|_{C^{0,1}} \leq A \min_{\Delta} \chi + B \theta_G.$$

Here the $C^{0,1}$ -norm is defined as the sum of the sup-norm and the Lipschitz constant; see §2.

The case $\dim \tau = 0$ is settled by (3.1) so let us assume that $\dim \tau > 0$. By Proposition 1.1 the restriction of χ to τ is piecewise affine and concave. It therefore admits directional derivatives, and we set as in (2.1)

$$D_v \chi(w) := \left. \frac{d}{dt} \right|_{t=0+} \chi((1-t)v + tw)$$

for $v, w \in \tau$.

Let us say that a codimension 1 face of τ is *opposite* a vertex when it is the convex hull of the remaining vertices of τ . This notion is well-defined since τ is a simplex.

Proposition 3.1. *We have*

$$D_v \chi(e) \leq A \min_{\Delta} \chi + B \theta_G$$

for any vertex e of τ , and any rational point v in the relative interior of the face σ of τ opposite to e , such that $\chi|_\sigma$ is affine near v .

Granting this result, let us explain how to conclude the proof of Theorem C. By induction we have $\sup_{\partial\tau} \chi \leq A \min_{\Delta} \chi + B\theta_G$. The fact that χ is concave and nonnegative implies that

$$D_v \chi(e) \geq \chi(e) - \chi(v) \geq -\min \chi$$

for any $e, v \in \partial\tau$. By Proposition 3.1 this gives (assuming, as we may, that $A \geq 1$)

$$|D_v \chi(e)| \leq A |\min_{\Delta} \chi| + B\theta_G \quad (\dagger)$$

for any vertex e of τ and any rational point v in the relative interior of the face σ opposite to e such that $\chi|_{\sigma}$ is affine near v .

By Lemma 2.1 applied to the convex function $-\chi$, the function $v \mapsto D_v \chi(e)$ is lower semicontinuous on σ . It follows by density that the upper bound (\dagger) holds for any v in the relative interior of σ . We conclude by Proposition 2.2 that the $C^{0,1}$ -norm of $\chi|_{\tau}$ is bounded by $A \min_{\Delta} \chi + B\theta_G$, completing the proof of Theorem C.

The rest of §3 is devoted to the proof of Proposition 3.1.

3.3. Special subdivisions. The *star* of a face σ of Δ is defined as usual as the subcomplex $\text{Star}(\sigma)$ of Δ made up of all the faces of Δ containing σ . A vertex e_i thus belongs to the star of a simplex σ_J iff E_i intersects E_J .

We shall need the following construction, see Figure 1. Let $\sigma = \sigma_J$ be a face of Δ and $L \subset I$ the set of vertices of Δ contained in $\text{Star}_{\Delta}(\sigma)$. Thus $j \in L$ if and only if $E_j \cap E_J \neq \emptyset$. Consider a rational point v in the relative interior of σ . Given $0 < \varepsilon < 1$ rational and $j \in L$ set $e_j^{\varepsilon} := \varepsilon e_j + (1 - \varepsilon)v$. We shall define a projective simplicial subdivision $\Delta' = \Delta'(\varepsilon, v)$ of Δ .

To define Δ' , first consider a polyhedral subdivision $\Delta^{\varepsilon} = \Delta^{\varepsilon}(v)$ of Δ leaving the complement of $\text{Star}_{\Delta}(\sigma)$ unchanged. The set of vertices of Δ^{ε} is precisely $(e_i)_{i \in I} \cup (e_j^{\varepsilon})_{j \in L}$. The faces of Δ^{ε} contained in $\text{Star}(\sigma)$ are of the following two types:

- if the convex hull $\text{Conv}(e_{j_1}, \dots, e_{j_m})$ is a face of Δ containing σ , then $\text{Conv}(e_{j_1}^{\varepsilon}, \dots, e_{j_m}^{\varepsilon})$ is a face of Δ^{ε} ;
- if $\text{Conv}(e_{j_1}, \dots, e_{j_m})$ is a face of Δ contained in $\text{Star}(\sigma)$ but not containing σ , then both $\text{Conv}(e_{j_1}, \dots, e_{j_m})$ and $\text{Conv}(e_{j_1}, \dots, e_{j_m}, e_{j_1}^{\varepsilon}, \dots, e_{j_m}^{\varepsilon})$ are faces of Δ^{ε} .

In a neighborhood of v , note that the subdivision Δ^{ε} is obtained by scaling Δ by a factor ε . More precisely, consider the affine map $\psi^{\varepsilon} : \text{Star}(\sigma) \rightarrow \text{Star}(\sigma)$ defined by $\psi^{\varepsilon}(w) = \varepsilon w + (1 - \varepsilon)v$. Then $\sigma^{\varepsilon} := \psi^{\varepsilon}(\sigma)$ is the face of Δ^{ε} containing v in its relative interior, and $\psi^{\varepsilon}(\text{Star}_{\Delta}(\sigma)) = \text{Star}_{\Delta^{\varepsilon}}(\sigma^{\varepsilon})$. In particular, even though Δ^{ε} is not simplicial in general, all faces of Δ^{ε} containing σ^{ε} are simplicial.

We claim that Δ^{ε} is projective. To see this, write $v = \sum_{j \in J} s_j e_j$, with $s_j > 0$ rational and $\sum s_j = 1$. For $j \in J$, define a linear function λ_j on $\sum_{i \in I} \mathbf{R}_+ e_i \supset \Delta$ by $\lambda_j(\sum t_i e_i) = -t_j/s_j$ and set $h = \max\{\max_{j \in J} \lambda_j, -(1 - \varepsilon)\}$. A suitable integer multiple of h^{-1} is then a strictly convex support function for Δ^{ε} in the sense of §1.4.

Now define $\Delta' = \Delta'(\varepsilon)$ as a simplicial subdivision of Δ^{ε} obtained using repeated barycentric subdivision in a way that leaves $\text{Star}_{\Delta^{\varepsilon}}(\sigma^{\varepsilon})$ unchanged. By [KKMS, pp.115–117], Δ' is still projective.

Note that $\sigma' := \sigma^{\varepsilon}$ is the face of Δ' containing v in its relative interior, For $j \in L$ set $e'_j = e_j^{\varepsilon}$. These are the vertices of Δ' contained in $\text{Star}_{\Delta'}(\sigma')$.

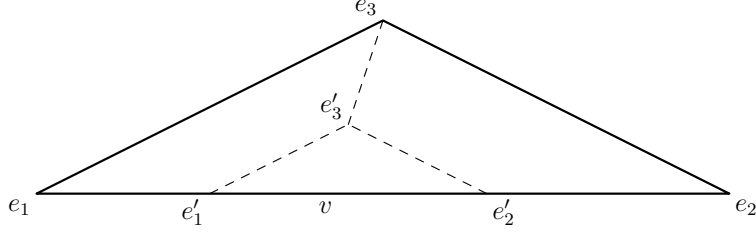


FIGURE 1. The subdivision of §3.3. Here v lies in the relative interior of the simplex σ of Δ with vertices e_1 and e_2 . The picture shows the intermediate subdivision Δ^ε , where v lies in the relative interior of the simplex σ' with vertices e'_1 and e'_2 . The final subdivision Δ' is obtained from Δ^ε by barycentric subdivision of the quadrilaterals $\text{Conv}(e_1, e_3, e'_1, e'_3)$ and $\text{Conv}(e_2, e_3, e'_2, e'_3)$

3.4. Proof of Proposition 3.1. Let I be the set of vertices in Δ , let $L \subset I$ be the set of vertices contained in $\text{Star}_\Delta(\sigma)$ and $J \subset L$ the set of vertices of σ . Thus $\sigma = \sigma_J$. The irreducible subvariety E_J has codimension $|J| = p \geq 1$.

Consider the simplicial projective subdivision $\Delta' = \Delta'(\varepsilon)$ constructed in §3.3. For $j \in L$, $e'_j := \varepsilon e_j + (1 - \varepsilon)v$ is a vertex of Δ' . Recall that $\sigma' = \sigma'_J$ is the face of Δ' containing v in its relative interior. Since $\chi|_\sigma$ is assumed affine in a neighborhood of v , we may choose $\varepsilon > 0$ small enough that:

- χ is affine on $\sigma' \subset \sigma$
- χ is affine on each segment $[v, e'_j]$, $j \in L$.

Let $\rho : X' \rightarrow X$ be the birational morphism corresponding to the subdivision Δ' of Δ as in Theorem 1.2. Note that ρ induces a generically finite map $E'_J \rightarrow E_J$ of projective k -varieties. Indeed, E_J (resp. E'_J) is the closure of the center of v on X (resp. X'), and both have codimension $|J| = p$ in view of Theorem 1.2.

The following result allows us to “linearize” the problem under consideration:

Lemma 3.2. *We have*

$$\rho^* \left(\chi(v)Z + \sum_{j \in L} D_v \chi(e_j) b_j E_j \right) \Big|_{E'_J} = \sum_{j \in L} \chi(e'_j) b'_j E'_j \Big|_{E'_J}$$

in $\text{Pic}(E'_J)\mathbf{Q}$, where we have set $b'_j := \text{ord}_{E'_j}(Z)$.

Grant this result for the moment. We then have

$$\rho^* G = \sum_{i \in I'} \chi(e'_i) b'_i E'_i + \tilde{G}',$$

where \tilde{G}' is an effective \mathbf{Q} -Cartier divisor on X' whose support does not contain any of the E'_i .

Lemma 3.3. *The support of \tilde{G}' does not contain E'_J . Hence $\tilde{G}'|_{E'_J}$ is effective.*

The proof is given below. Grant this result for the moment. Set $r = \deg(\rho|_{E'_J})$. Then we have

$$\begin{aligned}
r(G \cdot M^{n-p-1} \cdot E_J) &= \rho^* G \cdot (\rho^* M)^{n-p-1} \cdot E'_J \\
&= \left(\sum_{j \in L} \chi(e'_j) b'_j E'_j + \tilde{G}' \right) \cdot (\rho^* M)^{n-p-1} \cdot E'_J \\
&\geq \left(\sum_{j \in L} \chi(e'_j) b'_j E'_j \right) \cdot (\rho^* M)^{n-p-1} \cdot E'_J \\
&= \rho^* \left(\chi(v) Z + \sum_{j \in L} D_v \chi(e_j) b_j E_j \right) \cdot (\rho^* M)^{n-p-1} \cdot E'_J \\
&= r \left(\chi(v) Z + \sum_{j \in L} D_v \chi(e_j) b_j E_j \right) \cdot M^{n-p-1} \cdot E_J.
\end{aligned}$$

Here the first and last inequality follow from the projection formula. The inequality follows from Lemma 3.3. The second to last equality is a consequence of Lemma 3.2. We see that

$$\chi(v)(Z \cdot M^{n-p-1} \cdot E_J) + \sum_{j \in L} D_v \chi(e_j) b_j (E_j \cdot M^{n-p-1} \cdot E_J) \leq \theta_G. \quad (3.3)$$

By induction, the $C^{0,1}$ -norm of $\chi|_\sigma$ is under control. Since v belongs to $\sigma = \sigma_J$, this gives

$$\chi(v) + \max_{j \in J} |D_v \chi(e_j)| \leq A \min_{\Delta} \chi + B\theta_G, \quad (3.4)$$

which together with (3.3) yields an upper bound

$$\sum_{j \in L \setminus J} D_v \chi(e_j) b_j (E_J \cdot E_j \cdot M^{n-p-1}) \leq A \min_{\Delta} \chi + B\theta_G. \quad (3.5)$$

Now the fact that χ is nonnegative and concave shows that

$$\min_{j \in L \setminus J} D_v \chi(e_j) \geq \min_{j \in L \setminus J} (\chi(e_j) - \chi(v)) \geq -\chi(v) \geq -A \min_{\Delta} \chi - B\theta_G, \quad (3.6)$$

where the last inequality follows from the inductive assumption. Note that $E_j|_{E_J}$ is a non-zero effective divisor for $j \in L \setminus J$. As a consequence

$$(E_J \cdot E_j \cdot M^{n-p-1}) > 0$$

since M is ample. The inequalities (3.5) and (3.6) therefore imply that

$$\max_{j \in L \setminus J} D_v \chi(e_j) \leq A \min_{\Delta} \chi + B\theta_G,$$

which completes the proof, since $e = e_j$ for some $j \in L \setminus J$.

Proof of Lemma 3.2. We write $v = \sum_{j \in J} s_j e_j$ with $s_j > 0$ rational and $\sum_{j \in J} s_j = 1$. Set $s_i = 0$ for $i \in I \setminus J$. For $i \in I$ let χ_i be the function on Δ that is affine on each face of Δ

and satisfies $\chi_i(e_j) = \delta_{ij}$ for all $j \in I$. Since $e'_j = \varepsilon e_j + (1 - \varepsilon)v$ for $j \in L$ we get:

$$\chi_i(e'_j) = \begin{cases} \varepsilon + (1 - \varepsilon)s_i & \text{if } i = j \in J \\ (1 - \varepsilon)s_i & \text{if } i \neq j \in J \\ \varepsilon & \text{if } i = j \in L \setminus J \\ 0 & \text{if } i \neq j \in L \setminus J \end{cases}$$

By Theorem 1.2, E'_j intersects E'_j if and only if $j \in L$. We thus have

$$\rho^*(b_i E_i)|_{E'_j} = \sum_{j \in L} \chi_i(e'_j) b'_j E'_j|_{E'_j} \text{ for all } i \in I$$

and

$$\rho^* Z|_{E'_j} = \sum_{j \in L} b'_j E'_j|_{E'_j}$$

in $\text{Pic}(E'_j)_{\mathbf{Q}}$, where $b'_j = \text{ord}_{E'_j}(I_Z)$.

Recall also that χ is affine on each segment $[v, e'_i]$, so that $D_v \chi(e_i) = \varepsilon^{-1}(\chi(e'_i) - \chi(v))$ for $i \in L$. We can now compute in $\text{Pic}(E'_j)_{\mathbf{Q}}$

$$\begin{aligned} \rho^* \left(\sum_{i \in L} D_v \chi(e_i) b_i E_i \right) \Big|_{E'_j} &= \sum_{i \in L} \varepsilon^{-1} (\chi(e'_i) - \chi(v)) \left(\sum_{j \in L} \chi_i(e'_j) b'_j E'_j|_{E'_j} \right) = \\ &= \sum_{i \in J} \varepsilon^{-1} (\chi(e'_i) - \chi(v)) \left(\varepsilon b'_i E'_i|_{E'_j} + s_i \sum_{j \in J} (1 - \varepsilon) b'_j E'_j|_{E'_j} \right) + \\ &\quad + \sum_{i \in L \setminus J} \varepsilon^{-1} (\chi(e'_i) - \chi(v)) \varepsilon b'_i E'_i|_{E'_j} = \\ &= \sum_{i \in L} (\chi(e'_i) - \chi(v)) b'_i E'_i|_{E'_j} + \varepsilon^{-1} (1 - \varepsilon) \left(\sum_{i \in J} s_i (\chi(e'_i) - \chi(v)) \right) \left(\sum_{j \in J} b'_j E'_j|_{E'_j} \right) \\ &= \sum_{j \in L} (\chi(e'_j) - \chi(v)) b'_j E'_j \Big|_{E'_j}. \end{aligned}$$

The last equality follows from the fact that χ is affine on the simplex σ'_J of Δ' so that $\sum_{i \in J} s_i \chi(e'_i) = \chi(v) = \sum_{i \in J} s_i \chi(v)$. This concludes the proof. \square

Proof of Lemma 3.3. By assumption, the function $w \mapsto w(G)$ is affine on the face $\sigma' = \sigma'_J$ of Δ' . By Theorem 1.2, the same is true of the function $w \mapsto \sum_{i \in I'} \varphi(e'_i) b'_i w(E'_i)$. From this we see that the function $w \mapsto w(\tilde{G}')$ is also affine on σ' . But by construction, this function vanishes at the vertices of σ' and hence is identically zero on σ' . This implies that the support of \tilde{G}' does not contain E'_j , so that the \mathbf{Q} -Cartier divisor $\tilde{G}'|_{E'_j}$ is effective, as claimed. \square

4. CONSEQUENCES OF THEOREM B

In this final section we prove the various consequences of Theorem B, namely Theorems A and A', Izumi's Theorem (in characteristic zero) and Corollaries C, D and E.

4.1. Order functions, integral closure and Rees valuations. Let us return to the situation in the beginning of the introduction. Thus k is an algebraically closed field, Y is a normal variety over k and $0 \in Y$ is a closed point. We do not assume that Y is smooth outside 0 . Write \mathfrak{m}_0 for the maximal ideal of the local ring $\mathcal{O}_{Y,0}$ at 0 .

For any function $f \in \mathcal{O}_{Y,0}$ define

$$\text{ord}_0(f) := \max\{j \geq 0 \mid f \in \mathfrak{m}_0^j\}. \quad (4.1)$$

When 0 is a smooth point of Y , ord_0 is a divisorial valuation, associated to the exceptional divisor of the blowup of Y at 0 . In the singular case, however, ord_0 may not be a valuation. Indeed, the sequence $(\text{ord}_0(f^n))_{n \geq 1}$ which is clearly superadditive in the sense that

$$\text{ord}_0(f^{n+n'}) \geq \text{ord}_0(f^n) + \text{ord}_0(f^{n'}) \quad (4.2)$$

may fail to be additive, that is, strict inequality may hold in (4.2) for certain n, n' .

To remedy this particular fact, one defines

$$\widehat{\text{ord}}_0(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \text{ord}_0(f^n);$$

the limit exists as a standard consequence of (4.2). The function $\widehat{\text{ord}}_0$ is a special case of a construction introduced by Samuel [Sam52] and later studied extensively by Rees, see [Ree88] and also [HS, LT08, Swa11].

Recall that the *integral closure* $\overline{\mathfrak{b}}$ of an ideal $\mathfrak{b} \subset \mathcal{O}_{Y,0}$ is an ideal defined as the set of elements $f \in \mathcal{O}_{Y,0}$ that satisfy an equation

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0,$$

with $n \geq 1$ and $a_i \in \mathfrak{b}^i$ for $1 \leq i \leq n$. The following result is valid in a context far more general than what we state here, see [Hun92, Theorem 4.13] or [LT08, Proposition 1.14].

Theorem 4.1. *There exists an integer N such that*

$$\overline{\mathfrak{b}^n} \subset \mathfrak{b}^{n-N} \quad (4.3)$$

for any ideal $\mathfrak{b} \subset \mathcal{O}_{Y,0}$ and any $n \geq N$.

Let $\nu : Y^+ \rightarrow Y$ be the normalized blowup of \mathfrak{m}_0 and write

$$\mathfrak{m}_0 \cdot \mathcal{O}_{Y^+} = \mathcal{O}_{Y^+} \left(- \sum_{i=1}^k r_i E_i \right),$$

where the E_i are prime Weil divisors on Y^+ and $r_i \in \mathbf{Z}_{>0}$. For each i we have a divisorial valuation ord_{E_i} on $\mathcal{O}_{Y,0}$. We normalize these as follows.

Definition 4.2. *The divisorial valuations w_1, \dots, w_k defined by*

$$w_i := \frac{\text{ord}_{E_i}}{r_i} = \frac{\text{ord}_{E_i}}{\text{ord}_{E_i}(\mathfrak{m}_0)}$$

are called the Rees valuations of \mathfrak{m}_0 .

Theorem 4.3. *There exists an integer $N > 0$ such that the following conditions hold for any function $f \in \mathcal{O}_{Y,0}$ and any $n \geq 1$:*

- (i) $\widehat{\text{ord}}_0(f) = \min_i w_i(f)$;
- (ii) $f \in \overline{\mathfrak{m}}_0^n$ if and only if $\widehat{\text{ord}}_0(f) \geq n$;
- (iii) $\text{ord}_0(f) \leq \widehat{\text{ord}}_0(f) \leq \text{ord}_0(f) + N$;
- (iv) $\text{ord}_0(f) \leq \widehat{\text{ord}}_0(f) \leq (N + 1) \text{ord}_0(f)$.

Proof. Since ν is also the normalized blow-up of \mathfrak{m}_0^n for any $n \geq 1$, we have

$$\overline{\mathfrak{m}}_0^n = \nu_* \mathcal{O}_{Y+}(-\sum_i nr_i E_i),$$

see [Laz, Proposition 9.6.6]. Hence

$$f \in \overline{\mathfrak{m}}_0^n \quad \text{if and only if} \quad \min_i w_i(f) \geq n. \quad (4.4)$$

We first prove (i). Pick $\lambda \in \mathbf{Q}_{\geq 0}$. If $\min_i w_i(f) \geq \lambda$, then for p sufficiently divisible we have

$$f^p \in \overline{\mathfrak{m}}_0^{p\lambda} \subset \overline{\mathfrak{m}}_0^{p\lambda - N}$$

by (4.4) and (4.3), respectively. This gives $\text{ord}_0(f^p) \geq p\lambda - N$ and hence $\widehat{\text{ord}}_0(f) \geq \lambda$. On the other hand, suppose $\widehat{\text{ord}}_0(f) \geq \lambda$ and pick $0 < \mu < \lambda$. For p sufficiently divisible we then have $\text{ord}_0(f^p) \geq p\mu$, so that $f^p \in \overline{\mathfrak{m}}_0^{p\mu} \subset \overline{\mathfrak{m}}_0^{p\lambda}$. Using (4.4) we get $\min_i w_i(f) = p^{-1} \min_i w_i(f^p) \geq p\mu$ and hence $\min_i w_i(f) \geq \lambda$, proving (i).

Now (ii) follows immediately from (i) and from (4.4). As for (iii), the first inequality is obvious and the second results from (ii) and (4.3). Finally, (iv) is a direct consequence of (iii) when $\text{ord}_0(f) \geq 1$ and is trivial when $\text{ord}_0(f) < 1$ since in this case $f \notin \mathfrak{m}_0$ and $\text{ord}_0(f) = \widehat{\text{ord}}_0(f) = 0$. \square

Remark 4.4. *Theorem 4.3 is a special case of the strong valuation theorem due to Rees and is valid much more generally, see [HS, LT08, Ree88]. Our presentation follows [Laz, §9.6.A].*

4.2. Proof of Izumi's Theorem. Let v be any divisorial valuation of $k(Y)$ centered at 0. We may assume that v is normalized by $v(\mathfrak{m}_0) = 1$. It is then clear that $v \geq \text{ord}_0$.

It remains to prove that there exists a constant $C > 0$ such that $v(f) \leq C \text{ord}_0(f)$ for all $f \in \mathcal{O}_{Y,0}$. For this part, we assume that k has characteristic zero. Using Hironaka's theorem [Hir64] we can find a projective birational morphism $\pi : X \rightarrow Y$ with X smooth such that the scheme theoretic preimage $Z := \pi^{-1}(0)$ is a divisor (not necessarily reduced) with simple normal crossing support such that any nonempty intersection of irreducible components of Z is irreducible. Note that we do not assume that π is an isomorphism outside $|Z|$. We may also assume that the center of v has codimension 1 so that v is a vertex in the dual complex $\Delta = \Delta(X, Z)$ as in §1.2.

Given a function $f \in \mathcal{O}_{Y,0}$ define a continuous function $\chi = \chi_f$ on Δ by

$$\chi(v) = v(f).$$

It is clear that $\chi > 0$ on Δ if $f \in \mathfrak{m}_0$ and $\chi \equiv 0$ otherwise. Note that replacing Y by a suitable affine neighborhood of 0, we may view f as a section of the trivial line bundle \mathcal{O}_X .

We can therefore apply (3.1). We get that $v(f) \leq A \min_{\Delta} \chi_f$ for some constant $A > 0$ independent on f . It remains to relate $\min_{\Delta} \chi_f$ to $\text{ord}_0(f)$. To this end, we first prove

Lemma 4.5. *For any function $f \in \mathcal{O}_{Y,0}$ we have $\min_{\Delta} \chi_f = \widehat{\text{ord}}_0(f)$.*

Proof. If $\text{ord}_0(f) \geq n$, then $f \in \mathfrak{m}_0^n$ and hence $\min_{\Delta} \chi_f \geq \min_{v \in \Delta} v(\mathfrak{m}_0^n) = n$. Replacing f by a power, we get $\min_{\Delta} \chi_f \geq \widehat{\text{ord}}_0(f)$.

Since Z is a divisor and X is smooth, π must factor through the normalized blowup $\nu : Y^+ \rightarrow Y$ of 0. This implies that all the Rees valuations of 0 appear as (some of the) vertices of the dual complex Δ . This observation and Theorem 4.3 (i) now imply the reverse inequality. \square

Finally we have $v(f) \leq A \min_{\Delta} \chi_f = A \widehat{\text{ord}}_0(f) \leq A(N+1) \text{ord}_0(f)$ by Theorem 4.3 (i), and the proof of Izumi's theorem is complete.

Remark 4.6. *Observe that the proof does not rely on the Lipschitz estimates of Theorem B, and follows from a direct intersection theoretic computation which is similar to Izumi's original argument.*

4.3. Proof of Theorem A. Consider a projective birational morphism $\pi : X \rightarrow Y$ with X smooth such that $Z := \pi^{-1}(0)$ is a divisor with simple normal crossing support such that any nonempty intersection of irreducible components of Z is irreducible. Let $\Delta = \Delta(X, Z)$ be the dual complex.

Given a function $f \in \mathcal{O}_{Y,0}$, the function $\chi(v) := v(f)$ is continuous on Δ . It is clear that $\chi > 0$ on Δ if $f \in \mathfrak{m}_0$ and $\chi \equiv 0$ otherwise. As above, we may view f as a section of the trivial line bundle \mathcal{O}_X . We can therefore apply Theorem B and get that χ_f is concave on each face and Lipschitz continuous with Lipschitz constant at most $A \min_{\Delta} \chi_f$.

Thus the Lipschitz constant of χ_f is bounded from above by at most $A(N+1) \text{ord}_0(f)$ by Lemma 4.5 and Theorem 4.3 (iv), concluding the proof of Theorem A.

4.4. Mixed multiplicities. Let $(Y, 0)$ be as before. The Hilbert-Samuel multiplicity of an \mathfrak{m}_0 -primary ideal $\mathfrak{a} \subset \mathcal{O}_{Y,0}$ is defined as the limit

$$e(\mathfrak{a}) = \lim_{n \rightarrow \infty} \frac{m!}{n^m} \dim_k(\mathcal{O}_{Y,0}/\mathfrak{a}^n).$$

Recall that the mixed multiplicities of any two \mathfrak{m}_0 -primary ideals $\mathfrak{a}_1, \mathfrak{a}_2$ are a sequence of $m+1$ integers $e(\mathfrak{a}_1^{[0]}; \mathfrak{a}_2^{[m]}), e(\mathfrak{a}_1^{[1]}; \mathfrak{a}_2^{[m-1]}), \dots, e(\mathfrak{a}_1^{[m]}; \mathfrak{a}_2^{[0]})$ such that

$$e(\mathfrak{a}_1^r \cdot \mathfrak{a}_2^s) = \sum_{i=0}^m \binom{m}{i} e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]}) r^{m-i} s^i$$

for all $r, s \in \mathbf{Z}_+$, see [Tei72, §2] or [Laz, §1.6.B]. Observe that mixed multiplicities are symmetric in their argument $e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]}) = e(\mathfrak{a}_2^{[i]}; \mathfrak{a}_1^{[m-i]})$.

These multiplicities also have the following geometric interpretation, see [Laz, §1.6.B]. Let $\nu : Y^+ \rightarrow Y$ be any birational proper map that dominates the normalized blowups of \mathfrak{a}_1 and \mathfrak{a}_2 . For $\varepsilon = 1, 2$ write $\mathfrak{a}_{\varepsilon} \cdot \mathcal{O}_{Y^+} = \mathcal{O}_{Y^+}(-\sum_j r_{j,\varepsilon} E_j)$, with $r_{j,\varepsilon} \in \mathbf{Z}_{>0}$. Then

$$e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]}) = - \left(\sum_j r_{j,1} E_j \right)^{m-i} \cdot \left(\sum_j r_{j,2} E_j \right)^i \quad (4.5)$$

Since the antieffective divisors $-\sum_j r_{j,\varepsilon} E_j$ are π -exceptional and π -nef, it follows that mixed multiplicities are decreasing with respect to the inclusion of ideals:

$$\mathfrak{a}_1 \subset \mathfrak{a}'_1 \Rightarrow e(\mathfrak{a}_1^{[m-i]}; \mathfrak{a}_2^{[i]}) \geq e(\mathfrak{a}'_1^{[m-i]}; \mathfrak{a}_2^{[i]}) . \quad (***)$$

Pick any rank 1 valuation v on $\mathcal{O}_{Y,0}$ centered at 0. Then the sequence of valuation ideals

$$\mathfrak{a}(v, n) = \{f \in \mathcal{O}_{Y,0} \mid v(f) \geq n\}$$

forms a graded sequence in the sense that

$$\mathfrak{a}(v, n) \cdot \mathfrak{a}(v, n') \subset \mathfrak{a}(v, n + n')$$

for any n, n' . Recall that the volume of v is defined by

$$\text{Vol}(v) := \limsup_{n \rightarrow \infty} \frac{\dim_k(\mathcal{O}_{Y,0}/\mathfrak{a}(v, n))}{n^m/m!} \in [0, +\infty).$$

It is a theorem that the volume is actually defined as a limit, see [ELS03, LM09, Cut12].

Proposition 4.7. *For any integer $0 \leq i \leq m$, the sequence $\frac{1}{n^i} e(\mathfrak{a}(v, n)^{[i]}; \mathfrak{m}_0^{[m-i]})$ converges to a positive real number $\alpha_i(v)$.*

We have $\alpha_0(v) = e(\mathfrak{m})$ and $\alpha_m(v) = \text{Vol}(v)$. Moreover these numbers satisfy the Teissier inequalities

$$\alpha_i(v)^2 \leq \alpha_{i-1}(v) \alpha_{i+1}(v), \quad i = 1, \dots, m-1 . \quad (4.6)$$

Proof. Fix $0 \leq i \leq m$, and write $e_n := e(\mathfrak{a}(v, n)^{[i]}; \mathfrak{m}_0^{[m-i]})$. Since $\mathfrak{a}(v, 1)^n \subset \mathfrak{a}(v, n)$, we have

$$e(\mathfrak{a}(v, n)^{[i]}; \mathfrak{m}_0^{[m-i]}) \leq e(\mathfrak{a}(v, 1)^{n[i]}; \mathfrak{m}_0^{[m-i]}) = n^i e(\mathfrak{a}(v, 1)^{[i]}; \mathfrak{m}_0^{[m-i]}) ,$$

It follows that $\frac{e_n}{n^i} \leq e_1$ is bounded from above. Pick $\varepsilon > 0$ and choose N such that $\frac{e_N}{N^i} \leq \liminf_l \frac{e_l}{l^i} + \varepsilon$. For any $n \geq N$ write $n = pN + q$ with $p, q \in \mathbf{Z}_+$ and $0 \leq q \leq N-1$. Then $\mathfrak{a}(v, n) \supset \mathfrak{a}(v, N)^p \cdot \mathfrak{a}(v, q) \supset \mathfrak{a}(v, N)^{p+1}$ hence by monotonicity of mixed multiplicities

$$\frac{e_n}{n^i} \leq \frac{e_N}{N^i} \frac{N^i(p+1)^i}{n^i} \leq \left(\liminf_l \frac{e_l}{l^i} + \varepsilon \right) \left(\frac{N(p+1)}{pN+q} \right)^i .$$

If p is large enough, then we get $\frac{e_n}{n^i} \leq (1 - \varepsilon)(\liminf_l \frac{e_l}{l^i} + \varepsilon)$ which implies $\frac{e_n}{n^i} \rightarrow \liminf_l \frac{e_l}{l^i}$.

The fact that $\alpha_0(v) = e(\mathfrak{m}_0)$ follows from the definition, and the equality $\alpha_m(v) = \text{Vol}(v)$ is a theorem proved successively in greater generality in [ELS03, Mus02, LM09, Cut12]. The inequalities (4.6) follow from the usual Teissier inequalities for mixed multiplicities, see [Laz, Theorem 1.6.7 (iv)] and ultimately result from the Hodge index theorem.

Since $\alpha_i(v)$ is nonnegative, and $\alpha_0(v) > 0$, the Teissier inequalities imply that $\alpha_i(v) > 0$ for all i . \square

The invariant $\alpha_1(v)$ is closely related to the optimal Izumi constant. F The linking number [Huc70, Sam59] of any two rank 1 valuations on $\mathcal{O}_{Y,0}$ centered at 0 is defined by

$$\beta(v/w) := \sup_{f \in \mathfrak{m}_0} \frac{v(f)}{w(f)} \in (0, \infty] .$$

By Izumi's theorem, this number is finite whenever v and w are both quasimonomial.

Proposition 4.8. *Let $w_i = \frac{1}{r_i} \text{ord}_{E_i}$ be the Rees valuations normalized as in Definition 4.2. Then there exists integers $a_i \geq 1$ such that*

$$\alpha_1(v) = \sum_i \frac{a_i}{r_i} \beta(v/w_i)^{-1} . \quad (4.7)$$

In particular, there exists a constant $C > 0$ depending only on $(Y, 0)$ but not on v such that

$$C^{-1} \alpha_1(v) \leq \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1} \leq C \alpha_1(v) .$$

Finally, when $\mathcal{O}_{Y,0}$ admits a unique Rees valuation, there exists a positive rational number θ such that

$$\alpha_1(v) = \theta \left(\sup_{\mathfrak{m}_0} \frac{v}{\widehat{\text{ord}_0}} \right)^{-1}$$

for all v ; and $\theta = 1$ when 0 is a smooth point.

Remark 4.9. *Suppose $\dim(Y) = 2$ and the point 0 is smooth. Then [FJ04, Remark 3.33], Propositions 4.7 and 4.8 imply*

$$\alpha_1(v) = \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1} , \quad \text{and } \alpha_2(v)v(\mathfrak{m}_0) = \left(\sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} \right)^{-1} ,$$

It follows that $\alpha_1^2(v) = \alpha_0(v) \alpha_2(v)$ if and only if $\sup_{\mathfrak{m}_0} v/\text{ord}_0 = v(\mathfrak{m}_0)$. The latter condition is equivalent to v being proportional to ord_0 .

Proof. As in §4.1, let $\nu : Y^+ \rightarrow Y$ be the normalized blowup of \mathfrak{m}_0 , write $\mathfrak{m}_0 \cdot \mathcal{O}_{Y^+} = \mathcal{O}_{Y^+}(-\sum_i r_i E_i)$, so that $w_i = r_i^{-1} \text{ord}_{E_i}$.

Lemma 4.10. [JM10, Lemma 2.4] $\beta(v/w)^{-1} = \lim_n \frac{1}{n} w(\mathbf{a}(v, n))$.

By (4.5), we get

$$e(\mathbf{a}(v, n)^{[1]}; \mathfrak{m}_0^{[m-1]}) = \left(\sum_i \text{ord}_{E_i}(\mathbf{a}(v, n)) E_i \right) \cdot \left(-\sum_j r_j E_j \right)^{m-1} = \sum \frac{a_i}{r_i} w_i(\mathbf{a}(v, n))$$

with $a_i := E_i \cdot (\sum_j -r_j E_j)^{m-1}$. Then (4.7) follows from the previous lemma, by dividing by n and letting $n \rightarrow \infty$.

Pick any rank 1 valuation v on $\mathcal{O}_{Y,0}$ centered at 0 , and write

$$\beta(v) := \sup_{\mathfrak{m}_0} \frac{v}{\text{ord}_0} .$$

We have $v \leq \beta(v) \text{ord}_0 \leq \beta(v) w_i$ for all i , so that $\beta(v/w_i) \leq \beta(v)$, and $\alpha_1(v) \geq (\sum \frac{a_i}{r_i}) \beta(v)^{-1}$.

Conversely, $\alpha_1(v) \leq (\sum \frac{a_i}{r_i}) \beta(v/w_i)^{-1}$ for some i , whereas $\beta(v/w_i) \geq C^{-1} \sup_{\mathfrak{m}_0} v/\text{ord}_0 = C^{-1} \beta(v)$ by Izumi's theorem applied to w_i . This proves $\alpha_1(v) \leq C (\sum \frac{a_i}{r_i}) \beta(v)^{-1}$ as required.

Finally if $\mathcal{O}_{Y,0}$ has a unique Rees valuation w_i , then (4.7) and Theorem 4.3 (i) imply $\alpha_1(v) = \frac{a_i}{r_i} (\sup_{\mathfrak{m}_0} v/\widehat{\text{ord}_0})^{-1}$. Finally when 0 is a smooth point, it is easy to see that $a_i = r_i = 1$. This concludes the proof. \square

Proof of Lemma 4.10. We give a proof for completeness, see [JM10, Lemma 2.4]. Observe first that since $\mathfrak{a}(v, n)$ is a graded sequence of ideals, then the limit $\lim \frac{1}{n} w(\mathfrak{a}(v, n))$ exists as $n \rightarrow \infty$. Denote it by θ . For any n , we have $n = v(\mathfrak{a}(v, n)) \leq w(\mathfrak{a}(v, n)) \beta(v/w)$ hence $1 \leq \theta \beta(v/w)$. Conversely, pick any $f \in \mathfrak{m}_0$, and let $n := v(f)$. Then $f \in \mathfrak{a}(v, n)$ and $w(\mathfrak{a}(v, n)) \leq w(f)$ implies

$$\frac{1}{n} w(\mathfrak{a}(v, n)) v(f) \leq w(f)$$

Replacing f by f^l and letting $l \rightarrow \infty$ we get $v(f) \leq \theta^{-1} w(f)$ which implies $\beta(v/w) \leq \theta^{-1}$. This concludes the proof. \square

4.5. Proof of Corollaries D and E. We start by proving Corollary D. Fix $0 \leq i \leq m$. For any $v \in \Delta$ we have $v \geq \text{ord}_0$, so that $\mathfrak{a}(v, n) \supset \mathfrak{m}_0^n$ for any $n \geq 1$. This implies that

$$\alpha_i(v) \leq C \tag{4.8}$$

for all $v \in \Delta$.

Fix a metric on Δ compatible with the affine structure. Fix $v \in \Delta$ and an integer $n \geq 1$. It follows from Theorem A that the function $w \mapsto w(\mathfrak{a}(v, n))$ is Lipschitz continuous with Lipschitz constant at most

$$A \text{ord}_0(\mathfrak{a}(v, n)) \leq A v(\mathfrak{a}(v, n)).$$

Consider a valuation $w \in \Delta$ such that $\|v - w\| < \frac{1}{A}$. We then have

$$\begin{aligned} w(\mathfrak{a}(v, n)) &\geq v(\mathfrak{a}(v, n)) - A v(\mathfrak{a}(v, n)) \|v - w\| \\ &= (1 - A \|v - w\|) v(\mathfrak{a}(v, n)) \\ &\geq (1 - A \|v - w\|) n, \end{aligned}$$

so that

$$\mathfrak{a}(v, n) \subset \mathfrak{a}(w, n(1 - A \|v - w\|)). \tag{4.9}$$

and by (***)

$$\begin{aligned} \frac{e(\mathfrak{a}(v, n)^{[i]}, \mathfrak{m}_0^{[m-i]})}{n^i} &\geq \frac{e(\mathfrak{a}(w, n(1 - A \|v - w\|))^{[i]}, \mathfrak{m}_0^{[m-i]})}{n^i} \\ &= (1 - A \|v - w\|)^i \frac{e(\mathfrak{a}(w, n(1 - A \|v - w\|))^{[i]}, \mathfrak{m}_0^{[m-i]})}{n^i (1 - A \|v - w\|)^i}. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\alpha_i(v) \geq (1 - A \|v - w\|)^i \alpha_i(w),$$

so that

$$\begin{aligned} \alpha_i(w) - \alpha_i(v) &\leq (1 - (1 - A \|v - w\|)^i) \alpha_i(w) \\ &\leq C (1 - (1 - A \|v - w\|)^i) \\ &\leq i C A \|v - w\|, \end{aligned}$$

where we have used the inequality $1 - (1 - t)^i \leq it$ for $0 \leq t \leq 1$. Exchanging the roles of v and w we conclude that

$$|\alpha_i(v) - \alpha_i(w)| \leq i C A \|v - w\|$$

for all $v, w \in \Delta$ such that $\|v - w\| < \frac{1}{A}$. This completes the proof of Corollary D.

We now prove Corollary E. Pick four valuations $v, v', w, w' \in \Delta$. As in the proof of Corollary D we may assume $\max\{\|v - w\|, \|v' - w'\|\} \leq A$. By (4.9) we get

$$\mathfrak{a}(w, n) \subset \mathfrak{a}(v, n(1 - A\|v - w\|)) \text{ and } \mathfrak{a}(w', n) \leq \mathfrak{a}(v', n(1 - A\|v' - w'\|)).$$

In particular $n \leq w(\mathfrak{a}(w, n)) \leq w(\mathfrak{a}(v, n(1 - A\|v - w\|)))$ so that $\beta(v/w)^{-1} \geq (1 - A\|v - w\|)^{-1}$ by Lemma 4.10. Since

$$\beta(v/v') \leq \beta(v/w) \beta(w/v') \leq \beta(v/w) \beta(w'/v') \beta(w/w'),$$

we conclude

$$\beta(v/v') \leq (1 - A\|v - w\|)(1 - A\|v' - w'\|) \beta(w/w')$$

which implies the Lipschitz continuity of $(v, v') \mapsto \beta(v/v')$ as above.

4.6. Proof of Theorem A'. We keep the notation from the introduction. Thus we let (X, Z) , Δ , J , σ_J , ξ_J and $(z_j)_{j \in J}$ be as in the discussion before Theorem A'. By Cohen's Theorem there is an isomorphism $\widehat{\mathcal{O}_{X, \xi_J}} \simeq \kappa[[\xi_j]]_{j \in J}$, where $\kappa(\xi_J)$ is the residue field at ξ_J . Fix such an isomorphism. Given $f \in \mathcal{O}_{Y, 0}$ we can then write

$$f = \sum_{\beta \in \mathbf{Z}_{\geq 0}^J} a_\beta z^\beta$$

with $a_\beta \in \kappa(\xi_J)$. The Newton polyhedron $\text{Nw}(f, J)$ is then defined as

$$\text{Nw}(f, J) := \text{Conv} \left(\bigcup_{a_\beta \neq 0} (\beta + \mathbf{R}_{\geq 0}^J) \right) \subset \mathbf{R}_{\geq 0}^J.$$

Let us give an alternative description of the Newton polyhedron, which shows that it does not depend on the choice of coordinates z_j or the choice of isomorphism in Cohen's theorem. Consider σ_J as embedded as the unit simplex in $\sum_{j \in J} \mathbf{R}e_j \simeq \mathbf{R}^J$ and let $\langle \cdot, \cdot \rangle$ be the standard scalar product on \mathbf{R}^J . We then have

$$\begin{aligned} \beta \in \text{Nw}(f, J) &\Leftrightarrow \langle v, \beta \rangle \geq v(f) \text{ for all } v \in \sigma_J \\ &\Leftrightarrow \langle v, -\beta \rangle \leq \varphi(v) \text{ for all } v \in \sigma_J \\ &\Leftrightarrow -\beta \in \text{Nw}(\varphi), \end{aligned}$$

where $\text{Nw}(\varphi)$ denotes the Newton polyhedron of the piecewise affine convex function $\varphi = \log |f|$ on the simplex σ_J , as defined in §2.2.

Fix a norm on \mathbf{R}^J . By Theorem A, the Lipschitz constant of φ on σ is bounded by $A \text{ord}_0(\varphi)$. If $\beta \in \mathbf{R}^J$ is an extremal point of $\text{Nw}(f, J)$, then $-\beta$ is an extremal point of $\text{Nw}(\varphi)$ and we conclude from Proposition 2.3 that $\|\beta\| \leq AC \text{ord}_0(f)$, concluding the proof of Theorem A'.

4.7. Proof of Corollary C. As in the introduction, we fix an embedding $\mathbf{A}^m \hookrightarrow \mathbf{P}^m$ and call a smooth projective variety X an admissible compactification of \mathbf{A}^m if X admits a birational morphism $\pi : X \rightarrow \mathbf{P}^m$ that is an isomorphism above \mathbf{A}^m and such that the divisor $Z := \pi^{-1}(\mathbf{P}^m \setminus \mathbf{A}^m)$ has simple normal crossing support and that any nonempty intersection between irreducible components of Z is irreducible. Note that Z then has connected support as a consequence of Zariski's Main Theorem. Further, X contains \mathbf{A}^m as

a Zariski open subset. We view the elements of the dual complex $\Delta = \Delta(X, Z)$ as valuations on $k(X)$ normalized on $v(Z) = 1$.

Let $L_d = \pi^*\mathcal{O}(d)$ for $d \geq 1$ and let G be the pullback to X of the zero locus on P in \mathbf{P}^m . Thus $L_d = \mathcal{O}_X(G) = L_1^{\otimes d}$ and we have $v(G) = v(P) + d$ for every $v \in \Delta$. In particular, the functions $v \mapsto v(P)$ and $v \mapsto \chi_G(v) := v(G)$ on Δ have the same Lipschitz constant. Now $\min_{v \in \Delta} v(P) = -d$, with the minimum being obtained at the divisorial valuation corresponding to the divisor $\mathbf{P}^m \setminus \mathbf{A}^m$, so $\min_{\Delta} \chi_G = 0$. We therefore get from Theorem B that the Lipschitz constant of χ_G is bounded from above by

$$\begin{aligned} B \max_{J \subset I} |(G \cdot M^{n-|J|-1} \cdot E_J)| &= B \max_{J \subset I} |(L_d \cdot M^{n-|J|-1} \cdot E_J)| \\ &= Bd \max_{J \subset I} |(L_1 \cdot M^{n-|J|-1} \cdot E_J)|, \end{aligned}$$

which completes the proof.

REFERENCES

- [BR10] M. Baker, and R. Rumely. *Potential theory and dynamics on the Berkovich projective line*. Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010. 5
- [Bed09] C. Beddani. *Comparaison des valuations divisorielles*. Astérisque 323 (2009), 17–31. 2
- [Ber90] V. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. 5
- [BdFF10] S. Boucksom, T. de Fernex and C. Favre, *The volume of an isolated singularity*. Duke Math. J. **161** (2011), 1455–1520. 4, 5
- [BFJ08] S. Boucksom, C. Favre and M. Jonsson. *Valuations and plurisubharmonic singularities*. Publ. Res. Inst. Math. Sci. 44 (2008), 449–494. 5
- [BFJ09] S. Boucksom, C. Favre and M. Jonsson. *Differentiability of volumes of divisors and a problem of Teissier*. J. Algebraic Geom. 18 (2009), no. 2, 279308. 4
- [BFJ12a] S. Boucksom, C. Favre and M. Jonsson. *Singular semipositive metrics in non-Archimedean geometry*. [arXiv.org:1201.0187](https://arxiv.org/abs/1201.0187). 5, 8
- [BFJ12b] S. Boucksom, C. Favre and M. Jonsson. *Solution to a non-Archimedean Monge-Ampère equation*. [arXiv.org:1201.0188](https://arxiv.org/abs/1201.0188). 5
- [Cut12] S. D. Cutkosky. *Multiplicities Associated to Graded Families of Ideals*. [arXiv.org:1206.4077](https://arxiv.org/abs/1206.4077). 4, 18
- [CS93] S. D. Cutkosky, and V. Srinivas. *On a problem of Zariski on dimensions of linear systems*. Ann. of Math. (2) 137 (1993), no. 3, 531559. 4
- [ELS03] L. Ein, R. Lazarsfeld, and K. Smith. *Uniform approximation of Abhyankar valuation ideals in smooth function fields*. Amer. J. Math. 125 (2003), no. 2, 409–440. 2, 4, 18
- [Fav99] C. Favre. *Note on pull-back and Lelong number of currents*. Bull. Soc. Math. France 127 (1999), no. 3, 445–458. 2
- [FJ04] C. Favre and M. Jonsson. *The valuative tree*. Lecture Notes in Mathematics, 1853, Springer-Verlag, Berlin, 2004. 2, 5, 19
- [FJ07] C. Favre and M. Jonsson. *Eigenvaluations*. Ann. Sci. École Norm. Sup. **40** (2007), 309–349.
- [FJ11] C. Favre and M. Jonsson. *Dynamical compactifications of \mathbf{C}^2* . Ann. of Math. **173** (2011), 211–249. 3
- [FR10] C. Favre and J. Rivera-Letelier. *Théorie ergodique des fractions rationnelles sur un corps ultramétrique*. Proc. London Math. Soc. **100** (2010), 116–154. 5
- [Fu12] M. Fulger. *Local volumes on normal algebraic varieties*. [arXiv.org:1105.2981](https://arxiv.org/abs/1105.2981). 4
- [Hir64] H. Hironaka. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*. Ann. Math. 79 (1964), 109–203; *ibid.* 205–326. 2, 16
- [HS01] R. Hübl and I. Swanson. *Discrete valuations centered on local domains*. J. Pure Appl. Algebra 161 (2001), 145–166. 2

- [Huc70] J. Huckaba. *Some results on pseudo valuations*. Duke Math. J. **37** (1970), 19, 5, 18
- [Hun92] C. Huneke. *Uniform bounds in noetherian rings*. Invent. Math. **107** (1992), 203–223. 15
- [HS] C. Huneke and I. Swanson. *Integral closure of ideals in rings and modules*. London Mathematics Society Lecture Note Series, 336. Cambridge University Press. Cambridge, 2006. 15, 16
- [Izu81] S. Izumi. *Linear complementary inequalities for orders of germs of analytic functions*. Invent. Math. **65** (1981/82), no. 3, 459–471. 2
- [Izu85] S. Izumi. *A measure of integrity for local analytic algebras*. Publ. RIMS Kyoto Univ. **21** (1985), 719–735. 2
- [Jon12] M. Jonsson. *Dynamics on Berkovich spaces in low dimensions*. [arXiv:1201.1944](https://arxiv.org/abs/1201.1944). 3, 5
- [JM10] M. Jonsson and M. Mustață. *Valuations and asymptotic invariants for sequences of ideals*. [arXiv:1011.3967](https://arxiv.org/abs/1011.3967). To appear in Ann. Inst. Fourier. **2**, 6, 7, 19, 20
- [KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat. *Toroidal embeddings. I*. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973. 5, 7, 11
- [Kur03] A. Küronya. *A divisorial valuation with irrational volume*. J. Algebra **262** (2003), 413–423. 4
- [Laz] R. Lazarsfeld. *Positivity in algebraic geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 49. Springer-Verlag, Berlin, 2004. 4, 16, 17, 18
- [LM09] R. Lazarsfeld and M. Mustață. *Convex bodies associated to linear series*. Ann. Sci. École Norm. Sup. (4) **42** (2009), 783–835. 4, 18
- [LT08] M. Lejeune-Jalabert, B. Teissier. *Clôture intégrale des idéaux et équisingularité*. With an appendix by Jean-Jacques Risler. Ann. Fac. Sci. Toulouse Math. (6) **17** (2008), no. 4, 781–859. 15, 16
- [Mog11] M. Moghaddam. *On Izumi's theorem on comparison of valuations*. Kodai Math. J. **34** (2011), 16–30. 2
- [Mus02] M. Mustață. *On multiplicities of graded sequences of ideals*. J. Algebra **256** (2002), 229–249. 18
- [Ree88] D. Rees. *Lectures on the asymptotic theory of ideals*. London Mathematics Society Lecture Note Series, 113. Cambridge University Press. Cambridge, 1988. 15, 16
- [Ree89] D. Rees. *Izumi's theorem*. In Commutative algebra (Berkeley, CA, 1987), 407–416. Math. Sci. Res. Inst. Publ., 15. Springer, New York 1989. 2
- [Ron06] G. Rond. *Lemme d'Artin-Rees, théorème d'Izumi et fonction de Artin*. J. Algebra **299** (2006), 245–275. 2
- [Sam52] P. Samuel. *Some asymptotic properties of powers of ideals*. Ann. of Math. **56** (1952), 11–21. 15
- [Sam59] P. Samuel. *Multiplicités de certaines composantes singulières*. Illinois J. Math. **3** (1959), 319327. 5, 18
- [Swa11] I. Swanson. *Rees valuations*. In Commutative algebra—Noetherian and non-Noetherian perspectives, 421–440. Springer, New York 2011. 15
- [Tei72] B. Teissier. *Cycles évanescents, sections planes et conditions de Whitney*. Singularités à Cargèse, 1972. Astérisque 7-8 (1973), 285–362. 4, 17
- [Thu05] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*. Phd of the University of Rennes 1 (2005), viii + 184 p. <http://tel.archives-ouvertes.fr/docs/00/04/87/50/PDF/tel-00010990.pdf> 5
- [Tou72] J.-C. Tougeron. *Idéaux de fonctions différentiables*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71. Springer-Verlag, Berlin-New York, 1972. 2

CNRS-UNIVERSITÉ PARIS 6, INSTITUT DE MATHÉMATIQUES, F-75251 PARIS CEDEX 05, FRANCE
E-mail address: boucksom@math.jussieu.fr

CNRS-CMLS, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX FRANCE
E-mail address: favre@math.polytechnique.fr

DEPT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043, USA
E-mail address: mattiasj@umich.edu