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THE PSEUDO-EFFECTIVE CONE OF A COMPACT KÄHLER MANIFOLD AND VARIETIES OF NEGATIVE KODAIRA DIMENSION

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Abstract

We prove that a holomorphic line bundle on a projective manifold is pseudo-effective if and only if its degree on any member of a covering family of curves is non-negative. This is a consequence of a duality statement between the cone of pseudo-effective divisors and the cone of "movable curves", which is obtained from a general theory of movable intersections and approximate Zariski decomposition for closed positive $(1,1)$-currents. As a corollary, a projective manifold has a pseudo-effective canonical bundle if and only if it is not uniruled. We also prove that a 4-fold with a canonical bundle which is pseudo-effective and of numerical class zero in restriction to curves of a good covering family, has non-negative Kodaira dimension.

0. Introduction

One of the major open problems in the classification theory of projective or compact Kähler manifolds is the following geometric description of varieties of negative Kodaira dimension.

0.1. Conjecture. *A projective (or compact Kähler) manifold X has Kodaira dimension $\kappa(X) = -\infty$ if and only if X is uniruled.*

Received April 29, 2010 and, in revised form, July 15, 2010. The original version of the present paper was forwarded to arXiv (in May 2004), and has been revised several times since then. It had also been submitted to a journal in 2004, but then the submission was cancelled after the referee expressed concerns about certain parts of the paper, as they were written at that time. Although the results of sections 1-5 have been reproduced several times, e.g. in lecture notes of the second-named author or in Rob Lazarsfeld's book [La04], a complete version never appeared in a refereed journal. We thank the Journal of Algebraic Geometry for suggesting to repair this omission.

One direction is trivial, namely X uniruled implies $\kappa(X) = -\infty$. Also, the conjecture is known to be true for projective threefolds by [Mo88] and for non-algebraic Kähler threefolds by [Pe01], with the possible exception of simple threefolds (recall that a variety is said to be simple if there is no compact positive dimensional subvariety through a very general point of X). In the case of projective manifolds, the problem can be split into more tractable parts:

(A) *If the canonical bundle K_X is not pseudo-effective, i.e. not contained in the closure of the cone spanned by classes of effective divisors, then X is uniruled.*

(B) *If K_X is pseudo-effective, then $\kappa(X) \geq 0$.*

In the Kähler case, the statements should be essentially the same, except that effective divisors have to be replaced by closed positive $(1, 1)$-currents.

Part B again splits into two pieces:

(B1) *If K_X is pseudo-effective but not big, i.e. on the boundary of the pseudo-effective cone, then there exists a covering family of curves (C_t) such that $K_X \cdot C_t = 0$.*

(B2) *If K_X is pseudo-effective and there exists a covering family (C_t) of curves with $K_X \cdot C_t = 0$, then $\kappa(X) \geq 0$.*

In this paper we give a positive answer to (A) for projective manifolds of any dimension, and deal with (B2), mostly in dimension 4. Part (A) follows in fact from a much more general fact which describes the geometry of the pseudo-effective cone.

0.2. Theorem. *A line bundle L on a projective manifold X is pseudo-effective if and only if $L \cdot C \geq 0$ for all irreducible curves C which move in a family covering X.*

In other words, the dual cone to the pseudo-effective cone is the closure of the cone of "movable" curves. This should be compared with the duality between the nef cone and the cone of effective curves.

0.3. Corollary (Solution of (A))**.** *Let X be a projective manifold. If K_X is not pseudo-effective, then X is covered by rational curves.*

In fact, if K_X is not pseudo-effective, then by Theorem 0.2 there exists a covering family (C_t) of curves with $K_X \cdot C_t < 0$, so that Corollary 0.3 follows by a well-known characteristic p argument of Miyaoka and Mori [MM86] (the so called bend-and-break lemma essentially amounts to deform the C_t so that they break into pieces, one of which is a rational curve).

In the Kähler case both a suitable analogue to Theorem 0.2 and the theorem of Miyaoka-Mori are unknown. It should also be mentioned that the duality statement following Theorem 0.2 is actually Theorem 0.2 for $\mathbb{R}$-divisors. The proof is based on a use of "approximate Zariski decompositions" and an estimate for an intersection number related to this decomposition. A major tool is the volume of an $\mathbb{R}$-divisor which distinguishes big divisors (positive volume) from divisors on the boundary of the pseudo-effective cone (volume 0).

Concerning (B2), we need to distinguish between covering, but not connecting, families (C_t) on one side and connecting families on the other side. This latter term "connecting" means that any two points can be joined by a chain of curves C_t. However, for technical purposes it is better to consider *strongly connecting* families, i.e., any two sufficiently general points can be joined by a chain of irreducible C_t's. If X has a good minimal model via contractions and flips, then X clearly admits a covering non-connecting or a strongly connecting family (C_t) such that $K_X \cdot C_t = 0$; moreover, if X simply has a good minimal model, then at least after blowing up this will be the case. Let us say that (C_t) is a good covering family, if (C_t) is a covering, non-connecting family or a strongly connecting family. Then our remarks justify the division of problem B into the two parts (B1) and (B2), possibly by replacing "covering families" by "good covering families".

0.4. Theorem. *Let X be a smooth projective 4-fold. Assume that K_X is pseudo-effective and there is a good covering family (C_t) of curves such that $K_X \cdot C_t = 0$. Then $\kappa(X) \geq 0$.*

One important ingredient of the proof of Theorem 0.4 is the quotient defined by the family (C_t). The reason for the restriction to dimension 4 is that we use $C_{n,m}$ and the log minimal model program on the base of the quotient of the family (C_t). In one circumstance, however, we have a general result:

0.5. Theorem. *Let X be a projective manifold and (C_t) a strongly connecting family of curves. Let L be a pseudo-effective $\mathbb{R}$-divisor with $L \cdot C_t = 0$. Then the numerical dimension $\mathrm{nd}(L) = 0$. If L is Cartier, then L is numerically equivalent to a line bundle L' with $\kappa(L') = 0$.*

If $L = K_X$, then in connection with [CP09] we obtain $\kappa(X) = 0$. In order to obtain the answer to problem (B1) (e.g. in dimension 4), we would still need to prove that K_X is effective if K_X is positive on all good covering families of curves. In fact, in that case, K_X should be big, i.e. of maximal Kodaira dimension.

1. Positive cones in the spaces of divisors and of curves

In this section we introduce the relevant cones, both in the projective and Kähler contexts – in the latter case, divisors and curves should simply be replaced by positive currents of bidimension $(n-1,n-1)$ and $(1,1)$, respectively. We implicitly use that all (De Rham, respectively Dolbeault) cohomology groups under consideration can be computed in terms of smooth forms or currents, since in both cases we get resolutions of the same sheaf of locally constant functions (respectively of holomorphic sections).

1.1. Definition. Let X be a compact Kähler manifold.

(i) The Kähler cone is the set $\mathcal{K} \subset H^{1,1}_{\mathbb{R}}(X)$ of classes $\{\omega\}$ of Kähler forms (this is an open convex cone).

(ii) The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}_{\mathbb{R}}(X)$ of classes $\{T\}$ of closed positive currents of type $(1,1)$ (this is a closed convex cone). Clearly $\mathcal{E} \supset \overline{\mathcal{K}}$.

(iii) The Neron-Severi space is defined by

$$\mathrm{NS}_{\mathbb{R}}(X) := \big(H^{1,1}_{\mathbb{R}}(X) \cap H^2(X,\mathbb{Z})/\mathrm{tors}\big) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(iv) We set

$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \qquad \mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).$$

Algebraic geometers tend to restrict themselves to the algebraic cones generated by ample divisors and effective divisors, respectively. Using L^2 estimates for $\overline{\partial}$, one can show the following expected relations between the algebraic and transcendental cones (see [Dem90] and [Dem92]).

1.2. Proposition. *In a projective manifold X, $\mathcal{E}_{\mathrm{NS}}$ is the closure of the convex cone generated by effective divisors, and $\overline{\mathcal{K}_{\mathrm{NS}}}$ is the closure of the cone generated by nef $\mathbb{R}$-divisors.*

By extension, we will say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though they are not necessarily integral). We now turn ourselves to cones in cohomology of bidegree $(n-1,n-1)$.

1.3. Definition. Let X be a compact Kähler manifold.

(i) We define $\mathcal{N}$ to be the (closed) convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by classes of positive currents T of type $(n-1,n-1)$ (i.e., of bidimension $(1,1)$).

(ii) We define the cone $\mathcal{M} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ of **movable classes** to be the closure of the convex cone generated by classes of currents of the form

$$\mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ is an arbitrary modification (one could just restrict oneself to compositions of blow-ups with smooth centers), and the $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Clearly $\mathcal{M} \subset \mathcal{N}$.

(iii) Correspondingly, we introduce the intersections

$$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap N_1(X), \qquad \mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X),$$

in the space of integral bidimension $(1,1)$-classes

$$N_1(X) := (H^{n-1,n-1}_{\mathbb{R}}(X) \cap H^{2n-2}(X,\mathbb{Z})/\mathrm{tors}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(iv) If X is projective, we define $\mathrm{NE}(X)$ to be the convex cone generated by all effective curves. Clearly $\overline{\mathrm{NE}(X)} \subset \mathcal{N}_{\mathrm{NS}}$.

(v) If X is projective, we say that C is a **strongly movable curve** if

$$C = \mu_\star(\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_{n-1})$$

for suitable very ample divisors $\widetilde{A}_j$ on $\widetilde{X}$, where $\mu : \widetilde{X} \to X$ is a modification. We let $\mathrm{SME}(X)$ to be the convex cone generated by all strongly movable (effective) curves. Clearly $\overline{\mathrm{SME}(X)} \subset \mathcal{M}_{\mathrm{NS}}$.

(vi) We say that C is a **movable curve** if $C = C_{t_0}$ is a member of an analytic family $(C_t)_{t\in S}$ such that $\bigcup_{t\in S} C_t = X$ and, as such, is a reduced irreducible 1-cycle. We let $\mathrm{ME}(X)$ to be the convex cone generated by all movable (effective) curves.

The upshot of this definition lies in the following easy observation.

1.4. Proposition. *Let X be a compact Kähler manifold. Consider the Poincaré duality pairing*

$$H^{1,1}_{\mathbb{R}}(X) \times H^{n-1,n-1}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta) \longmapsto \int_X \alpha \wedge \beta.$$

Then the duality pairing takes non-negative values

(i) *for all pairs $(\alpha,\beta) \in \overline{\mathcal{K}} \times \mathcal{N}$;*

(ii) *for all pairs $(\alpha,\beta) \in \mathcal{E} \times \mathcal{M}$.*

(iii) *for all pairs (α,β) where $\alpha \in \mathcal{E}$ and $\beta = [C_t] \in \mathrm{ME}(X)$ is the class of a movable curve.*

Proof. (i) is obvious. In order to prove (ii), we may assume that $\beta = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$ for some modification $\mu : \widetilde{X} \to X$, where $\alpha = \{T\}$ is the class of a positive $(1,1)$-current on X and $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Then

$$\int_X \alpha \wedge \beta = \int_X T \wedge \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1}) = \int_X \mu^* T \wedge \widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1} \geq 0.$$

Here, we have used the fact that a closed positive $(1,1)$-current T always has a pull-back $\mu^\star T$, which follows from the fact that if $T = i\partial\overline{\partial}\varphi$ locally for some plurisubharmonic function in X, we can set $\mu^\star T = i\partial\overline{\partial}(\varphi\circ\mu)$. For (iii), we suppose $\alpha = \{T\}$ and $\beta = \{[C_t]\}$. Then we take an open covering (U_j) on X such that $T = i\partial\overline{\partial}\varphi_j$ with suitable plurisubharmonic functions φ_j on U_j. If we select a smooth partition of unity $\sum\theta_j = 1$ subordinate to (U_j), we then get

$$\int_X \alpha\wedge\beta = \int_{C_t} T_{|C_t} = \sum_j \int_{C_t\cap U_j} \theta_j i\partial\overline{\partial}\varphi_{j|C_t} \geq 0.$$

For this to make sense, it should be noticed that $T_{|C_t}$ is a well-defined closed positive $(1,1)$-current (i.e. measure) on C_t for almost every $t\in S$, in the sense of Lebesgue measure. This is true only because (C_t) covers X, thus $\varphi_{j|C_t}$ is not identically $-\infty$ for almost every $t\in S$. The equality in the last formula is then shown by a regularization argument for T, writing $T = \lim T_k$ with $T_k = \alpha + i\partial\overline{\partial}\psi_k$ and a decreasing sequence of smooth almost plurisubharmonic potentials $\psi_k\downarrow\psi$ such that the Levi forms have a uniform lower bound $i\partial\overline{\partial}\psi_k \geq -C\omega$ (such a sequence exists by [Dem92]). Then, writing $\alpha = i\partial\overline{\partial}v_j$ for some smooth potential v_j on U_j, we have $T = i\partial\overline{\partial}\varphi_j$ on U_j with $\varphi_j = v_j + \psi$, and this is the decreasing limit of the smooth approximations $\varphi_{j,k} = v_j + \psi_k$ on U_j. Hence $T_{k|C_t}\to T_{|C_t}$ for the weak topology of measures on C_t. □

If $\mathcal{C}$ is a convex cone in a finite dimensional vector space E, we denote by $\mathcal{C}^\vee$ the dual cone, i.e. the set of linear forms $u\in E^\star$ which take nonnegative values on all elements of $\mathcal{C}$. By the Hahn-Banach theorem, we always have $\mathcal{C}^{\vee\vee} = \overline{\mathcal{C}}$.

Proposition 1.4 leads to the natural question whether the cones $(\mathcal{K},\mathcal{N})$ and $(\mathcal{E},\mathcal{M})$ are dual under Poincaré duality. This question is addressed in the next section. Before doing so, we observe that the algebraic and transcendental cones of $(n-1,n-1)$ cohomology classes are related by the following equalities (similar to what we already noticed for $(1,1)$-classes; see Proposition 1.2).

1.5. Theorem. *Let X be a projective manifold. Then*

(i) $\overline{\mathrm{NE}(X)} = \mathcal{N}_{\mathrm{NS}}$.
(ii) $\overline{\mathrm{SME}(X)} = \overline{\mathrm{ME}(X)} = \mathcal{M}_{\mathrm{NS}}$.

Proof. (i) It is a standard result of algebraic geometry (see e.g. [Ha70]), that the cone of effective cone $\mathrm{NE}(X)$ is dual to the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors, hence

$$\mathcal{N}_{\mathrm{NS}} \supset \overline{\mathrm{NE}(X)} = \mathcal{K}^\vee.$$

On the other hand, Proposition 1.4(i) implies that $\mathcal{N}_{\mathrm{NS}}\subset\mathcal{K}^\vee$, so we must have equality and (i) follows.

Similarly, (ii) requires a duality statement which will be established only in the next sections, so we postpone the proof. □

2. Main results and conjectures

First, the already mentioned duality between nef divisors and effective curves extends to the Kähler case and to transcendental classes. More precisely, [DPa04] gives

2.1. Theorem (Demailly-Păun, 2001)**.** *If X is Kähler, then the cones $\overline{\mathcal{K}} \subset H^{1,1}_{\mathbb{R}}(X)$ and $\mathcal{N} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ are dual by Poincaré duality, and $\mathcal{N}$ is the closed convex cone generated by classes $[Y] \wedge \omega^{p-1}$ where $Y \subset X$ ranges over p-dimensional analytic subsets, $p = 1, 2, \ldots, n$, and ω ranges over Kähler forms.*

Proof. Indeed, Proposition 1.4 shows that the dual cone $\mathcal{K}^\vee$ contains $\mathcal{N}$ which itself contains the cone $\mathcal{N}'$ of all classes of the form $\{[Y] \wedge \omega^{p-1}\}$. The main result of [DPa04] conversely shows that the dual of $(\mathcal{N}')^\vee$ is equal to $\overline{\mathcal{K}}$, so we must have

$$\mathcal{K}^\vee = \overline{\mathcal{N}'} = \mathcal{N}.$$

□

The main new result of this paper is the following characterization of pseudo-effective classes (in which the "only if" part already follows from Proposition 1.4(iii)).

2.2. Theorem. *If X is projective, then a class $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$ is pseudo-effective if* (*and only if*) *it is in the dual cone of the cone* $\mathrm{SME}(X)$ *of strongly movable curves.*

In other words, a line bundle L is pseudo-effective if (and only if) $L \cdot C \geq 0$ for all *movable curves*, i.e., $L \cdot C \geq 0$ for every very generic curve C (not contained in a countable union of algebraic subvarieties). In fact, by definition of $\mathrm{SME}(X)$, it is enough to consider only those curves C which are images of generic complete intersection of very ample divisors on some variety $\widetilde{X}$, under a modification $\mu : \widetilde{X} \to X$.

By a standard blowing-up argument, it also follows that a line bundle L on a normal Moishezon variety is pseudo-effective if and only if $L \cdot C \geq 0$ for every movable curve C.

The Kähler analogue should be:

2.3. Conjecture. *For an arbitrary compact Kähler manifold X, the cones $\mathcal{E}$ and $\mathcal{M}$ are dual.*

The relation between the various cones of movable curves and currents in Theorem 1.5 is now a rather direct consequence of Theorem 2.2. In fact, using ideas hinted in [DPS96], we can say a little bit more. Given an irreducible curve $C \subset X$, we consider its normal "bundle" $N_C = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf of C. If C is a general member of a covering family (C_t), then N_C is nef. Now [DPS96] says that the dual cone of the pseudo-effective cone of X contains the closed cone spanned by curves with nef normal bundle, which in turn contains the cone of movable curves. In this way we get:

2.4. Theorem. *Let X be a projective manifold. Then the following cones coincide:*

(i) *the cone $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$;*
(ii) *the closed cone $\overline{\mathrm{SME}(X)}$ of strongly movable curves;*
(iii) *the closed cone $\overline{\mathrm{ME}(X)}$ of movable curves;*
(iv) *the closed cone $\overline{\mathrm{ME}_{\mathrm{nef}}(X)}$ of curves with nef normal bundle.*

Proof. We have already seen that

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathrm{ME}_{\mathrm{nef}}(X) \subset (\mathcal{E}_{\mathrm{NS}})^\vee$$

and

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathcal{M}_{\mathrm{NS}} \subset (\mathcal{E}_{\mathrm{NS}})^\vee$$

by Proposition 1.4(iii). Now Theorem 2.2 implies $(\mathcal{M}_{\mathrm{NS}})^\vee = \overline{\mathrm{SME}(X)}$, and Theorem 2.4 follows. □

2.5. Corollary. *Let X be a projective manifold and L a line bundle on X.*

(i) *L is pseudo-effective if and only if $L \cdot C \geq 0$ for all curves C with nef normal sheaf N_C.*
(ii) *If L is big, then $L \cdot C > 0$ for all curves C with nef normal sheaf N_C.*

Corollary 2.5(i) strengthens results from [PSS99]. It is, however, not yet clear whether $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$ is equal to the closed cone of curves with *ample* normal bundle (although we certainly expect this to be true).

The most important special case of Theorem 2.2 is the following.

2.6. Theorem. *If X is a projective manifold and is not uniruled, then K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.*

Proof. This is merely a restatement of Corollary 0.3, which was proved in the introduction (as a consequence of the results of [MM86]).

Theorem 2.6 can be generalized as follows.

2.7. Theorem. *Let X be a projective manifold (or a normal projective variety). Let $\mathcal{F} \subset T_X$ be a coherent subsheaf. If $\det \mathcal{F}^*$ is not pseudo-effective,* *then X is uniruled. In other words, if X is not uniruled and $\Omega_X^1 \to \mathcal{G}$ is generically surjective, then* $\det \mathcal{G}$ *is pseudo-effective.*

Proof. In fact, since $\det \mathcal{F}^*$ is not pseudo-effective, there exists by Theorem 2.2 a covering family (C_t) such that $c_1(\mathcal{F}) \cdot C_t > 0$. Hence, X is uniruled by [Mi87] and [SB92]. □

2.8. Remark.

(1) In [CP09], Theorem 2.7 is generalized to subsheaves $\mathcal{F} \subset T_X^{\otimes m}$.
(2) Suppose in Theorem 2.7 that only $\kappa(\det \mathcal{F}^*) = -\infty$. Is X still uniruled? What can be said if $c_1(\mathcal{F}^*)$ is on the boundary of the pseudo-effective cone?

Turning to varieties with pseudo-effective canonical bundles, we have the following.

2.9. Conjecture (part of the "abundance conjecture")**.** *If K_X is pseudo-effective, then $\kappa(X) \geq 0$.*

This problem splits into two parts:

(1) *If K_X is pseudo-effective but not big, i.e. on the boundary of the pseudo-effective cone, then there exists a (good) covering family of curve (C_t) such that $K_X \cdot C_t = 0$.*
(2) *If K_X is pseudo-effective and there exists a good covering family (C_t) of curves with $K_X \cdot C_t = 0$, then $\kappa(X) \geq 0$.*

In the last section we will prove (2) in dimension 4 and even parts of it in any dimension.

3. Zariski decomposition and movable intersections

Let X be compact Kähler and let $\alpha \in \mathcal{E}^\circ$ be in the *interior* of the pseudo-effective cone. In analogy with the algebraic context such a class α is called "big", and it can then be represented by a *Kähler current* T, i.e. a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth Hermitian metric ω and a constant $\delta \ll 1$.

3.1. Theorem [Demailly [Dem92], [Bou02b], and 3.1.24]**.** *If T is a Kähler current, then one can write $T = \lim T_m$ for a sequence of Kähler currents T_m which have logarithmic poles with coefficients in $\frac{1}{m}\mathbb{Z}$, i.e. there are modifications $\mu_m : X_m \to X$ such that*

$$\mu_m^\star T_m = [E_m] + \beta_m$$

where E_m is an effective $\mathbb{Q}$-divisor on X_m with coefficients in $\frac{1}{m}\mathbb{Z}$ (the "fixed part") and β_m is a closed semi-positive form (the "movable part").

Proof. Since this result has already been studied extensively, we just recall the main idea. Locally we can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ. By a Bergman kernel trick and the Ohsawa-Takegoshi L^2 extension theorem, we get local approximations

$$\varphi = \lim \varphi_m, \qquad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space of holomorphic functions which are L^2 with respect to the weight $e^{-2m\varphi}$. This Hilbert basis is also a family of local generators of the globally defined multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. Then $\mu_m : X_m \to X$ is obtained by blowing-up this ideal sheaf, so that

$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

We should notice that by approximating $T - \frac{1}{m}\omega$ instead of T, we can replace β_m by $\beta_m + \frac{1}{m}\mu^\star\omega$ which is a big class on X_m ; by playing with the multiplicities of the components of the exceptional divisor, we could even achieve that β_m is a Kähler class on X_m, but this will not be needed here. □

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \text{ effective},\ D_m \text{ free}.$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.

3.2. Definition. We define the **volume**, or **movable self-intersection** of a big class $\alpha \in \mathcal{E}^\circ$ to be

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^\star T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$.

By Fujita [Fuj94] and Demailly-Ein-Lazarsfeld [DEL00], if L is a big line bundle, we have

$$\mathrm{Vol}(c_1(L)) = \lim_{m\to+\infty} D_m^n = \lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL),$$

and in these terms, we get the following statement.

3.3. Proposition. *Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \leq \mathrm{Vol}(\beta) \leq \mathrm{Vol}(L).$$

It is very useful to observe that the supremum in Definition 3.2 can actually be computed by a collection of currents whose singularities satisfy a filtering property. Namely, if $T_1 = \alpha + i\partial\overline{\partial}\varphi_1$ and $T_2 = \alpha + i\partial\overline{\partial}\varphi_2$ are two Kähler currents with logarithmic poles in the class of α, then

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \max(\varphi_1, \varphi_2) \tag{3.4}$$

is again a Kähler current with weaker singularities than T_1 and T_2. One could define as well

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \frac{1}{2m}\log(e^{2m\varphi_1} + e^{2m\varphi_2}), \tag{3.4$'$}$$

where $m = \mathrm{lcm}(m_1, m_2)$ is the lowest common multiple of the denominators occuring in T_1, T_2. Now, take a simultaneous log-resolution $\mu_m : X_m \to X$ for which the singularities of T_1 and T_2 are resolved as $\mathbb{Q}$-divisors E_1 and E_2. Then clearly the associated divisor in the decomposition $\mu_m^\star T = [E] + \beta$ is given by $E = \min(E_1, E_2)$. By doing so, the volume $\int_{X_m} \beta^n$ gets increased, as we shall see in the proof of Theorem 3.5 below.

3.5. Theorem [Boucksom [Bou02b]]**.** *Let X be a compact Kähler manifold. We denote here by $H^{k,k}_{\geq 0}(X)$ the cone of cohomology classes of type (k,k) which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k, n-k)$.*

(i) *For each integer $k = 1, 2, \ldots, n$, there exists a canonical "movable intersection product"*

$$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever α is a big class.

(ii) *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

$$\langle \alpha_1 \cdots (\alpha'_j + \alpha''_j) \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha'_j \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha''_j \cdots \alpha_k \rangle.$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

(iii) *The movable intersection product satisfies the Teissier-Hovanskii inequalities*

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \geq (\langle \alpha_1^n \rangle)^{1/n} \ldots (\langle \alpha_n^n \rangle)^{1/n} \qquad (\textit{with } \langle \alpha_j^n \rangle = \mathrm{Vol}(\alpha_j)\,).$$

(iv) *For $k = 1$, the above "product" reduces to a (non-linear) projection operator*

$$\mathcal{E} \to \mathcal{E}_1, \qquad \alpha \to \langle \alpha \rangle$$

onto a certain convex subcone $\mathcal{E}_1$ of $\mathcal{E}$ such that $\overline{\mathcal{K}} \subset \mathcal{E}_1 \subset \mathcal{E}$. Moreover, there is a "divisorial Zariski decomposition"

$$\alpha = \{N(\alpha)\} + \langle \alpha \rangle$$

where $N(\alpha)$ is a uniquely defined effective divisor which is called the "negative divisorial part" of α. The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive, and $N(\alpha) = 0$ if and only if $\alpha \in \mathcal{E}_1$.

(v) *The components of $N(\alpha)$ always consist of divisors whose cohomology classes are linearly independent, especially $N(\alpha)$ has at most $\rho = \operatorname{rank}_{\mathbb{Z}} \mathrm{NS}(X)$ components.*

Proof. We essentially repeat the arguments developed in [Bou02b], with some simplifications arising from the fact that X is supposed to be Kähler from the start.

(i) First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and a simultaneous log-resolution $\mu : \widetilde{X} \to X$ such that

$$\mu^\star T_j = [E_j] + \beta_j.$$

We consider the direct image current $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ (which is a closed positive current of bidegree (k,k) on X) and the corresponding integrals

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \geq 0.$$

If we change the representative T_j with another current T'_j, we may always take a simultaneous log-resolution such that $\mu^\star T'_j = [E'_j] + \beta'_j$, and by using $(3.4')$ we can always assume that $E'_j \leq E_j$. Then $D_j = E_j - E'_j$ is an effective divisor and we find $[E_j] + \beta_j \equiv [E'_j] + \beta'_j$, hence $\beta'_j \equiv \beta_j + [D_j]$. A substitution in the integral implies

$$\begin{aligned}
&\int_{\widetilde{X}} \beta'_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad = \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u + \int_{\widetilde{X}} [D_1] \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad \geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.
\end{aligned}$$

Similarly, we can replace successively all forms β_j by the β'_j, and by doing so, we find

$$\int_{\widetilde{X}} \beta'_1 \wedge \beta'_2 \wedge \ldots \wedge \beta'_k \wedge \mu^\star u \geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.$$

We claim that the closed positive currents $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ are uniformly bounded in mass. In fact, if ω is a Kähler metric in X, there exists a constant $C_j \geq 0$ such that $C_j\{\omega\} - \alpha_j$ is a Kähler class. Hence $C_j\omega - T_j \equiv \gamma_j$ for some Kähler form γ_j on X. By pulling back with μ, we find $C_j\mu^\star\omega - ([E_j] + \beta_j) \equiv \mu^\star\gamma_j$, hence

$$\beta_j \equiv C_j\mu^\star\omega - ([E_j] + \mu^\star\gamma_j).$$

By performing again a substitution in the integrals, we find

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \leq C_1 \ldots C_k \int_{\widetilde{X}} \mu^\star\omega^k \wedge \mu^\star u = C_1 \ldots C_k \int_X \omega^k \wedge u$$

and this is true especially for $u = \omega^{n-k}$. We can now arrange that for each of the integrals associated with a countable dense family of forms u, the supremum is achieved by a sequence of currents $(\mu_m)_\star(\beta_{1,m} \wedge \ldots \wedge \beta_{k,m})$ obtained as direct images by a suitable sequence of modifications $\mu_m : \widetilde{X}_m \to X$. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m\to+\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

(the monotonicity is not in terms of the currents themselves, but in terms of the integrals obtained when we evaluate against a smooth closed semi-positive form u). By evaluating against a basis of positive classes $\{u\} \in H^{n-k,n-k}(X)$, we infer by Poincaré duality that the class of $\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle$ is uniquely defined (although, in general, the representing current is not unique).

(ii) It is indeed clear from the definition that the movable intersection product is homogeneous, increasing and superadditive in each argument, at least when the α_j's are in $\mathcal{E}^\circ$. However, we can extend the product to the closed cone $\mathcal{E}$ by monotonicity, by setting

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{\delta\downarrow 0} \downarrow \langle (\alpha_1 + \delta\omega) \cdot (\alpha_2 + \delta\omega) \cdots (\alpha_k + \delta\omega) \rangle$$

for arbitrary classes $\alpha_j \in \mathcal{E}$ (again, monotonicity occurs only where we evaluate against closed semi-positive forms u). By weak compactness, the movable intersection product can always be represented by a closed positive current of bidegree (k, k).

(iii) The Teissier-Hovanskii inequalities are a direct consequence of the fact that they hold true for nef classes, so we just have to apply them to the classes $\beta_{j,m}$ on $\widetilde{X}_m$ and pass to the limit.

(iv) When $k = 1$ and $\alpha \in \mathcal{E}^0$, we have

$$\alpha = \lim_{m\to+\infty} \{(\mu_m)_\star T_m\} = \lim_{m\to+\infty} (\mu_m)_\star[E_m] + \{(\mu_m)_\star\beta_m\}$$

and $\langle\alpha\rangle = \lim_{m\to+\infty}\{(\mu_m)_\star\beta_m\}$ by definition. However, the images $F_m = (\mu_m)_\star E_m$ are effective $\mathbb{Q}$-divisors in X, and the filtering property implies that F_m is a decreasing sequence. It must therefore converge to a (uniquely defined) limit $F = \lim F_m := N(\alpha)$ which is an effective $\mathbb{R}$-divisor, and we get the asserted decomposition in the limit.

Since $N(\alpha) = \alpha - \langle\alpha\rangle$ we easily see that $N(\alpha)$ is subadditive and that $N(\alpha) = 0$ if α is the class of a smooth semi-positive form. When α is no longer a big class, we define

$$\langle\alpha\rangle = \lim_{\delta\downarrow 0} \downarrow \langle\alpha + \delta\omega\rangle, \qquad N(\alpha) = \lim_{\delta\downarrow 0} \uparrow N(\alpha + \delta\omega)$$

(the subadditivity of N implies $N(\alpha+(\delta+\varepsilon)\omega) \le N(\alpha+\delta\omega)$). The divisorial Zariski decomposition follows except maybe for the fact that $N(\alpha)$ might be a convergent countable sum of divisors. However, this will be ruled out when (v) is proved. As $N(\bullet)$ is subadditive and homogeneous, the set $\mathcal{E}_1 = \{\alpha \in \mathcal{E}\ ;\ N(\alpha) = 0\}$ is a closed convex cone, and we find that $\alpha \mapsto \langle\alpha\rangle$ is a projection of $\mathcal{E}$ onto $\mathcal{E}_1$ (according to [Bou02b], $\mathcal{E}_1$ consists of those pseudo-effective classes which are "nef in codimension 1").

(v) Let $\alpha \in \mathcal{E}^\circ$, and assume that $N(\alpha)$ contains linearly dependent components F_j. Then already all currents $T \in \alpha$ should be such that $\mu^\star T = [E] + \beta$ where $F = \mu_\star E$ contains those linearly dependent components. Write $F = \sum \lambda_j F_j$, $\lambda_j > 0$ and assume that

$$\sum_{j\in J} c_j F_j \equiv 0$$

for a certain non-trivial linear combination. Then some of the coefficients c_j must be negative (and some other positive). Then E is numerically equivalent to

$$E' \equiv E + t\mu^\star\Big(\sum \lambda_j F_j\Big),$$

and by choosing $t > 0$ appropriate, we obtain an effective divisor E' which has a zero coefficient on one of the components $\mu^\star F_{j_0}$. By replacing E with $\min(E, E')$ via (3.4′), we eliminate the component $\mu^\star F_{j_0}$. This is a contradiction since $N(\alpha)$ was supposed to contain F_{j_0}. $\square$

3.6. Definition. For a class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$, we define the **numerical dimension** $\mathrm{nd}(\alpha)$ to be $\mathrm{nd}(\alpha) = -\infty$ if α is not pseudo-effective, and

$$\mathrm{nd}(\alpha) = \max\{p \in \mathbb{N}\,;\ \langle\alpha^p\rangle \neq 0\}, \qquad \mathrm{nd}(\alpha) \in \{0, 1, \ldots, n\}$$

if α is pseudo-effective.

By the results of [DPa04], a class is big ($\alpha \in \mathcal{E}^\circ$) if and only if $\mathrm{nd}(\alpha) = n$. Classes of numerical dimension 0 can be described much more precisely, again following Boucksom [Bou02b].

3.7. Theorem. *Let X be a compact Kähler manifold. Then the subset $\mathcal{D}_0$ of irreducible divisors D in X such that $\mathrm{nd}(D) = 0$ is countable, and these divisors are rigid as well as their multiples. If $\alpha \in \mathcal{E}$ is a pseudo-effective class of numerical dimension 0, then α is numerically equivalent to an effective $\mathbb{R}$-divisor $D = \sum_{j\in J} \lambda_j D_j$, for some finite subset $(D_j)_{j\in J} \subset \mathcal{D}_0$ such that the cohomology classes $\{D_j\}$ are linearly independent and some $\lambda_j > 0$. If such a linear combination is of numerical dimension 0, then so is any other linear combination of the same divisors.*

Proof. It is immediate from the definition that a pseudo-effective class is of numerical dimension 0 if and only if $\langle\alpha\rangle = 0$; in other words, if $\alpha = N(\alpha)$. Thus $\alpha \equiv \sum \lambda_j D_j$ as described in Theorem 3.7, and since $\lambda_j \langle D_j\rangle \leq \langle\alpha\rangle$, the divisors D_j must themselves have numerical dimension 0. There is, at most, one such divisor D in any given cohomology class in $NS(X) \cap \mathcal{E} \subset H^2(X, \mathbb{Z})$; otherwise, two such divisors $D \equiv D'$ would yield a blow-up $\mu : \widetilde{X} \to X$ resolving the intersection, and by taking $\min(\mu^\star D, \mu^\star D')$ via (3.4′), we would find $\mu^\star D \equiv E + \beta$, $\beta \neq 0$, so that $\{D\}$ would not be of numerical dimension 0. This implies that there are at most countably many divisors of numerical dimension 0, and that these divisors are rigid as well as their multiples. □

The above general concept of numerical dimension leads to a very natural formulation of the abundance conjecture for non-minimal (Kähler) varieties.

3.8. Generalized abundance conjecture. *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension:*

$$\kappa(X) = \mathrm{nd}(X) := \mathrm{nd}(c_1(K_X)).$$

This appears to be a fairly strong statement. In fact, it is not difficult to show that the generalized abundance conjecture would contain the $C_{n,m}$ conjectures.

3.9. Remark. Using the Iitaka fibration, it is immediate to see that $\kappa(X) \leq \mathrm{nd}(X)$.

3.10. Remark. It is known that abundance holds in case $\mathrm{nd}(X) = -\infty$ (if K_X is not pseudo-effective, no multiple of K_X can have sections), or in case $\mathrm{nd}(X) = n$. The latter follows from the solution of the Grauert-Riemenschneider conjecture in the form proven in [Dem85] (see also [DPa04]).

In the remaining cases, the most tractable situation is the case when $\mathrm{nd}(X) = 0$. In fact, Theorem 3.7 then gives $K_X \equiv \sum \lambda_j D_j$ for some effective divisor with numerically independent components, $\mathrm{nd}(D_j) = 0$. It follows that the λ_j are rational and therefore

$$(*)\quad K_X \equiv \sum \lambda_j D_j + F \qquad \text{where } \lambda_j \in \mathbb{Q}^+,\ \mathrm{nd}(D_j) = 0 \text{ and } F \in \mathrm{Pic}^0(X).$$

By [CP09] it now follows that $\kappa(X) \geq 0$. Thus we obtain:

3.11. Proposition. *Let X be a smooth projective manifold with K_X pseudo-effective. If $\mathrm{nd}(X) = 0$, then $\kappa(X) = 0$.*

We will come back to abundance on 4-folds in section 9.

The arguments given in Remark 3.10 are actually not restricted to the canonical bundle and show the following.

3.12. Proposition. *Let X be a projective manifold and L a pseudo-effective $\mathbb{R}$-divisor on X.*

(i) *If $\mathrm{nd}(L) = 0$, then $L \equiv \sum \lambda_j D_j$ with λ_j positive real numbers and D_j irreducible divisors. If L is Cartier, the λ_j are rational.*
(ii) *If L is moreover nef in codimension 1 and if $\mathrm{nd}(L) = 0$, then $L \equiv 0$.*

4. The orthogonality estimate

The goal of this section is to show that, in an appropriate sense, approximate Zariski decompositions are almost orthogonal.

4.1. Theorem. *Let X be a projective manifold, and let $\alpha = \{T\} \in \mathcal{E}^{\circ}_{\mathrm{NS}}$ be a big class represented by a Kähler current T. Consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m].$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \leq 20\,(C\omega)^n \big(\mathrm{Vol}(\alpha) - D_m^n\big),$$

where $\omega = c_1(H)$ is a Kähler form and $C \geq 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

Proof. For every $t \in [0,1]$, we have

$$\mathrm{Vol}(\alpha) = \mathrm{Vol}(E_m + D_m) \geq \mathrm{Vol}(tE_m + D_m).$$

Now, by our choice of C, we can write E_m as a difference of two nef divisors

$$E_m = \mu^\star\alpha - D_m = \mu_m^\star(\alpha + C\omega) - (D_m + C\mu_m^\star\omega).$$

4.2. Lemma. *For all nef $\mathbb{R}$-divisors A, B we have*

$$\mathrm{Vol}(A - B) \geq A^n - nA^{n-1}\cdot B$$

as soon as the right hand side is positive.

Proof. In case A and B are integral (Cartier) divisors, this is a consequence of the holomorphic Morse inequalities, [Dem01, 8.5]. If A and B are $\mathbb{Q}$-Cartier, we conclude by the homogeneity of the volume. The general case of $\mathbb{R}$-divisors follows by approximation using the upper semi-continuity of the volume [Bou02b, 3.1.26]. □

4.3. Remark. We hope that Lemma 4.2 also holds true on an arbitrary Kähler manifold for arbitrary nef (non-necessarily integral) classes. This would follow from a generalization of holomorphic Morse inequalities to non-integral classes. However, the proof of such a result seems technically much more involved than in the case of integral classes.

4.4. Lemma. *Let $\beta_1, \ldots, \beta_n$ and $\beta'_1, \ldots, \beta'_n$ be nef classes on a compact Kähler manifold $\widetilde{X}$ such that each difference $\beta'_j - \beta_j$ is pseudo-effective. Then the n-th intersection products satisfy*

$$\beta_1 \cdots \beta_n \leq \beta'_1 \cdots \beta'_n.$$

Proof. We can proceed step-by-step and replace just one β_j by $\beta'_j \equiv \beta_j + T_j$ where T_j is a closed positive $(1,1)$-current and the other classes $\beta'_k = \beta_k$, $k \neq j$ are limits of Kähler forms. The inequality is then obvious. □

End of proof of Theorem 4.1. In order to exploit the lower bound of the volume, we write

$$tE_m + D_m = A - B, \qquad A = D_m + t\mu_m^\star(\alpha + C\omega), \quad B = t(D_m + C\mu_m^\star\omega).$$

By our choice of the constant C, both A and B are nef. Lemma 4.2 and the binomial formula imply

$$\begin{aligned}
&\mathrm{Vol}(tE_m + D_m)\\
&\qquad \geq A^n - nA^{n-1}\cdot B\\
&\qquad = D_m^n + nt\,D_m^{n-1}\cdot\mu_m^\star(\alpha+C\omega) + \sum_{k=2}^n t^k\binom{n}{k}D_m^{n-k}\cdot\mu_m^\star(\alpha+C\omega)^k\\
&\qquad\qquad - nt\,D_m^{n-1}\cdot(D_m + C\mu_m^\star\omega)\\
&\qquad\qquad - nt^2\sum_{k=1}^{n-1}t^{k-1}\binom{n-1}{k}D_m^{n-1-k}\cdot\mu_m^\star(\alpha+C\omega)^k\cdot(D_m+C\mu_m^\star\omega).
\end{aligned}$$

Now, we use the obvious inequalities

$$D_m \leq \mu_m^\star(C\omega), \qquad \mu_m^\star(\alpha+C\omega)\leq 2\mu_m^\star(C\omega), \qquad D_m + C\mu_m^\star\omega \leq 2\mu_m^\star(C\omega)$$

in which all members are nef (and where the inequality $\leq$ means that the difference of classes is pseudo-effective). We use Lemma 4.4 to bound the last summation in the estimate of the volume, and in this way we get

$$\mathrm{Vol}(tE_m + D_m) \geq D_m^n + ntD_m^{n-1}\cdot E_m - nt^2\sum_{k=1}^{n-1}2^{k+1}t^{k-1}\binom{n-1}{k}(C\omega)^n.$$

We will always take t smaller than $1/10n$ so that the last summation is bounded by $4(n-1)(1+1/5n)^{n-2} < 4ne^{1/5} < 5n$. This implies

$$\mathrm{Vol}(tE_m + D_m) \geq D_m^n + nt\,D_m^{n-1}\cdot E_m - 5n^2t^2(C\omega)^n.$$

Now, the choice $t = \frac{1}{10n}(D_m^{n-1}\cdot E_m)((C\omega)^n)^{-1}$ gives by substituting

$$\frac{1}{20}\frac{(D_m^{n-1}\cdot E_m)^2}{(C\omega)^n} \leq \mathrm{Vol}(E_m + D_m) - D_m^n \leq \mathrm{Vol}(\alpha) - D_m^n$$

(and we have indeed $t \leq \frac{1}{10n}$ by Lemma 4.4), whence Theorem 4.1. Of course, the constant 20 is certainly not optimal. □

4.5. Corollary. *If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = N(\alpha) + \langle\alpha\rangle$ is such that*

$$\langle\alpha^{n-1}\rangle\cdot N(\alpha) = 0.$$

Proof. By replacing α by $\alpha+\delta c_1(H)$, one sees that it is sufficient to consider the case where α is big. Then the orthogonality estimate implies

$$\begin{aligned}
(\mu_m)_\star(D_m^{n-1})\cdot(\mu_m)_\star E_m = D_m^{n-1}\cdot(\mu_m)^\star(\mu_m)_\star E_m &\leq D_m^{n-1}\cdot E_m\\
&\leq C(\mathrm{Vol}(\alpha) - D_m^n)^{1/2}.
\end{aligned}$$

Since $\langle\alpha^{n-1}\rangle = \lim(\mu_m)_\star(D_m^{n-1})$, $N(\alpha) = \lim(\mu_m)_\star E_m$ and $\lim D_m^n = \mathrm{Vol}(\alpha)$, we get the desired conclusion in the limit. $\square$

5. Proof of the duality theorem

We want to prove that $\mathcal{E}_{\mathrm{NS}}$ and $\mathrm{SME}(X)$ are dual (Theorem 2.2). By Proposition 1.4(iii) we have in any case

$$\mathcal{E}_{\mathrm{NS}} \subset (\mathrm{SME}(X))^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$ on the boundary of $\mathcal{E}_{\mathrm{NS}}$ which is in the interior of $\mathrm{SME}(X)^\vee$.

Let $\omega = c_1(H)$ be an ample class. Since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class $\alpha+\delta\omega$ is big for every $\delta > 0$, and since $\alpha \in ((\mathrm{SME}(X))^\vee)^\circ$ we still have $\alpha - \varepsilon\omega \in (\mathrm{SME}(X))^\vee$ for $\varepsilon > 0$ small. Therefore

$$\alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma \tag{5.1}$$

for every movable curve Γ. We are going to contradict (5.1). Since $\alpha + \delta\omega$ is big, we have an approximate Zariski decomposition

$$\mu_\delta^\star(\alpha + \delta\omega) = E_\delta + D_\delta.$$

We pick $\Gamma = (\mu_\delta)_\star(D_\delta^{n-1})$. By the Hovanskii-Teissier concavity inequality

$$\omega \cdot \Gamma \geq (\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}.$$

On the other hand

$$\begin{aligned}
\alpha \cdot \Gamma &= \alpha \cdot (\mu_\delta)_\star(D_\delta^{n-1}) \\
&= \mu_\delta^\star\alpha \cdot D_\delta^{n-1} \leq \mu_\delta^\star(\alpha + \delta\omega) \cdot D_\delta^{n-1} \\
&= (E_\delta + D_\delta) \cdot D_\delta^{n-1} = D_\delta^n + D_\delta^{n-1} \cdot E_\delta.
\end{aligned}$$

By the orthogonality estimate, we find

$$\begin{aligned}
\frac{\alpha \cdot \Gamma}{\omega \cdot \Gamma} &\leq \frac{D_\delta^n + \big(20(C\omega)^n(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)\big)^{1/2}}{(\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}} \\
&\leq C'(D_\delta^n)^{1/n} + C''\frac{(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)^{1/2}}{(D_\delta^n)^{(n-1)/n}}.
\end{aligned}$$

However, since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class α cannot be big so

$$\lim_{\delta\to 0} D_\delta^n = \mathrm{Vol}(\alpha) = 0.$$

We can also take D_δ to approximate $\mathrm{Vol}(\alpha+\delta\omega)$ in such a way that $(\mathrm{Vol}(\alpha+\delta\omega)-D_\delta^n)^{1/2}$ tends to 0 much faster than D_δ^n. Notice that $D_\delta^n \geq \delta^n\omega^n$, so in fact it is enough to take

$$\mathrm{Vol}(\alpha+\delta\omega)-D_\delta^n \leq \delta^{2n}.$$

This is the desired contradiction by (5.1). □

5.2. Remark. If holomorphic Morse inequalities were known also in the Kähler case, we would infer by the same proof that "α not pseudo-effective" implies the existence of a blow-up $\mu : \widetilde{X} \to X$ and a Kähler metric $\widetilde{\omega}$ on $\widetilde{X}$ such that $\alpha\cdot\mu_\star(\widetilde{\omega})^{n-1} < 0$. In the special case when $\alpha = K_X$ is not pseudo-effective, we would expect the Kähler manifold X to be covered by rational curves. The main trouble is that characteristic p techniques are no longer available. On the other hand it is tempting to approach the question via techniques of symplectic geometry:

5.3. Question. *Let (M,ω) be a compact real symplectic manifold. Fix an almost complex structure J compatible with ω, and for this structure, assume that $c_1(M)\cdot\omega^{n-1} > 0$. Does it follow that M is covered by rational J-pseudo-holomorphic curves?*

6. Non-nef loci

Following [Bou02b], we introduce the concept of non-nef locus of an arbitrary pseudo-effective class. The details differ a little bit here (and are substantially simpler) because the scope is limited to compact Kähler manifolds.

6.1. Definition. Let X be a compact Kähler manifold, ω a Kähler metric, and $\alpha \in \mathcal{E}$ a pseudo-effective class. We define the non-nef locus of α to be

$$L_{\mathrm{nonnef}}(\alpha) = \bigcup_{\delta>0}\bigcap_T \mu(|E|)$$

for all log resolutions $\mu^\star T = [E]+\beta$ of positive currents $T \in \{\alpha+\delta\omega\}$ with logarithmic singularities, $\mu : \widetilde{X} \to X$, and $\mu(|E|)$ is the set-theoretic image of the support of E.

It should be noticed that the union in the above definition can be restricted to any sequence δ_k converging to 0, hence $L_{\mathrm{nonnef}}(\alpha)$ is either an analytic set or a countable union of analytic sets. The results of [Dem92] and [Bou02b] show that

$$L_{\mathrm{nonnef}}(\alpha) = \bigcup_{\delta>0} \bigcap_{T} E_+(T) \tag{6.1'}$$

where T runs over the set $\alpha[-\delta\omega]$ of all d-closed real $(1,1)$-currents $T \in \alpha$ such that $T \geq -\delta\omega$, and $E_+(T)$ denotes the locus where the Lelong numbers of T are strictly positive. The latter definition (6.1′) works even in the non-Kähler case, taking ω an arbitrary positive Hermitian form on X. By [Bou02b], there is always a current $T_{\min}$ which achieves minimum singularities and minimum Lelong numbers among all members of $\alpha[-\delta\omega]$, hence $\bigcap_T E_+(T) = E_+(T_{\min})$.

6.2. Theorem. *Let $\alpha \in \mathcal{E}$ be a pseudo-effective class. Then $L_{\mathrm{nonnef}}(\alpha)$ contains the union of all irreducible algebraic curves C such that $\alpha \cdot C < 0$.*

Proof. If C is an irreducible curve not contained in $L_{\mathrm{nonnef}}(\alpha)$, the definition implies that for every $\delta > 0$ we can choose a positive current $T \in \{\alpha + \delta\omega\}$ and a log-resolution $\mu^\star T = [E] + \beta$ such that $C \not\subset \mu(|E|)$. Let $\widetilde{C}$ be the strict transform of C in $\widetilde{X}$, so that $C = \mu_\star \widetilde{C}$. We then find

$$(\alpha + \delta\omega) \cdot C = ([E] + \beta) \cdot \widetilde{C} \geq 0$$

since $\beta \geq 0$ and $\widetilde{C} \not\subset |E|$. This is true for all $\delta > 0$ and the claim follows.

6.3. Remark. One may wonder, at least when X is projective and $\alpha \in \mathcal{E}_{\mathrm{NS}}$, whether $L_{\mathrm{nonnef}}(\alpha)$ is actually equal to the union of curves C such that $L \cdot C < 0$ (or the "countable Zariski closure" of such a union). Unfortunately, this is not true, even on surfaces. The following simple example was shown to us by E. Viehweg. Let Y be a complex algebraic surface possessing a big line bundle F with a curve C such that $F \cdot C < 0$ as its base locus (e.g. $F = \pi^\star\mathcal{O}(1) + E$ for the blow-up $\pi : Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ in one point, and $C = E =$ exceptional divisor). Then take finitely many points $p_j \in C$, $1 \leq j \leq N$, and blow-up these points to get a modification $\mu : X \to Y$. We select

$$L = \mu^\star F + \widehat{C} + 2\sum E_j = \mu^\star(F + C) + \sum E_j$$

where $\widehat{C}$ is the strict transform of C and $E_j = \mu^{-1}(p_j)$. It is clear that the non-nef locus of $\alpha = c_1(L)$ must be equal to $\widehat{C} \cup \bigcup E_j$, although

$$L \cdot \widehat{C} = (F + C) \cdot C + N > 0$$

for N large. This example shows that the set of α-negative curves is not the appropriate tool to understand the non-nef locus.

7. Pseudo-effective vector bundles

In this section we consider pseudo-effective and almost nef vector bundles as introduced in [DPS01]. As an application, we obtain interesting informations concerning the tangent bundle of Calabi-Yau manifolds. First we recall the relevant definitions.

7.1. Definition. Let X be a compact Kähler manifold and E a holomorphic vector bundle on X. Then E is said to be *pseudo-effective* if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$ of hyperplanes of E, and if the projection $\pi(L_{\mathrm{nonnef}}(\mathcal{O}_{\mathbb{P}(E)}(1)))$ of the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ onto X does not cover all of X.

This definition would even make sense on a general compact complex manifold, using the general definition of the non-nef locus in [Bou02b]. On the other hand, the following proposition gives an algebraic characterization of pseudo-effective vector bundles in the projective case.

7.2. Proposition. *Let X be a projective manifold. A holomorphic vector bundle E on X is pseudo-effective if and only if for any given ample line bundle A on X and any positive integers m_0, p_0, the vector bundle*

$$S^p((S^m E)\otimes A)$$

is generically generated (i.e. generated by its global sections on the complement $X \smallsetminus Z_{m,p}$ of some algebraic set $Z_{m,p} \neq X$) for some [respectively every] $m \geq m_0$ and $p \geq p_0$.

Proof. If global sections as in the statement of Proposition 7.2 exist, they can be used to define a singular Hermitian metric $h_{m,p}$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ which has poles contained in $\pi^{-1}(Z_{m,p})$ and whose curvature form satisfies $\Theta_{h_{m,p}}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geq -\frac{1}{m}\pi^*\Theta(A)$. Hence, by selecting suitable integers $m = M(m_0,p_0)$ and $p = P(m_0,p_0)$, we find that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective (its first Chern class is a limit of pseudo-effective classes), and that

$$\pi(L_{\mathrm{nonnef}}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subset \bigcup_{m_0}\bigcap_{p_0} Z_{m,p} \subsetneq X.$$

Conversely, assume that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and admits singular Hermitian metrics h_δ such that $\Theta_{h_\delta}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geq -\delta\widetilde{\omega}$ and $\pi(\mathrm{Sing}(h_\delta)) \subset Z_\delta \subsetneq X$ (for some Kähler metric $\widetilde{\omega}$ on $\mathbb{P}(E)$ and arbitrary small $\delta > 0$). We can actually take $\omega = \Theta(A)$ and $\widetilde{\omega} = \varepsilon_0\Theta_{h_0}(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^*\omega$ with a given smooth Hermitian metric h_0 on E and $\varepsilon_0 \ll 1$. An easy calculation shows that the linear combination $h'_\delta = h_\delta^{1/(1+\delta\varepsilon_0)} h_0^{\delta\varepsilon_0}$ yields a metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that

$$\Theta_{h'_\delta}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geq -\delta\pi^*\Theta(A).$$

By taking $\delta = 1/2m$ and multiplying by m, we find

$$\Theta(\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A) \geq \frac{1}{2}\pi^*\Theta(A)$$

for some metric on $\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A$ which is smooth over $\pi^{-1}(X \smallsetminus Z_\delta)$. The standard theory of L^2 estimates for bundle-valued $\overline{\partial}$-operators can be used to produce the required sections, after we multiply $\Theta(A)$ by a sufficiently large integer p to compensate the curvature of $-K_X$. The sections possibly still have to vanish along the poles of the metric, but they are unrestricted on fibers of $\mathbb{P}(S^m E) \to X$ which do not meet the singularities. $\square$

Note that if E is pseudo-effective, then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and E is *almost nef* in the following sense which is just the straightforward generalization from the line bundle case.

7.3. Definition. Let X be a projective manifold and E a vector bundle on X. Then E is said to be *almost nef*, if there is a countable family A_i of proper subvarieties of X such that $E|C$ is nef for all $C \not\subset \bigcup_i A_i$. Alternatively, E is almost nef if there is no covering family of curves such that E is non-nef on the general member of the family.

Observe that E is almost nef if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is almost nef and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on the general member of any family of curves in $\mathbb{P}(E)$ whose images cover X. Hence Theorem 2.2 yields the following corollary.

7.4. Corollary. *Let X be a projective manifold and E a holomorphic vector bundle on X. If E is almost nef, then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective. Thus for some [or any] ample line bundle A, there are positive numbers m_0 and p_0 such that*

$$H^0(X, S^p((S^m E) \otimes A)) \neq 0$$

for all $m \geq m_0$ and $p \geq p_0$.

One should notice that it makes a big difference to assert just the existence of a non-zero section, and to assert the existence of sufficiently many sections guaranteeing that the fibers are generically generated. It is therefore natural to raise the following question.

7.5. Question. *Let X be a projective manifold and E a vector bundle on X. Suppose that E is almost nef. Is E always pseudo-effective in the sense of Definition* 7.1*?*

This was stated as a theorem in [DPS01, 6.3], but the proof given there was incomplete. The result now appears quite doubtful to us. However, we give below a positive answer to Question 7.5 in case of a rank 2-bundle E

with $c_1(E) = 0$ (conjectured in [DPS01]), and then apply it to the study of tangent bundles of K3-surfaces.

7.6. Theorem. *Let E be an almost nef vector bundle of rank at most 3 on a projective manifold X. Suppose that $\det E \equiv 0$. Then E is numerically flat.*

Proof. Recall (cf. [DPS94]) that a vector bundle E is said to be numerically flat if it is nef as well as its dual (or, equivalently, if E is nef and $\det E$ numerically trivial); also, E is numerically flat if and only if E admits a filtration by subbundles such that the graded pieces are unitary flat vector bundles. By [Ko87, p. 115], E is unitary flat as soon as E is stable for some polarization and $c_1(E) = c_2(E) = 0$.

Under our assumptions, E is necessarily semi-stable since semi-stability with respect to a polarization H can be tested against a generic complete intersection curve, and we know that E is nef, hence numerically flat, on such a curve. Therefore (see also [DPa04, 6.8]) we can assume without loss of generality that $\dim X = 2$ and that E is stable with respect to all polarizations, and it is enough to show in that case that $c_2(E) = 0$. Since E is almost nef, E is nef, hence numerically flat, on all curves except for at most a countable number of curves, say $(\Gamma_j)_{j \in \mathbb{N}}$.

First suppose that E has rank 2. Then the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ is immediately seen to be nef on all but a countable number of curves. In fact, the only curves on which $\mathcal{O}(1)$ is negative are the sections over the curves Γ_j with negative self-intersection in $\mathbb{P}(E|\Gamma_j)$. Now take a general hyperplane section H on $\mathbb{P}(E)$. Then H does not contain any of these bad curves and therefore $\mathcal{O}(1)$ is nef on H. Hence,

$$c_1(\mathcal{O}(1))^2 \cdot H \geq 0.$$

Now, up to a multiple, H is of the form $H = \mathcal{O}(1) \otimes \pi^*(G)$ so that

$$c_1(\mathcal{O}(1))^3 + c_1(\mathcal{O}(1))^2 \cdot \pi^*(G) \geq 0.$$

Since $c_1(\mathcal{O}(1))^3 = c_1(E)^2 - c_2(E) = -c_2(E)$ and $c_1(\mathcal{O}(1))^2 \cdot \pi^*(G) = c_1(E) \cdot G = 0$, we conclude $c_2(E) = 0$.

If E has rank 3, we need to argue more carefully, because now $\mathcal{O}(1)$ is non-nef on the surfaces $S_j = \mathbb{P}(E|C_j)$ so that $\mathcal{O}(1)$ might be non-nef on a general hyperplane section H. We will, however, show that this can be avoided by choosing carefully the linear system $|H|$. To be more precise, we fix G ample on X and look for

$$H \in |\mathcal{O}(1) + \pi^*(mG)|$$

with $m \gg 0$, so that $\mathcal{O}(1)$ is nef on $H \cap S_j$ for all j. Given that $\mathcal{O}(1)|H$ and we can argue as in the previous case to obtain $c_2(E) = 0$. Of course for a general choice of H, all curves $H \cap S_j$ will be irreducible (but possibly singular since

C_j might be singular). Now fix j and set $\tilde{C} = H \cap S_j$, a section over $C = C_j$. Let $V \subset E_C$ be the maximal ample subsheaf (see [PS02]). Then we obtain a vector bundle sequence

$$0 \to V \to E_C \to F \to 0$$

and we may assume that F has rank 2, because otherwise $\mathcal{O}(1)$ is not nef only on one curve over C. Now $\tilde{C}$ induces an exact sequence

$$0 \to \mathcal{O}_C(-mG) \to F \to F' \to 0$$

and therefore $\mathcal{O}(1)|\tilde{C}$ is nef iff $c_1(F') \geq 0$. This translates into $c_1(F) + m(G \cdot C) \geq 0$. Now let t_0 be the nef value of E with respect to G, i.e. $E(t_0G)$ is nef but not ample. Then $F(t_0G)$ is nef, too, so that $c_1(F) \geq -2t_0(G \cdot C)$. In total

$$c_1(F') \geq (m - 2t_0)(G \cdot C),$$

hence we choose $m \geq 2t_0$ and for this choice $\mathcal{O}(1)|H$ is nef. □

As a corollary we obtain the following theorem.

7.7. Theorem. *Let X be a projective K3-surface or a Calabi-Yau 3-fold. Then the tangent bundle T_X is not almost nef, and there exists a covering family (C_t) of (generically irreducible) curves such that $T_X|C_t$ is not nef for general t.*

In other words, if $c_1(X) = 0$ and T_X is almost nef, then a finite étale cover of X is abelian. One should compare this with Miyaoka's theorem that $T_X|C$ is nef for a smooth curve C cut out by hyperplane sections of sufficiently large degree. Note also that $T_X|C$ being not nef is equivalent to say that $T_X|C$ is not semi-stable. We expect that Theorem 7.7 holds in general for Calabi-Yau manifolds of any dimension.

Proof. Assume that T_X is almost nef. Then by Theorem 7.6, T_X is numerically flat. In particular, $c_2(X) = 0$ and hence X is an étale quotient of a torus. □

We will now improve Theorem 7.7 for K3-surfaces; namely, if X is a projective K3-surface, then already $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ should be non-pseudo-effective. In other words, let A be a fixed ample divisor on X. Then for all positive integers m there exists a positive integer p such that

$$H^0(X, S^p((S^m T_X) \otimes A)) = 0.$$

This has been verified in [DPS01] for the general quartic in $\mathbb{P}_3$ and below for any K3-surface. The next theorem is also proved in [Na04].

7.8. Theorem. *Let X be a projective K3-surface and $L = \mathcal{O}_{\mathbb{P}(T_X)}(1)$. Then L is not pseudo-effective.*

Proof. Suppose that L is pseudo-effective and consider the divisorial Zariski decomposition ([Bou02b], cf. also Theorem 3.5(iv))

$$L = N + Z$$

with N an effective $\mathbb{R}$-divisor and Z nef in codimension 1. Write $N = aL + \pi^*(N')$ and $Z = bL + \pi^*(Z')$. Let H be very ample on S. By restricting to a general curve C in $|H_H|$ and observing that $T_X|C$ is numerically flat, we see that $L|\pi^{-1}(C)$ is nef (but not ample), hence

$$N' \cdot C = 0.$$

Thus $N' = 0$ since H is arbitrary. If $a > 0$, then some mL would be effective, i.e. $S^m T_X$ would have a section, which is known not to be the case. Hence, $a = 0$ and L is nef in codimension 1 so that L can be negative only on finitely many curves. This contradicts Theorem 7.7. □

8. Nef reduction relative to a covering family of curves

In this section we construct reduction maps for pseudo-effective line bundles which have vanishing intersection numbers on large families of curves. This will be applied in the next section in connection with the abundance problem.

8.1. Notation. Let $(C_t)_{t\in T}$ be a covering family of (generically irreducible) curves (in particular, T is irreducible and compact). Then (C_t) is said to be a *connecting family* if and only if two general points x, y can be joined by a chain of C_t's. We also say that X is (C_t)-connected.

By T^* we denote the Zariski-open (non-empty) set of those t for which C_t is irreducible.

Using Campana's reduction theory [Ca81], [Ca94], and [Ca04], we immediately obtain the following theorem.

8.2. Theorem. *Let X be a projective manifold and L a pseudo-effective line bundle on X. Let (C_t) be a covering family with $L \cdot C_t = 0$. Then there exists an almost holomorphic surjective meromorphic map $f : X \dashrightarrow Y$ with $\dim Y < \dim X$ such that the general (compact) fiber of f is (C_t)-connected. f is called the nef reduction of L relative to (C_t).*

Recall that a meromorphic map is *almost holomorphic*, if there is an open non-empty set on which the map is holomorphic and proper. Of course, there might be other families (C'_s) with $L \cdot C'_s = 0$ leading to different quotients.

8.3. Definition. Let L be a pseudo-effective line bundle on X. The minimal number which can be realised as $\dim Y$ with a nef reduction $f : X \dashrightarrow Y$

relative to L is denoted $p(L)$. If there is no covering family (C_t) with $L \cdot C_t = 0$, then we set $p(L) = \dim X$.

8.4. Remark. The equality $p(L) = 0$ holds if and only if there exists a *connecting* family (C_t) such that $L \cdot C_t = 0$. If, moreover, L is nef, then $p(L) = 0$ if and only if $L \equiv 0$ [workshop].

For computing Kodaira dimensions, sometimes the notion of a connecting family has to be strenghtened:

8.5. Definition. A covering family $(C_t)_{t \in T}$ is strongly connecting if any two sufficiently general points x and y can be joined by a chain of irreducible C_t, i.e. $t \in T^*$, avoiding any given analytic set A of codimension at least 2.

8.6. Theorem. *Let X be a projective manifold, L a pseudo-effective $\mathbb{R}$-divisor. Let (C_t) be a strongly connected family of curves. If $L \cdot C_t = 0$, then* $\mathrm{nd}(L) = 0$. *If L is Cartier, then L is numerically equivalent to a Cartier divisor L' with $\kappa(L') = 0$.*

Equivalently:

8.7. Theorem. *Let X be a projective manifold of any dimension n, (C_t) a strongly connecting family and L an $\mathbb{R}$-divisor that is nef in codimension 1. If $L \cdot C_t = 0$, then $L \equiv 0$.*

Proof (of Theorem 8.6 from Theorem 8.7). Consider the divisorial Zariski decomposition $L = N + Z$ with N an $\mathbb{R}$-effective divisor and Z nef in codimension 1. Then $N \cdot C_t \geq 0$ and $Z \cdot C_t \geq 0$, so that $L \cdot C_t = 0$ forces $Z \cdot C_t = 0$. Hence, $Z \equiv 0$ by Theorem 8.7 so that $\mathrm{nd}(L) = 0$. For the second statement we refer to Proposition 3.12. □

Proof (of Theorem 8.7) (I). Let T be the parameter space of the family (C_t); we may assume $\dim T = n - 1$. In a first step, we reduce to the case that through every point $x \in X$ there are only finitely many C_t. In fact, in the general situation, take a birational map $\sigma : \tilde{X} \to X$ from a projective manifold $\tilde{X}$, such that the induced family $\tilde{C}_t$ has the finiteness property (this can be achieved, e.g., by flattening the projection map from the graph of the family to X). Then let $\tilde{L}$ be the mobile part of $\sigma^*(L)$, i.e. the part in the divisorial Zariski decomposition which is nef in codimension 1. Since we know that the general C_t avoids codimension 2 sets, we obtain $\tilde{L} \cdot \tilde{C}_t = 0$. By our assumption that Theorem 8.7 already holds if the finiteness condition is verified, we conclude that $\tilde{L} \equiv 0$. Hence, $L \equiv 0$.

(II) From now on we may assume that only finitely many C_t pass through a fixed point of X. Therefore the following holds. If $C \to T$ is the parameter space of the family (C_t) and if $Y \subset X$ is any analytic set of codimension at least 2, then the set of points $t \in T$ such that $\dim(C_t \cap Y) = 1$ has codimension at least 2 in T.

Let $B \subset X$ be a general curve. We follow the arguments in [workshop] and fix a general point x. We choose a family $(B_s)_{s\in S}$ joining x with the curve B by chains of irreducible curves C_t. Let $p : S \to X$ be the graph of the family with parameter space $q : C \to T$. Let S_j be the irreducible components of C. The codimension argument above says that we may assume that $\dim B_s \cap A \leq 0$, where A is the non-nef locus of L. Thus $p^*(L)|S_j$ is nef in codimension 1 for all j, hence nef. But now the arguments of [workshop] work, and consequently $L \cdot B = 0$. Taking, e.g., B by complete intersection curves cut out by arbitrary hyperplane sections, we conclude that $L \equiv 0$. □

Of course the question arises whether Theorem 8.6 holds for all connecting families. Unfortunately, this is not true, as demonstrated by the following example.

8.8. Example. We produce a smooth projective threefold X, a line bundle L on X which is nef in codimension 1, in particular, pseudo-effective, and a connecting family (C_t) such that $h^0(L) = 2$ and $\kappa(L) = 1$, but

$$L \cdot C_t = 0.$$

We start with the $\mathbb{P}_2$-bundle

$$p : X_1 := \mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}_1.$$

Consider the section

$$B_1 = \mathbb{P}(\mathcal{O}(-1)) \subset X_1$$

with normal bundle $N_{B_1/X_1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Now we flop the curve B_1 : we first consider the blow-up $\tau : X_2 \to X_1$ of X_1 along B_1. The exceptional divisor $E \simeq \mathbb{P}_1 \times \mathbb{P}_1$ has normal bundle $\mathcal{O}(-1,-1)$ and therefore X_2 can be blown-down along the other projection to obtain $\sigma : X_2 \to X$. Let $B = \sigma(E)$ and

$$L = (\sigma_* \tau^* p^*(\mathcal{O}(1)))^{**}.$$

Then clearly $h^0(L) = 2$ and $\kappa(L) = 1$. Let F_s be the strict transform of $p^{-1}(s)$, $s \in \mathbb{P}_1$ in X so that $L = \mathcal{O}_X(F_s)$. Observe that F_s is $\mathbb{P}_2$ blown up in one point, that B is the base locus of $H^0(L)$, and that $B \subset F_s$ is the (-1)-curve. In order to establish the connecting family (C_t) with $L \cdot C_t = 0$, consider a section $D' = \mathbb{P}(\mathcal{O})$, disjoint from B_1 and let D be its strict transform in X. In every fiber F_s we can consider the family of "lines" meeting D', i.e., containing the point $p_s = D \cap F_s$. A "line" is of course a curve whose images in $\mathbb{P}_2$ is a line in the usual sense. The family of "lines" inside F_s has exactly one splitting element containing the (-1)-curve B. The other component is the strict transform of the line in $\mathbb{P}_2$ joining p_s and the point to be blown up. Varying s, so we obtain a family (C_t), which is clearly connecting because of the curve B. Moreover, it is clear that $L \cdot C_t = 0$. $\square$

It is still possible to say something for general connecting families. First observe the following lemma.

8.9. Lemma. *Suppose $\kappa(L) \geq 0$ and let $D \in |mL|$ for some positive integer m. If (C_t) is a covering family such that $L \cdot C_t = 0$, then*

$$\operatorname{supp}(D) \cap C_t = \emptyset$$

for general t.

Proof. Just choose t general so that $C_t \not\subset \operatorname{supp}(D)$. Then $D \cdot C_t$ implies the claim. $\square$

8.10. Proposition. *Let X be a projective manifold of dimension n and L a line bundle on X such that $L \cdot C_t = 0$ for a connecting family (C_t). Then $\kappa(L) \leq n - 2$.*

Proof. Suppose the contrary and choose m such that $H^0(mL)$ defines a map $f : X \dashrightarrow Y$ to a variety of dimension at least $n - 1$. By Lemma 8.9, f is holomorphic near the general C_t. This already reduces to $\dim Y = n - 1$. Since we may assume that f has connected fibers, the C_t are just the fibers of f (at least for general t), so that f is almost holomorphic, and it is immediate that the family (C_t) cannot be connecting. $\square$

Although Theorem 8.6 fails in general for connecting families and arbitrary line bundles L, one might hope more in case $L = K_X$.

8.11. Proposition. *Let X be a smooth projective threefold and (C_t) be a connecting family such that $K_X \cdot C_t = 0$. Then $\kappa(X) \leq 0$ unless we are in the following special situation:*

(i) *K_X is not nef, $\kappa(X) = 1$.*
(ii) *There is a sequence $\phi : X \dashrightarrow X'$ of Mori contractions and flips such that*

$$K_{X'} \cdot C'_t = 0$$

for the induced family (C'_t) (ϕ is holomorphic near the general C_t).
(iii) *On X' we have a flip*

$$X' \dashrightarrow X^+,$$

the induced family (C^+_t) is no longer connecting; moreover, $K_{X^+} \cdot C^+ = 0$.

(iv) *The Iitaka fibration is a holomorphic map* $f : X^+ \to Z^+ \simeq \mathbb{P}_1$ *and* f *is a quotient for the family* (C_t^+).

(v) *The exceptional locus* $E^+ \subset X^+$ *of the flip* $X' \dashrightarrow X^+$ *dominates* Z^+.

Proof. Assume $\kappa(X) \geq 1$, hence $\kappa(X) = 1$ (see Example 8.8). By [workshop], K_X cannot be nef. Let $\phi : X \to X_1$ be a Mori contraction, necessarily birational; let E be the exceptional divisor. Then $E \cdot C_t = 0$; otherwise, $K_{X_1} \cdot \phi(C_t) < 0$, and X_1 would be uniruled. Then we get a connecting family (C_t^1) in X_1 and proceed inductively. So we arrive at a flip

$$X' \dashrightarrow X^+.$$

Let $E' \subset X'$ respectively $E^+ \subset X^+$ be the 1-dimensional "exceptional" sets. From Lemma 8.9 and the fact that K_X is negative on all components of E', we deduce that

$$C_t' \cap E' = \emptyset$$

for general t. Hence,

$$K_{X^+} \cdot C_t^+ = 0.$$

If (C_t^+) is again connecting, then we proceed with X^+. After finitely many steps we must arrive at the case where (C_t^+) is no longer connecting, because by [workshop] we cannot arrive at some X^+ with K_{X^+} nef and the induced family being connecting. Therefore, we consider the quotient

$$f : X^+ \dashrightarrow Z^+.$$

Then E^+ must map onto Z^+; otherwise, (C_t') would not be connecting. Thus $Z^+ \simeq \mathbb{P}_1$ and the almost holomorphic map f must be holomorphic. By Example 8.8, the general fiber F of f has $\kappa(F) = 0$. Hence, f is the Iitaka fibration. □

It is not completely clear whether the situation in Proposition 8.11 really occurs.

It is also interesting to look at covering families (C_t) of *ample* curves. Here "ample" means that the dual of the conormal sheaf modulo torsion is ample (say on the normalization). Then we have the same result as in Theorem 8.6 which is prepared by the following lemma.

8.12. Lemma. *Let* X *be a projective manifold,* $C \subset X$ *an irreducible curve with normalization* $f : \tilde{C} \to C$ *and ideal sheaf* $\mathcal{I}$. *Let* L *be a line bundle on* X. *Then there exists a positive number* c *such that for all* $t \geq 0$*:*

$$h^0(X, L^t) \leq \sum_{k=0}^{ct} h^0(f^*(S^k(\mathcal{I}/\mathcal{I}^2/\mathrm{tor}) \otimes L^t)).$$

Proof. Easy adaptation of the proof of Theorem 2.1 in [PSS99]. □

8.13. Corollary. *Let X be a projective manifold and $C \subset X$ be an irreducible curve with normalisation $f : \tilde{C} \to C$ such that $f^*(\mathcal{I}/\mathcal{I}^2)^*$ is ample. Let L be a line bundle with $L \cdot C_t = 0$. Then $\kappa(L) \leq 0$. In particular, this holds for the general member of an ample covering family.*

Proof. By Lemma 8.12 it suffices to show that

$$h^0(f^*(S^k(\mathcal{I}/\mathcal{I}^2/\mathrm{tor}) \otimes L^t)) = 0$$

for all $k \geq 1$. This is, however, clear since by assumption $f^*(\mathcal{I}/\mathcal{I}^2)^*$ is an ample bundle. □

8.14. Corollary. *Let X be a smooth projective threefold with K_X pseudo-effective. If there is a ample covering family or a connecting family (C_t) such that $K_X \cdot C_t = 0$, then $\kappa(X) = 0$.*

Proof. By Corollary 8.13 we have $\kappa(X) \leq 0$. Suppose that $\kappa(X) = -\infty$. Then X is uniruled by Miyaoka's theorem. Thus K_X is not pseudo-effective. □

To complete the picture, we will construct a nef reduction for pseudo-effective line bundles, generalizing to a certain extent the result of [workshop] for nef line bundle (however, the result is weaker). A different type of reduction was constructed in [TS00] and [Ec02].

8.15. Theorem. *Let L be a pseudo-effective line bundle on a projective manifold X. Then there exists an almost holomorphic meromorphic map $f : X \dashrightarrow Y$ such that:*

(i) *General points on the general fiber of F can be connected by a chain of L-trivial irreducible curves.*
(ii) *If $x \in X$ is general and C is an irreducible curve through x with* $\dim f(C)$
> 0*, then* $L \cdot C > 0$.

Proof. Start with a covering L-trivial family (C_t) and build the relative nef reduction $h : X \dashrightarrow Z$ (if the family does not exist, put $f = id$). Now take another covering L-trivial family (B_s) (if this does not exist, just stop) with relative nef reduction g. For general $z \in Z$, let F_z be the set of all $x \in X$ which can be joined with the fiber X_z by a chain of curves B_s. In other words, F_z is the closure of $g^{-1}(g(X_z))$. Now the F_z define a covering family (of higher-dimensional subvarieties) which defines by Campana's theorem a new reduction map. After finitely many steps, we arrive at the map we are looking for. □

Finally, we give a criterion when a covering family is actually connecting:

8.16. Theorem. *Let X be a projective manifold and (C_t) a covering family. Suppose that $[C_t]$ is an interior point of the movable cone $\mathcal{M}$. Then (C_t) is connecting.*

Proof. Let $f : X \dashrightarrow Z$ be the reduction of the family (C_t). If the family is not connecting, then $\dim Z > 0$. Let $\pi : \tilde{X} \to X$ be a modification such that the induced map $\tilde{f} : \tilde{X} \to Z$ is holomorphic. Let A be very ample on Z and put $L = \pi_*(\tilde{f}^*(A))^{**}$. Then L is an effective line bundle on X with $L \cdot C_t = 0$ since L is trivial on the general fiber of f, this map being almost holomorphic. Hence, $[C_t]$ must be on the boundary of $\mathcal{M}$. □

The converse of Theorem 8.16 is of course false: consider the family of lines l in $\mathbb{P}_2$ and let X be the blow-up of some point in $\mathbb{P}_2$. Let (C_t) be the closure of the family of preimages of general lines. This is a connecting family, but if E is the exceptional divior, then $E \cdot C_t = 0$. So (C_t) cannot be in the interior of $\mathcal{M}$.

9. Towards abundance

In this section we prove that a smooth projective 4-fold X with K_X pseudo-effective and with the additional property that $K_X \cdot C_t = 0$ for some good covering family of curves (C_t), has $\kappa(X) \geq 0$. In other words, we deal with problem (B2) from the introduction in dimension 4. In the remaining case, that K_X is positive on all covering and non-connecting or strongly connecting families of curves, one expects that K_X is big.

9.1. Proposition. *Let X be a smooth projective 4-fold with K_X pseudo-effective. Suppose that there exists a dominant rational map $f : X \dashrightarrow Y$ to a projective manifold Y with $\kappa(Y) \geq 0$ (and $0 < \dim Y < 4$). Then $\kappa(X) \geq 0$.*

Proof. We may assume f holomorphic with general fiber F. If $\kappa(F) = -\infty$, then F would be uniruled, hence X would be uniruled. Hence, $\kappa(F) \geq 0$. Now $C_{n,n-3}$, $C_{n,n-2}$ and $C_{n,n-1}$ hold true; see e.g. [Mo87] for further references. This gives

$$\kappa(X) \geq \kappa(F) + \kappa(Y) \geq 0$$

and therefore our claim. □

9.2. Corollary. *Let X be a smooth projective 4-fold with K_X pseudo-effective. Let $f : X \dashrightarrow Y$ be a dominant rational map $(0 < \dim Y < 4)$ with Y not rationally connected. Then $\kappa(X) \geq 0$.*

Proof. If $\dim Y \leq 2$, this is immediate from Proposition 9.1. So let $\dim Y = 3$. Since we may assume $\kappa(Y) = -\infty$, the threefold Y is uniruled. Let $h : Y \dashrightarrow Z$ be the rational quotient; we may assume that h is holomorphic and Z smooth. Since Y is not rationally connected, we have $\dim Z \geq 1$ and, moreover, Z is not uniruled by Colliot-Thélène [CT86]; see also Graber-Harris-Starr [GHS03]. Hence, $\kappa(Z) \geq 0$ and we conclude by Proposition 9.1. $\square$

9.3. Conclusion. *In order to prove $\kappa(X) \geq 0$ in case of a dominant rational map $f : X_4 \dashrightarrow Y$, we may assume that Y is a rational curve, a rational surface or a rationally connected 3-fold.*

9.4. Proposition. *Let X be a smooth projective 4-fold with K_X pseudo-effective. If $p(K_X) = 1$, then $\kappa(X) \geq 0$.*

Proof. By assumption we have a covering family (C_t) with $K_X \cdot C_t = 0$, so that the relative nef reduction is a holomorphic map $f : X \to Y$ to a curve Y. By Conclusion 9.3 we may assume $Y = \mathbb{P}_1$. We already saw that $\kappa(F) \geq 0$. By Proposition 8.11 "mostly" we even have $\kappa(F) = 0$ unless we are in a very special situation.

We begin treating the case $\kappa(F) = 0$. Choose m such that $h^0(mK_F) \neq 0$ for the general fiber F of f. Thus $f_*(mK_X)$ is a line bundle on Y, and we can write

$$mK_X = f^*(A) + \sum a_i F_i + E$$

where F_i are fiber components and E surjects onto Y with $h^0(\mathcal{O}_X(E)) = 1$. The divisor E comes from the fact that F is not necessarily minimal; actually $E|F = mK_F$. By enlarging m, we may also assume that the support of $\sum a_i F_i$ does not contain any fiber and also that mK_X is Cartier. We consider the divisorial Zariski decomposition

$$mK_X = \tilde{N} + \tilde{Z}$$

with $\tilde{N}$ being $\mathbb{R}$-effective and $\tilde{Z}$ nef in codimension 1. Then clearly $E \subset \tilde{N}$ so that

$$(*) \qquad f^*(A) + \sum a_i F_i = \tilde{N}' + \tilde{Z}$$

with $\tilde{N}'$ again $\mathbb{R}$-effective.

Now let $S \subset X$ be a surface cut out by 2 general hyperplane sections. Let $L = mK_X|S$ and $E' = E|S$. Denoting $G_i = F_i|S$, $g = f|S$ and $N = \tilde{N}'|,\ Z = \tilde{Z}$, we obtain

$$(**) \qquad L = g^*(A) + \sum a_i G_i + E'$$

and from $(*)$

$$(***) \qquad g^*(A) + \sum a_i G_i = N' + Z.$$

Then N' is $\mathbb{R}$-effective and not nef, and Z is nef. However, N' might a priori have a nef part; so we consider the divisorial Zariski decomposition $N' = N_0 + Z_0$ and set $Z_1 = Z_0 + Z$. Let l be a general fiber of g. Then we conclude from $(***)$ that $(N' + Z) \cdot l = (N_0 + Z_0) \cdot l = 0$ and thus

$$N_0 \cdot l = Z_0 \cdot l = 0.$$

So N_0 is contained in fibers of g and $Z_0 = f^*(\mathcal{O}_Y(a))$, [workshop, 2.11]; moreover, $a \geq 0$. Comparing with $(***)$ and using the fact that $\sum a_i G_i$ does not contain the support of a full fiber, A must be nef. Hence $(*)$ gives $\kappa(X) \geq 0$.

It remains to treat the case $\kappa(F) = 1$. Then we can use the relative Iitaka fibration of f and obtain a birational model $\hat{X}$ of X such that the induced map $\hat{f} : \hat{X} \to Y$ factors as $\hat{f} = h \circ g$, where $g|\hat{X}_y$ is the Iitaka fibration of $\hat{X}_y$, so that the general fiber F_g of g has $\kappa(F_g) = 0$. Now we conclude by Proposition 9.7. $\square$

9.5. Proposition. *Let X be a smooth projective 4-fold with K_X pseudo-effective. If $p(K_X) = 3$, then $\kappa(X) \geq 0$.*

Proof. Here any reduction is a (possibly meromorphic) elliptic fibration. We choose a holomorphic birational model $f : X \longrightarrow Y$ (with X and Y smooth), such that:

(a) f is smooth over Y_0 and $Y \setminus Y_0$ is a divisor with simple normal crossings only.
(b) The j-function extends to a holomorphic map $J : Y \longrightarrow \mathbb{P}_1$.

By the first property, $f_*(K_X)$ is locally free [Ko86], and we obtain the well-known formula of $\mathbb{Q}$-divisors

$$(*) \qquad K_X = f^*(K_Y + \Delta) + E - G.$$

Here E is an effective divisor such that $f_*(\mathcal{O}_X(E)) = \mathcal{O}_Y$ and G is an effective divisor such that $\dim f(G) \leq 1$. Moreover,

$$\Delta = \Delta_1 + \Delta_2$$

with

$$\Delta_1 = \sum (1 - \frac{1}{m_i}) F_i + \sum a_k$$

and

$$\Delta_2 \sim \frac{1}{12} J^*(\mathcal{O}(1)).$$

Here F_i are the components over which we have multiple fibers and D_k are the other divisor components over which there are singular fibers. The $a_k \in \frac{1}{12}\mathbb{N}$ according to Kodaira's list. Then by a general choice of the divisor Δ_2, the pair $(Y, \Delta_1 + \Delta_2)$ is klt. Now $K_Y + \Delta$ is pseudo-effective. In fact, by Theorem 2.2, it suffices to show that $(K_Y + \Delta) \cdot B_s \geq 0$ for every covering family (B_s) of curves. But this is checked very easily by restricting to $f^{-1}(C_t)$. Hence, the log Minimal Model Program [Ko92] in dimension 3 implies that $K_Y + \Delta$ is effective. If $G = 0$, then we could conclude immediately $\kappa(X) \geq 0$ by (*). In general we argue as follows. Consider the divisorial Zariski decomposition

$$K_X = f^*(K_Y + \Delta) + E - G = N + Z$$

with Z nef in codimension 1 and N the "exceptional" part. Using the log minimal model of (Y, Δ) we can write

$$K_Y + \Delta = N' + Z',$$

where Z' is the movable part and N' the fixed part; this is automatically the divisorial Zariski decomposition of $K_Y + \Delta$. Since $f^*(Z')$ might not be nef in codimension 1 due to the large fibers of f, we consider the decomposition

$$f^*(Z') = N_0 + Z_0$$

into the movable part Z_0 and the fixed part N_0 which is again the divisorial Zariski decomposition. Notice that N', E, N_0, Z_0 and G are $\mathbb{Q}$-divisors. We obtain

$$f^*(N') + N_0 + E + Z_0 = N + G + Z.$$

Then $f^*(N') + N_0 + E$ is the exceptional part of the left hand side while $N + G$ is the exceptional part of the right hand side. Thus $Z = Z_0$, so that Z is an effective $\mathbb{Q}$-divisor. Then also N is an effective $\mathbb{Q}$-divisor, and we conclude. □

9.6. Proposition. *Let X be a smooth projective 4-fold with K_X pseudo-effective. If $p(K_X) = 2$, then $\kappa(X) \geq 0$.*

Proof. Again we may assume that we have a holomorphic relative nef reduction $f : X \to Y$ with Y smooth and that f is smooth over Y_0 and $Y \setminus Y_0$ is a divisor with simple normal crossings only. By Propositions 8.10 and 8.11, $\kappa(F) = 0$ for the general fiber F of f. As in Proposition 9.5, we can write - possibly after birational transformation -

$$K_X = f^*(K_Y + \Delta) + E - G$$

with E effective, $f_*(\mathcal{O}_X(E)) = \mathcal{O}_Y$, G consisting of three-dimensional fiber components and (Y, Δ) klt. This is proved in [Am04], proof of (5.1). Then we can proceed as in Proposition 9.5. □

The arguments of Proposition 9.6 actually show the following.

9.7. Proposition. *Let X be a smooth projective 4-fold with K_X pseudo-effective. Suppose X has a holomorphic surjective map (with connected fibers) $X \to Y$ whose general fiber F has $\kappa(F) = 0$. Then $\kappa(X) \geq 0$.*

Proposition 9.4 could have been proved in the same way as Proposition 9.6; but maybe the ad hoc proof given above is instructive. The case $p(K_X) = 0$ is partially settled by Theorem 8.6 together with Proposition 3.11:

9.8. Theorem. *Let X be a projective manifold such that K_X is pseudo-effective. If there is a strongly connecting family (C_t) such that $K_X \cdot C_t = 0$, then $\kappa(X) = 0$.*

9.9. Remark. Let X be a projective manifold of dimension n and (C_t) a connecting family which is not strongly connecting. Assume $L \cdot C_t =$ for some pseudo-effective line bundle L on X. One may wonder whether L is numerically equivalent to an effective divisor. Let us concentrate on the case $n = 4$ and $L = K_X$. Fix a general curve C_t, consider the possibly non-compact subspace of X filled up by the chains of irreducible C_s meeting C_t and take closure. If this subspace is reducible, pick some irreducible component, say Y. Since we assume the family not to be strongly connecting, $Y \neq X$, so Y is either a surface or a divisor. It is not difficult to see (using Example 8.8) that

$$\mathrm{nd}(K_X|Y) = 0.$$

Let us assume that $\dim Y = 3$. Varying the curve C_t, we obtain a family (Y_s) of divisors. We may assume that the parameter space S of the family is 1-dimensional. If it is not connecting, we obtain a holomorphic map $f : X \to S$. Let F be the general fiber; then $\mathrm{nd}(K_F) = 0$ and by abundance for threefolds, $\kappa(F) = 0$. Hence, we conclude by Proposition 9.5.

If the family is connecting, we have to pass to the graph: $p : \mathcal{C} \to X$ with projection $q : \mathcal{C} \to S$. In that case, it might happen that $\mathrm{nd}(p^*(K_X)) = 1$ and it seems likely that the part of $p^*(K_X)$ which is nef in codimension 1 comes from S, hence is effective. Then we obtain $\kappa(X) \geq 0$. Details are left to a future paper.

9.10. Definition. Let X be a projective manifold and (C_t) a covering family of curves. Then (C_t) is a good covering family if it is either non-connecting or strongly connecting.

Using this notation, we may summarize our results as follows.

9.11. Theorem. *Let X be a smooth projective 4-fold (or a normal projective 4-fold with only canonical singularities). If K_X is pseudo-effective and* *if there is a good covering family (C_t) of curves such that $K_X \cdot C_t = 0$, then $\kappa(X) \geq 0$.*

9.12. Remark. Let X be a projective manifold of dimension n.

(1) Suppose $1 \leq \kappa(X) \leq n-1$. If X has a good minimal model via contractions and flips, then X carries a covering non-connecting family of curves. If we require only that X has a good minimal model, then at least some blow-up of X carries a covering non-connecting family of curves.

(2) Suppose $\kappa(X) = 0$. If X has a good minimal model via contractions and flips, then X carries a strongly connecting family of curves. Again, if we require only that X has a good minimal model, then at least some blow-up of X carries a covering non-connecting family of curves.

(3) This shows that the program of proving Problem (B) via (B1) and (B2), in the version with good connecting families, is really meaningful, i.e., we do not try to prove too much.

The remaining task is essentially to consider 4-folds X with $K_X \cdot C_t > 0$ for all good covering families (C_t) (but a priori it might happen that $K_X \cdot C_t = 0$ for a connecting, but not strongly connecting family (C_t). In that case one expects that X is of general type. It is easy to see that every proper subvariety S of X passing through a very general point of X is of general type, i.e. its desingularisation is of general type; see Proposition 9.13 below. But it is not at all clear whether $K_X|S$ is big, which is of course still not enough to conclude.

9.13. Proposition. *Let X be a smooth projective 4-fold with $p(K_X) = 0$. Then every proper subvariety $S \subset X$ through a very general point of X is of general type.*

Proof. Supposing the contrary, we find a covering family (S_t) of subvarieties such that the general S_t, hence every S_t, is not of general type. Consider the desingularised graph $p : \mathcal{C} \to X$ of this family; by passing to a subfamily we may assume p generically finite. Denoting $q : \mathcal{C} \to T$ the parametrising projection, the general fiber $\hat{S}_t$ is a smooth variety of dimension at most 3 and not of general type. We have an equation

$$K_{\mathcal{C}} = p^*(K_X) + E$$

with an effective divisor E. Notice that E must dominate T, otherwise q induces (up to finite étale cover of X) an almost holomorphic map $X \dashrightarrow T$ which would give rise to a covering non-connecting family (C_t) with $K_X \cdot C_t = 0$. By further blowing-up and using [KM92], we may assume that the general $\hat{S}_t$ dominates holomorphically a minimal model. We now easily find a covering family (C_s) of curves (sitting in q-fibers) such that $K_{\mathcal{C}} \cdot C_s = 0$, but $E \cdot C_s > 0$. Hence, $K_X \cdot p_*(C_s) < 0$; a contradiction. $\square$

Using the Iitaka fibration we obtain the following.

9.14. Proposition. *Let X be a smooth projective 4-fold with $p(K_X) = 0$. Then $\kappa(X) \neq 1, 2, 3$.*

10. Appendix: towards transcendental Morse inequalities

As already pointed out, for the general case of Conjecture 2.3, a transcendental version of the holomorphic Morse inequalities would be needed. The expected statements are contained in the following conjecture, which is also discussed in [Dem10].

10.1. Conjecture. *Let X be a compact complex manifold, and $n = \dim X$.*

(i) *Let α be a closed, $(1,1)$-form on X. We denote by $X(\alpha, \leq 1)$ the set of points $x \in X$ such that α_x has at most one negative eigenvalue. If $\int_{X(\alpha,\leq 1)} \alpha^n > 0$, the class $\{\alpha\}$ contains a Kähler current and*

$$\mathrm{Vol}(\alpha) \geq \int_{X(\alpha,\leq 1)} \alpha^n.$$

(ii) *Let $\{\alpha\}$ and $\{\beta\}$ be nef cohomology classes of type $(1,1)$ on X satisfying the inequality $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$. Then $\{\alpha - \beta\}$ contains a Kähler current and*

$$\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Remarks about the conjecture. If $\alpha = c_1(L)$ for some holomorphic line bundle L on X, then the inequality (i) was established in [Bou02a] as a consequence of the results of [Dem85]. In general, (ii) is a consequence of (i). In fact, if α and β are smooth positive definite $(1,1)$-forms and

$$\lambda_1 \geq \ldots \geq \lambda_n > 0$$

are the eigenvalues of β with respect to α, then $X(\alpha - \beta, \leq 1) = \{x \in X\,;\ \lambda_2(x) < 1\}$ and

$$\mathbf{1}_{X(\alpha-\beta,\leq 1)}(\alpha - \beta)^n = \mathbf{1}_{X(\alpha-\beta,\leq 1)}(1 - \lambda_1) \ldots (1 - \lambda_n) \geq 1 - (\lambda_1 + \ldots + \lambda_n)$$

everywhere on X. This is proved by an easy induction on n. An integration on X yields inequality (ii). In case α and β are just nef but not necessarily positive definite, we argue by considering $(\alpha + \varepsilon\omega) - (\beta + \varepsilon\omega)$ with a positive Hermitian form ω and $\varepsilon > 0$ small. $\square$

The full force of the conjecture is not needed here. First of all, we need only the case when X is compact Kähler. Let us consider a big class $\{\alpha\}$, and a sequence of Kähler currents $T_m \in \{\alpha\}$ with logarithmic poles, such that there exists a modification $\mu_m : X_m \mapsto X$, with the properties:

(10.2′) $\mu_m^* T_m = \beta_m + [E_m]$ where β_m is a semi-positive $(1,1)$-form, and E_m is an effective $\mathbb{Q}$-divisor on X_m;

(10.2″) $\mathrm{Vol}(\{\alpha\}) = \lim_{m\mapsto\infty} \int_X \beta_m^n$

(see Definition 3.2).

A first trivial observation is that the following uniform upper bound for $c_1(E_m)$ holds.

10.3. Lemma. *Let ω be a Kähler metric on X, such that $\{\omega-\alpha\}$ contains a smooth, positive representative. Then for each $m \in \mathbb{Z}_+$, the $(1,1)$-class $\mu_m^*\{\omega\} - c_1(E_m)$ on X_m is nef.*

Proof. If γ is a smooth positive representative in $\{\omega - \alpha\}$, then $\mu_m^\star\gamma + \beta_m$ is a smooth semi-positive representative of $\mu_m^*\{\omega\} - c_1(E_m)$. □

A second remark is that in order to prove the duality statement, Conjecture 2.3, for *projective* manifolds, it is enough to establish the estimate

$$(*) \qquad \mathrm{Vol}(\omega - A) \geq \int_X \omega^n - n\int_X \omega^{n-1} \wedge c_1(A)$$

where ω is a Kähler metric, and A is an ample line bundle on X. Indeed, if $\{\alpha\}$ is a big cohomology class, we use the above notation and we can write

$$\beta_m + tE_m = \beta_m + t\mu_m^* A - t(\mu_m^* A - E_m)$$

where A is an ample line bundle on X such that $c_1(A) - \{\alpha\}$ contains a smooth, positive representative. The arguments of the proof of Theorem 4.1 will give the orthogonality estimate, provided that we are able to establish $(*)$.

In this direction, we can get only a weaker statement with a suboptimal constant c_n.

10.4. Theorem (analogue of Lemma 4.2)**.** *Let X be a projective manifold of dimension n. Then there exists a constant c_n depending only on dimension (actually one can take $c_n = (n+1)^2/4$), such that the inequality*

$$\mathrm{Vol}(\omega - A) \geq \int_X \omega^n - c_n\int_X \omega^{n-1} \wedge c_1(A)$$

holds for every Kähler metric ω and every ample line bundle A on X.

Proof. Without loss of generality, we can assume that A is very ample (otherwise multiply ω and A by a large positive integer). Pick generic sections $\sigma_0, \sigma_1, \ldots, \sigma_n \in |A|$ so that one gets a finite map

$$F : X \to \mathbb{P}^n_{\mathbb{C}}, \qquad x \mapsto [\sigma_0(x) : \sigma_1(x) : \ldots : \sigma_n(x)].$$

We let $\theta = F^*\omega_{\mathrm{FS}} \in c_1(A)$ be the pull-back of the Fubini-Study metric on $\mathbb{P}^n_{\mathbb{C}}$ (in particular, $\theta \geq 0$ everywhere on X), and put

$$\psi = \log \frac{|\sigma_0|^2}{|\sigma_0|^2 + |\sigma_1|^2 + \ldots + |\sigma_n|^2}.$$

We also use the standard notation $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. Then

$$dd^c\psi = [H] - \theta$$

where H is the hyperplane section $\sigma_0 = 0$ and $[H]$ is the current of integration over H (for simplicity, we may further assume that H is smooth and reduced, although this is not required in what follows). The set $U_\varepsilon = \{\psi \leq 2\log\varepsilon\}$ is an ε-tubular neighborhood of H. Take a convex increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(t) = t$ for $t \geq 0$ and $\chi(t) =$ constant on some interval $]-\infty, t_0]$. We put $\psi_\varepsilon = \psi - 2\log\varepsilon$ and

$$\alpha_\varepsilon := dd^c\chi(\psi_\varepsilon) + \theta = (1 - \chi'(\psi_\varepsilon))\theta + \chi''(\psi_\varepsilon) d\psi_\varepsilon \wedge d^c\psi_\varepsilon \geq 0.$$

Thanks to our choice of χ, this is a smooth form with support in U_ε. In particular, we find

$$\int_{U_\varepsilon} \alpha_\varepsilon^n = \int_{U_\varepsilon} \alpha_\varepsilon \wedge \theta^{n-1} = \int_X \theta^n = c_1(A)^n.$$

It follows from these equalities that we have $\lim_{\varepsilon\to 0} \alpha_\varepsilon = [H]$ in the weak topology of currents. Now, for each choice of positive parameters ε, δ, we consider the Monge-Ampère equation

$$(\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = (1-\delta)\omega^n + \delta \frac{\int_X \omega^n}{c_1(A)^n}\, \alpha_\varepsilon^n. \tag{10.5}$$

By the theorem of S.-T. Yau [Yau78], there exists a smooth solution φ_ε, unique up to normalization by an additive constant, such that $\omega_\varepsilon := \omega + i\partial\overline{\partial}\varphi_\varepsilon > 0$. Since $\int_X \omega_\varepsilon \wedge \omega^{n-1} = \int_X \omega^n$ remains bounded, we can extract a weak limit T out of the family ω_ε ; then T is a closed positive current, and the arguments in [Bou02a] show that its absolutely continuous part satisfies

$$\int_X T_{ac}^n \geq (1-\delta) \int_X \omega^n.$$

We are going to use the same ideas as in [DPa04], in order to estimate the singularity of the current T on the hypersurface H. For this, we estimate the integral $\int_{U_\varepsilon} \omega_\varepsilon \wedge \theta^{n-1}$ on the tubular neighborhood U_ε of H. Let us denote by $\rho_1 \leq \ldots \leq \rho_n$ the eigenvalues of ω_ε with respect to α_ε, computed on the open set $U'_\varepsilon \subset U_\varepsilon$ where α_ε is positive definite. The Monge-Ampère equation (10.5) implies

$$\rho_1\rho_2\ldots\rho_n \geq \delta\frac{\int_X \omega^n}{c_1(A)^n}.$$

On the other hand, we find $\omega_\varepsilon \geq \rho_1\alpha_\varepsilon$ on U'_ε, hence

$$\int_{U_\varepsilon} \omega_\varepsilon \wedge \theta^{n-1} \geq \int_{U'_\varepsilon} \rho_1\alpha_\varepsilon \wedge \theta^{n-1} \geq \delta\frac{\int_X \omega^n}{c_1(A)^n}\int_{U'_\varepsilon} \frac{1}{\rho_2\ldots\rho_n}\alpha_\varepsilon \wedge \theta^{n-1}. \tag{10.6}$$

In order to estimate the last integral in the right hand side, we apply the Cauchy-Schwarz inequality to get

$$\Big(\int_{U'_\varepsilon} (\alpha_\varepsilon^n)^{1/2}(\alpha_\varepsilon\wedge\theta^{n-1})^{1/2}\Big)^2 \leq \int_{U'_\varepsilon} \rho_2\ldots\rho_n\alpha_\varepsilon^n \int_{U'_\varepsilon} \frac{1}{\rho_2\ldots\rho_n}\alpha_\varepsilon\wedge\theta^{n-1}. \tag{10.7}$$

By definition of the eigenvalues ρ_j, we have

$$\int_{U'_\varepsilon} \rho_2\ldots\rho_n\alpha_\varepsilon^n \leq n\int_X \omega_\varepsilon^{n-1}\wedge\alpha_\varepsilon = n\int_X \omega^{n-1}\wedge c_1(A). \tag{10.8}$$

On the other hand, an explicit calculation shows that

$$\begin{aligned}
&\alpha_\varepsilon^n \geq n(1-\chi'(\psi_\varepsilon))^{n-1}\chi''(\psi_\varepsilon)\, d\psi_\varepsilon \wedge d^c\psi_\varepsilon \wedge \theta^{n-1},\\
&\alpha_\varepsilon \wedge \theta^{n-1} \geq \chi''(\psi_\varepsilon)\, d\psi_\varepsilon \wedge d^c\psi_\varepsilon \wedge \theta^{n-1},
\end{aligned}$$

hence

$$\int_{U'_\varepsilon} (\alpha_\varepsilon^n)^{1/2}(\alpha_\varepsilon\wedge\theta^{n-1})^{1/2} \geq n^{1/2}\int_X (1-\chi'(\psi_\varepsilon))^{(n-1)/2}\chi''(\psi_\varepsilon)\, d\psi_\varepsilon\wedge d^c\psi_\varepsilon\wedge\theta^{n-1}$$

(we can integrate on X since the integrand is zero anyway outside U'_ε). Now, we have

$$\begin{aligned}
&\frac{n+1}{2}(1-\chi'(\psi_\varepsilon))^{(n-1)/2}\chi''(\psi_\varepsilon)\, d\psi_\varepsilon \wedge d^c\psi_\varepsilon\\
&\qquad = -d\Big((1-\chi'(\psi_\varepsilon))^{(n+1)/2}d^c\psi_\varepsilon\Big) + (1-\chi'(\psi_\varepsilon))^{(n+1)/2}dd^c\psi_\varepsilon\\
&\qquad = -d\Big((1-\chi'(\psi_\varepsilon))^{(n+1)/2}d^c\psi_\varepsilon\Big) + [H] - (1-\chi'(\psi_\varepsilon))^{(n+1)/2}\theta
\end{aligned}$$

and from this we infer

$$\begin{aligned}
&\frac{n+1}{2}\int_X (1-\chi'(\psi_\varepsilon))^{(n-1)/2}\chi''(\psi_\varepsilon)\, d\psi_\varepsilon \wedge d^c\psi_\varepsilon \wedge \theta^{n-1}\\
&\qquad = \int_X [H]\wedge\theta^{n-1} - \int_X (1-\chi'(\psi_\varepsilon))^{(n+1)/2}\theta^n\\
&\qquad \to c_1(A)^n \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

We thus obtain

$$(10.9)\qquad \int_{U'_\varepsilon} (\alpha_\varepsilon^n)^{1/2}(\alpha_\varepsilon \wedge \theta^{n-1})^{1/2} \geq \frac{2\sqrt{n}}{n+1} c_1(A)^n - o(1) \qquad \text{as } \varepsilon \to 0.$$

The reader will notice, and this looks at first a bit surprising, that the final lower bound does not depend at all on the choice of χ. This seems to indicate that our estimates are essentially optimal and will be hard to improve. Putting together (10.7), (10.8) and (10.9) we find the lower bound

$$(10.10)\qquad \int_{U'_\varepsilon} \frac{1}{\rho_2 \dots \rho_n} \alpha_\varepsilon \wedge \theta^{n-1} \geq \frac{4\delta}{(n+1)^2} \frac{(c_1(A)^n)^2}{\int_X \omega^{n-1} \wedge c_1(A)} - o(1).$$

Finally, (10.6) and (10.10) yield

$$\int_{U_\varepsilon} \omega_\varepsilon \wedge \theta^{n-1} \geq \frac{4\delta}{(n+1)^2} \frac{\int_X \omega^n}{\int_X \omega^{n-1} \wedge c_1(A)} c_1(A)^n - o(1).$$

As $\bigcap U_\varepsilon = H$, the standard support theorems for currents imply that the weak limit $T = \lim \omega_\varepsilon$ carries a divisorial component $c[H]$ with

$$\int_X c[H] \wedge \theta^{n-1} \geq \frac{4\delta}{(n+1)^2} \frac{\int_X \omega^n}{\int_X \omega^{n-1} \wedge c_1(A)} c_1(A)^n.$$

Therefore, as $[H] \equiv \theta \in c_1(A)$, we infer

$$c \geq \frac{4\delta}{(n+1)^2} \frac{\int_X \omega^n}{\int_X \omega^{n-1} \wedge c_1(A)}.$$

The difference $T - c[H]$ is still a positive current and has the same absolutely continuous part as T. Hence,

$$\mathrm{Vol}(T - c[H]) \geq \int_X T_{ac}^n \geq (1-\delta) \int_X \omega^n.$$

The specific choice

$$\delta = \frac{(n+1)^2}{4} \frac{\int_X \omega^{n-1} \wedge c_1(A)}{\int_X \omega^n}$$

gives $c \geq 1$, hence

$$\mathrm{Vol}(T - [H]) \geq \int_X \omega^n - \frac{(n+1)^2}{4} \int_X \omega^{n-1} \wedge c_1(A).$$

Theorem 10.4 follows from this estimate. □

10.11. Remark. By using similar methods, we could also obtain an estimate for the volume of the difference of two Kähler classes on a general compact Kähler manifold, by using the technique of concentrating the mass on the diagonal of $X \times X$ (see [DPa04]). However, the constant c implied by this technique also depends on the curvature of the tangent bundle of X.

We show below that the answer to Conjecture 10.1 is positive at least when X is a compact hyperkähler manifold (= compact irreducible holomorphic symplectic manifold). The same proof would work for a compact Kähler manifold which is a limit by deformation of projective manifolds with Picard number $\rho = h^{1,1}$.

10.12. Theorem. *Let X be a compact hyperkähler manifold, and let α be a closed, $(1,1)$-form on X. Then we have*

$$\mathrm{Vol}(\alpha) \geq \int_{X(\alpha,\leq 1)} \alpha^n.$$

Proof. We follow closely the approach of D. Huybrechts in [Huy98]. Consider $\mathcal{X} \mapsto \mathrm{Def}(X)$ the universal deformation of X, such that $\mathcal{X}_0 = X$. If $\beta \in H^2(X,\mathbb{R})$ is a real cohomology class, then we denote by S_β the set of points $t \in \mathrm{Def}(X)$ such that the restriction $\beta_{|\mathcal{X}_t}$ is of $(1,1)$-type.

Next, we take a sequence of rational classes $\{\alpha_k\} \in H^2(X,\mathbb{Q})$, such that $\alpha_k \to \alpha$ on $\mathcal{X}$ as $k \mapsto \infty$. As $\{\alpha_k\} \to \{\alpha\}$, the hypersurface S_{α_k} converge to S_α; in particular, we can take $t_k \in S_{\alpha_k}$ such that $t_k \to 0$. In this way, the rational $(1,1)$-forms $\alpha_{k|\mathcal{X}_{t_k}}$ will converge to our form α on X.

We have

$$\begin{aligned}\mathrm{Vol}(\alpha) &\geq \limsup_{k\mapsto\infty} \mathrm{Vol}(\alpha_{k|\mathcal{X}_{t_k}}) \\ &\geq \limsup_{k\mapsto\infty} \int_{\mathcal{X}_{t_k}(\alpha_{t_k},\leq 1)} \alpha_{t_k}^n \\ &= \int_{X(\alpha,\leq 1)} \alpha^n\end{aligned}$$

where the first inequality is a consequence of the semi-continuity of the volume obtained in [Bou02b], and the second one is a consequence of the convergence statement above.

10.13. Corollary. *If X is a compact hyperkähler manifold, or more generally, a limit by deformation of projective manifolds with Picard number $\rho = h^{1,1}$, then the cones $\mathcal{E}$ and $\mathcal{M}$ are dual.*

11. Concluding remarks

We would like to conclude with some new developments and applications of the main theorem. For a general overview article we refer to [Dem07] and also to the Bourbaki talk by Debarre [Deb06].

(1) In the paper [BFJ09], Boucksom, Favre and Jonsson consider the volume function on the big cone of a projective manifold X of dimension n. They showed (Theorem A) that the volume function vol is differentible of class $\mathcal{C}^1$ and computed the derivative. Namely, given a big class α and $\gamma \in N^1(X)$, then, using the notation of section 3,

$$\left(\frac{d}{dt}\right)_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle\alpha^{n-1}\rangle \cdot \gamma.$$

In particular, setting $\gamma = \alpha$, one obtains

$$\langle\alpha^n\rangle = \langle\alpha^{n-1}\rangle \cdot \alpha,$$

which implies Corollary 4.5.

(2) Hacon-McKernan [HM07] showed, solving a conjecture of Shokurov, that given a divisorial log terminal pair (X, Δ) and a birational map $f : Y \to X$, then all fibers $f^{-1}(x)$ are rationally chain connected. A crucial point in the proof is to check that the images of rational maps from fibers of f are uniruled. This is done using the uniruledness criterion Theorem 2.6.

(3) As indicated in section 2, given a surjective map

$$(\Omega^1_X)^{\otimes m}_X \to Q \to 0,$$

then by [CP09], $\det Q$ is pseudo-effective unless X is uniruled. This generalises a theorem of Miyaoka and again Theorem 2.6 is indispensable.

At this point we add a remark concerning general complete intersection curves. For many purposes, e.g., in connection with stability, when one wants to use the theorem of Mehta-Ramanathan, those curves play a central role. The movable cone was defined as the closed cone of curves generated by complete intersection curves on birational models of the variety X which dominate X holomorphically. One might ask whether the movable cone actually agrees with the complete intersection on X, i.e., with the cone generated by curves cut out by sufficiently high multiples of ample divisors. Very unfortunately however, these curves do not coincide in general. Some example was already exhibited in [DPS96], Example 4.8 (and then forgotten, so that we worked also an example of a $\mathbb{P}_1$-bundle over $\mathbb{P}_2$).

(4) As already mentioned, Theorem 2.6 in the Kähler case is wide open. However in dimension 3, Brunella [Br06] was able to establish the uniruledness of a (non-algebraic) Kähler manifold X with K_X not nef. In fact, if X is a non-algebraic Kähler threefold, then X carries a holomorphic 2-form which can be seen as a foliation $\mathcal{F} = K_X \subset T_X$, possible with singularities in codimension 1. Now, K_X being not pseudo-effective, then the canonical bundle of the foliation (by curves) $\mathcal{F}$ is not pseudo-effective. In this situation, Brunella shows that $\mathcal{F}$ is a foliation by *rational* curves (actually in any dimension).

(5) There is also a version of Theorem 2.6 for the cotangent bundle. In fact, fix an ample line bundle A and suppose that

$$H^0(X, ((\Omega^1_X)^{\otimes m} \otimes A)^{\otimes N}) = 0$$

for m, N sufficiently large, then X is rationally connected. A conjecture of Mumford says that one can actually omit the ample line bundle A. See [Pe06] for details; the proof uses once more Theorem 2.6 to get uniruledness.

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