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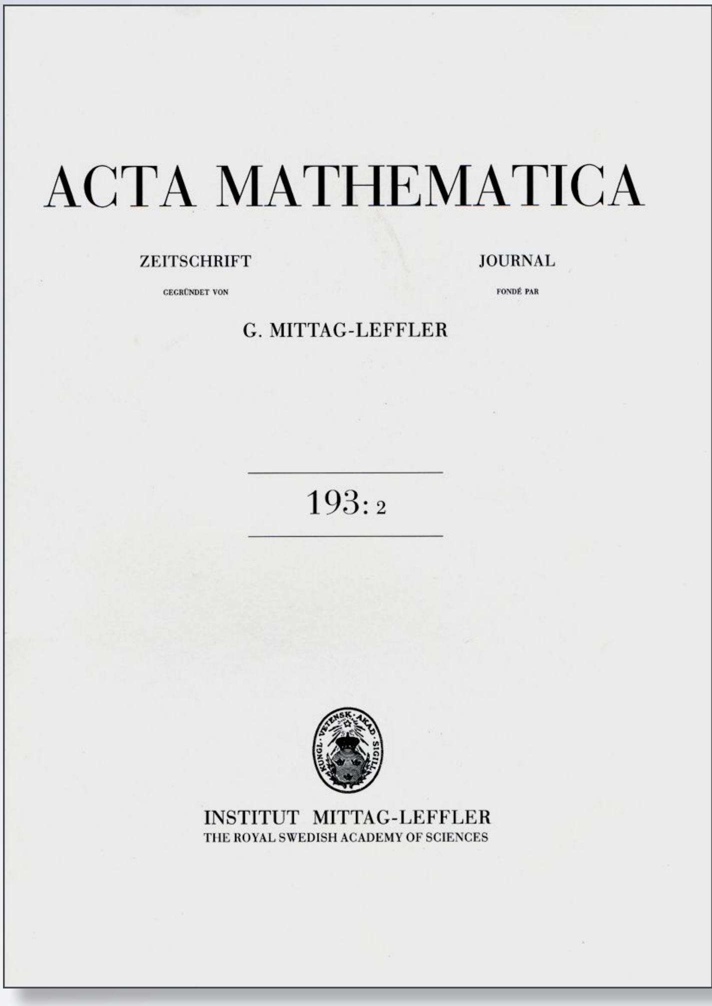

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Fekete points and convergence towards equilibrium measures on complex manifolds

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Introduction

The setting

Let L be a holomorphic line bundle on a compact complex manifold X of complex dimension n. Let (K, ϕ) be a *weighted compact subset*, i.e. K is a non-pluripolar compact subset K of X and ϕ is (the weight of) a continuous Hermitian metric on L. If s is a section of $kL:=L^{\otimes k}$, we denote the corresponding pointwise length function by

$$|s|_{k\phi} = |s|e^{-k\phi}.$$

We refer to [BB1, §2.1 and §2.2] for more details on the terminology and notation. Finally let μ be a probability measure on K.

The asymptotic study as $k \to \infty$ of the spaces of global sections $s \in H^0(X, kL)$ endowed with either the L^2-norm

$$\|s\|^2_{L^2(\mu,k\phi)} := \int_X |s|^2_{k\phi}\, d\mu$$

or the L^∞-norm

$$\|s\|_{L^\infty(K,k\phi)} := \sup_K |s|_{k\phi}$$

is a natural generalization of the classical theory of orthogonal polynomials. The latter indeed corresponds to the case

$$K \subset \mathbb{C}^n \subset \mathbb{P}^n =: X$$

equipped with the ample tautological line bundle $\mathcal{O}(1) =: L$. It is of course well known that $H^0(\mathbb{P}^n, \mathcal{O}(k))$ identifies with the space of polynomials on $\mathbb{C}^n$ of total degree at most k. The section of $\mathcal{O}(1)$ cutting out the hyperplane at infinity induces a flat Hermitian metric on L over $\mathbb{C}^n$, so that a continuous weight ϕ on $\mathcal{O}(1)|_K$ is naturally identified with a function in $C^0(K)$. On the other hand, a plurisubharmonic function on $\mathbb{C}^n$ with at most logarithmic growth at infinity gets identified with the weight ϕ of a non-negatively curved (singular) Hermitian metric on $\mathcal{O}(1)$. In the general case we will say that a weight ϕ on L is *plurisubharmonic* (*psh*, for short) if the associated (singular) Hermitian metric is non-negatively curved (in the sense of currents).

Our geometric setting is therefore seen to be a natural (and more symmetric) extension of the so-called *weighted potential theory* (cf. [ST] and in particular Bloom's appendix therein). It also contains the case of *spherical polynomials* on the round sphere $S^n \subset \mathbb{R}^{n+1}$, as studied e.g. in [M], [MO] and [SW] (we are grateful to N. Levenberg for having pointed this out to us). Indeed, the space of spherical polynomials of total degree at most k is by definition the image by restriction to S^n of the space of all polynomials on $\mathbb{R}^{n+1}$ of degree at most k. It thus coincides with (the real points of) $H^0(X, kL)$ with X being the smooth quadric hypersurface

$$\{[X_0 : X_1 : \ldots : X_n] : X_1^2 + \ldots + X_n^2 = X_0^2\} \subset \mathbb{P}^{n+1}$$

endowed with the ample line bundle $L := \mathcal{O}(1)|_X$. Here we take $K := S^n = X(\mathbb{R})$, and the section cutting out the hyperplane at infinity again identifies weights on L with certain functions on the affine piece of X.

In view of the above dictionary, one is naturally led to introduce the *equilibrium weight* of (K, ϕ) as

$$\phi_K := \sup\{\psi \text{ psh weight on } L : \psi \leqslant \phi \text{ on } K\}, \tag{0.1}$$

whose upper semi-continuous (usc, for short) regularization ϕ_K^* is a psh weight on L since K is non-pluripolar (cf. §1.1).

The *equilibrium measure* of (K,ϕ) is then defined as the Monge–Ampère measure of ϕ_K^* normalized to unit mass:

$$\mu_{\mathrm{eq}}(K,\phi) := V^{-1}\,\mathrm{MA}(\phi_K^*), \quad \text{with } V := \int_X \mathrm{MA}(\phi_K^*).$$

This measure is concentrated on K and $\phi=\phi_K^*$ holds $\mu_{\mathrm{eq}}(K,\phi)$-a.e.

This approach is least technical when L is *ample*, but the natural setting appears to be the more general case of a *big* line bundle, which is the one considered in the present paper, following our preceding work [BB1]. As was shown there, the Monge–Ampère measure $\mathrm{MA}(\psi)$ of a psh weight ψ with minimal singularities, defined as the Beford–Taylor top-power $(dd^c\psi)^n$ of the curvature $dd^c\psi$ on its bounded locus, is well behaved. Its total mass V is in particular an invariant of the big line bundle L, and in fact coincides with the *volume* $\mathrm{vol}(L)$, characterized by

$$N_k := \dim H^0(kL) = \mathrm{vol}(L)\frac{k^n}{n!} + o(k^n).$$

Note that the case of a big line bundle covers in particular the case where X is allowed to be singular, since the pull-back of a big line bundle to a resolution of singularities remains big.

The main goal of the present paper is to give a general criterion involving spaces of global sections that ensures convergence of certain sequences of probability measures on K of Bergman-type towards the equilibrium measure $\mu_{\mathrm{eq}}(K,\phi)$.

Fekete configurations

Let (K,ϕ) be a weighted compact subset as above. A *Fekete configuration* is a finite subset of points in K maximizing the determinant in the interpolation problem. More precisely, let $N := \dim H^0(L)$ and

$$P = (x_1, ..., x_N) \in K^N$$

be a configuration of points in the given compact subset K. Then P is said to be a Fekete configuration for (K,ϕ) if it maximizes the determinant of the evaluation operator

$$\mathrm{ev}_P \colon H^0(L) \longrightarrow \bigoplus_{j=1}^N L_{x_j} \tag{0.2}$$

with respect to a given basis $s_1, ..., s_N$ of $H^0(L)$, i.e. the Vandermonde-type determinant

$$|\det(s_i(x_j))|e^{-(\phi(x_1)+...+\phi(x_N))}.$$

This condition is independent of the choice of the basis $(s_i)_{i=1}^N$.

For each configuration $P=(x_1, ..., x_N)\in X^N$ we let

$$\delta_P := \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$$

be the averaging measure along P. Our first main result is an equidistribution result for Fekete configurations.

Theorem A. *Let (X, L) be a compact complex manifold equipped with a big line bundle. Let K be a non-pluripolar compact subset of X and ϕ be a continuous weight on L. For each k let $P_k\in K^{N_k}$ be a Fekete configuration for $(K, k\phi)$. Then the sequence P_k equidistributes towards the equilibrium measure as $k\to\infty$, that is*

$$\lim_{k\to\infty} \delta_{P_k} = \mu_{\mathrm{eq}}(K, \phi)$$

holds in the weak topology of measures.

Theorem A first appeared in the preprint [BB2] by Berman–Boucksom. It will be obtained here as a consequence of a more general convergence result (Theorem C below).

In $\mathbb{C}$ this result is well known (cf. [ST] for a modern reference and [Dei] for the relation to Hermitian random matrices). In $\mathbb{C}^n$ this result has been conjectured for quite some time, probably going back to the pioneering work of Leja in the late 1950s. See for instance Levenberg's survey on approximation theory in $\mathbb{C}^n$ [L, p. 120] and Bloom's appendix to [ST].

As explained above, the spherical polynomials situation corresponds to the round sphere S^n embedded in its complexification X, the complex quadric hypersurface in $\mathbb{P}^{n+1}$ (with L being the restriction of $\mathcal{O}(1)$ to X). This special case of Theorem A thus yields the following result.

Corollary A. *Let $K\subset S^n$ be a compact subset of the round n-sphere and assume that K is non-pluripolar in the complexification of S^n. For each k let $P_k\in K^{N_k}$ be a Fekete configuration of degree k for K (also called* extremal fundamental system *in this setting). Then δ_{P_k} converges to the equilibrium measure $\mu_{\mathrm{eq}}(K)$ of K.*

This is a generalization of the recent result of Morza and Ortega-Cerdà [MO] on equidistribution of Fekete points on the sphere. Their result corresponds to the case $K=S^n$ whose equilibrium measure $\mu_{\mathrm{eq}}(S^n)$ coincides with the rotation invariant probability measure on S^n for symmetry reasons.

Bernstein–Markov measures

Let as before (K,ϕ) be a weighted compact subset and let μ be a probability measure on K. The distortion between the natural L^2 and L^∞ norms on $H^0(L)$ introduced above is locally accounted for by the *distortion function* $\varrho(\mu,\phi)$, whose value at $x\in X$ is defined by

$$\varrho(\mu,\phi)(x)=\sup_{\|s\|_{L^2(\mu,\phi)}=1}|s(x)|_\phi^2, \tag{0.3}$$

the squared norm of the evaluation operator at x.

The function $\varrho(\mu,\phi)$ is known as the *Christoffel–Darboux function* in the orthogonal polynomials literature and may also be represented as

$$\varrho(\mu,\phi)=\sum_{i=1}^N|s_i|_\phi^2 \tag{0.4}$$

in terms of any given orthonormal basis $(s_i)_{i=1}^N$ for $H^0(L)$ with respect to the L^2-norm induced by (μ,ϕ). In this latter form, it sometimes also appears under the name *density of states function.* Integrating (0.4) over X shows that the corresponding *probability* measure

$$\beta(\mu,\phi):=\frac{1}{N}\varrho(\mu,\phi)\mu, \tag{0.5}$$

which will be referred to as the *Bergman measure*, can indeed be interpreted as a dimensional density for $H^0(L)$.

When μ is a smooth positive volume form on X and ϕ is smooth and strictly psh, the celebrated Bouche–Catlin–Tian–Zelditch theorem ([Bou], [C], [T], [Z]) asserts that $\beta(\mu,k\phi)$ admits a full asymptotic expansion in the space of smooth volume forms as $k\to\infty$, with $V^{-1}(dd^c\phi)^n$ as the dominant term.

As was shown by Berman (in [B1] for the $\mathbb{P}^n$ case and in [B2] for the general case), part of this result still holds when μ is a smooth positive volume form and ϕ is smooth but without any a priori curvature sign. More specifically, the norm distortion still satisfies

$$\sup_X\varrho(\mu,k\phi)=O(k^n) \tag{0.6}$$

and the Bergman measures still converge towards the equilibrium measure:

$$\lim_{k\to\infty}\beta(\mu,k\phi)=\mu_{\mathrm{eq}}(X,\phi) \tag{0.7}$$

now in the weak topology of measures.

Both of these results fail when K, μ and ϕ are more general. However *sub-exponential* growth of the distortion between $L^2(\mu, k\phi)$ and $L^\infty(K, k\phi)$ norms, that is

$$\sup_K \varrho(\mu, k\phi) = O(e^{\varepsilon k}) \quad \text{for all } \varepsilon > 0, \tag{0.8}$$

appears to be a much more robust condition. Following standard terminology (cf. [NZ] and [L, p. 120]), we will say that the measure μ is *Bernstein–Markov* for (K, ϕ) when (0.8) holds.

When $K = X$ any measure dominating Lebesgue measure is Bernstein–Markov for (X, ϕ) by the mean-value inequality. In §1.2 we give more generally a characterization of a stronger Bernstein–Markov property, with respect to psh weights instead of holomorphic sections, generalizing classical results of Nguyen–Zeriahi [NZ] and Siciak [S]. The result shows in particular that Bernstein–Markov measures for (K, ϕ) always exist when (K, ϕ) is *regular* in the sense of pluripotential theory, i.e. when ϕ_K is usc. Regularity holds for instance when K is a smoothly bounded domain in X.

Our second main result asserts that the convergence of Bergman measures to the equilibrium measure as in (0.7) holds for arbitrary Bernstein–Markov measures.

Theorem B. *Let (X, L) be a compact complex manifold equipped with a big line bundle. Let K be a non-pluripolar compact subset of X and ϕ be a continuous weight on L. Let μ be a Bernstein–Markov measure for (K, ϕ). Then*

$$\lim_{k \to \infty} \beta(\mu, k\phi) = \mu_{\text{eq}}(K, \phi)$$

holds in the weak topology of measures.

In the classical one-variable setting, this theorem was obtained using completely different methods by Bloom and Levenberg [BL2], who also conjectured the several variable case in [BL3]. A slightly less general version of Theorem B (dealing only with *stably* Bernstein–Markov measures) was first obtained by Berman–Witt Nyström in the preprint [BW]. Theorem B will here be obtained as a special case of Theorem C below.

Donaldson's $\mathcal{L}$-functionals and a general convergence criterion

We now state our third main result, which is a general criterion ensuring convergence of Bergman measures to equilibrium in terms of $\mathcal{L}$-functionals, first introduced by Donaldson [D1], [D2]. This final result actually implies Theorems A and B above, as well as a convergence result for so-called *optimal measures* first obtained in [BBLW] by reducing the result to the preprint [BB2].

The L^2 and L^∞ norms on $H^0(kL)$ introduced above are described geometrically by their unit balls, which will be denoted respectively by

$$\mathcal{B}^\infty(K, k\phi) \subset \mathcal{B}^2(\mu, k\phi) \subset H^0(kL).$$

We fix a reference weighted compact subset (K_0, ϕ_0) and a probability measure μ_0 on K_0 which is Bernstein–Markov with respect to (K_0, ϕ_0). This data should be taken to be the Haar measure of the compact unit torus endowed with the standard flat weight in the $\mathbb{C}^n$ case. We can then normalize the Haar measure vol on $H^0(kL)$ by

$$\operatorname{vol} \mathcal{B}^2(K_0, k\phi_0) = 1,$$

and we introduce the following slight variants of Donaldson's $\mathcal{L}$-functional [D1]

$$\mathcal{L}_k(\mu, \phi) := \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^2(\mu, k\phi)$$

and

$$\mathcal{L}_k(K, \phi) := \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^\infty(K, k\phi).$$

By [BB1, Theorem A], we have

$$\lim_{k \to \infty} \mathcal{L}_k(K, \phi) = \mathcal{E}_{\mathrm{eq}}(K, \phi), \tag{0.9}$$

where

$$\mathcal{E}_{\mathrm{eq}}(K, \phi) := \frac{1}{M} \mathcal{E}(\phi_K^*)$$

denotes the *energy at equilibrium* of (K, ϕ) (with respect to (K_0, ϕ_0)) and $\mathcal{E}(\psi)$ stands for the *Monge–Ampère energy* of a psh weight ψ with minimal singularities, characterized as the primitive of the Monge–Ampère operator:

$$\frac{d}{dt}\bigg|_{t=0_+} \mathcal{E}(t\psi_1 + (1-t)\psi_2) = \int_X (\psi_1 - \psi_2) \operatorname{MA}(\psi_2)$$

normalized by

$$\mathcal{E}(\phi_{0,K_0}^*) = 0.$$

Since $\mathcal{L}_k(\mu, \phi) \geqslant \mathcal{L}_k(K, \phi)$ for any probability measure μ on K, (0.9) shows in particular that the energy at equilibrium $\mathcal{E}_{\mathrm{eq}}(K, \phi)$ is an a priori asymptotic lower bound for $\mathcal{L}_k(\mu, \phi)$. Our final result describes the limiting distribution of the Bergman measures of asymptotically minimizing sequences.

Theorem C. *Let μ_k be a sequence of probability measures on K such that*

$$\lim_{k\to\infty} \mathcal{L}_k(\mu_k,\phi)=\mathcal{E}_{\mathrm{eq}}(K,\phi).$$

Then the associated Bergman measures satisfy

$$\lim_{k\to\infty} \beta(\mu_k,k\phi)=\mu_{\mathrm{eq}}(K,\phi)$$

in the weak topology of measures.

The condition bearing on the sequence $(\mu_k)_{k=1}^{\infty}$ in Theorem C is independent of the choice of the reference weighted compact subset (K_0,ϕ_0). In fact (0.9) shows that it can equivalently be written as the condition

$$\log\frac{\operatorname{vol}\mathcal{B}^2(\mu_k,k\phi)}{\operatorname{vol}\mathcal{B}^\infty(K,k\phi)}=o(kN_k),$$

which can be understood as a *weak Bernstein–Markov condition* on the sequence $(\mu_k)_{k=1}^{\infty}$, relative to (K,ϕ), cf. Lemma 3.2 below.

For measures of the form $\mu_k=\delta_{P_k}$ the weak Bernstein–Markov condition reads

$$\lim_{k\to\infty}\frac{1}{kN_k}\log|(\det S_k)(P_k)|_{k\phi}^{-1}=\mathcal{E}_{\mathrm{eq}}(K,\phi), \tag{0.10}$$

where S_k is an $L^2(\mu_0,k\phi)$-orthonormal basis for $H^0(kL)$, which thus means that the sequence of configurations $(P_k)_{k=1}^{\infty}$ is *asymptotically Fekete* for $(K,k\phi)$. In order to get Theorem A, we then use the simple fact that

$$\beta(\mu,\phi)=\mu \tag{0.11}$$

for measures μ of the form δ_P.

The proof of Theorem C is closely related to the generalization of Yuan's equidistribution theorem for algebraic points [Y] obtained in [BB1].

Applications to interpolation

Optimal configurations

Next, we will consider an application of Theorem C to a general *interpolation problem* for sections of kL. The problem may be formulated as follows: given a weighted set (K,ϕ), what is the distribution of N_k (nearly) optimal *interpolation nodes* on K for elements in $H^0(X,kL)$? Of course, for any generic configuration P_k the evaluation operator ev_{P_k}

in (0.2) is invertible and interpolation is thus possible. But the problem is to find the distribution of *optimal* interpolation nodes, in the sense that P_k minimizes a suitable operator norm of the interpolation operator $(\mathrm{ev}_{P_k})^{-1}$ over all configurations of N_k points in K.

We fix a weight ϕ on L. Given a measure μ on K we say that a configuration $P\in K^{N_k}$ is (p,q)-*optimal* for $1\leqslant q\leqslant\infty$ and $1\leqslant p<\infty$ (resp. $p=\infty$) if it minimizes the $L^p(\mu)$-$L^q(\delta_P)$ distortion

$$\sup_{s\in H^0(kL)} \frac{\|s\|_{L^p(\mu,k\phi)}}{\|s\|_{L^q(\delta_{P_k},k\phi)}} \tag{0.12}$$

(resp. the $L^\infty(K)$-$L^q(\delta_P)$ distortion). In the orthogonal polynomials literature, (∞,∞)-optimal configurations are usually called *Lebesgue points*, whereas $(\infty,2)$-optimal configurations are known as *Fejér points*.

In practice it is virtually impossible to find such optimal configurations numerically. But the next corollary gives necessary conditions for any sequence of configurations to have *sub-exponential* distortion and in particular to be optimal.

Corollary C. *Let μ be a Bernstein–Markov measure for the weighted compact set (K,ϕ) and let $1\leqslant p,q\leqslant\infty$. For any sequence $P_k\in K^{N_k}$ of (p,q)-optimal configurations the $L^\infty(K)$-$L^\infty(\delta_{P_k})$ distortion has subexponential growth in k, and the latter condition in turn implies that $(P_k)_{k=1}^\infty$ is asymptotically equilibrium distributed, i.e.*

$$\lim_{k\to\infty}\delta_{P_k}=\mu_{\mathrm{eq}}(K,\phi)$$

holds in the weak topology of measures.

For Lebesgue points, i.e. for $(p,q)=(\infty,\infty)$, the result was shown in [GMS] when K is a compact subset of the real line $\mathbb{R}\subset\mathbb{C}\subset\mathbb{P}^1=X$, and in [BL1] when X is a compact Riemann surface and $L=\mathcal{O}(1)|_X$ for a given projective embedding $X\subset\mathbb{P}^N$.

Remark 0.1. For a numerical study in the setting of Corollary A and with μ_0 being the invariant measure on S^2 see [SW], where the cases $(p,q)=(\infty,\infty)$ and $(p,q)=(2,2)$ are considered.

Optimal measures

Given a weighted subset (K,ϕ), the measures μ on K which minimize the $L^2(\mu,k\phi)$-$L^\infty(K,k\phi)$ distortion among all probability probability measures on K are called *optimal measures* (for $(K,k\phi)$) in [BBLW]. Such measures appear naturally in the context of *optimal experimental designs* (see [BBLW] and references therein).

It was shown in [BBLW] (by reducing to the convergence of Fekete configurations obtained in the preprint [BB2]) that any sequence of $(K, k\phi)$-optimal measures μ_k converges to $\mu_{\mathrm{eq}}(K,\phi)$ as $k\to\infty$. But optimal measures satisfy (0.11) and yield probability measures on K that minimize the functional $\mathcal{L}_k(\cdot\,,\phi)$—see [KW] and Proposition 2.9 in our setting. This latter property implies in turn that any sequence of $(K, k\phi)$-optimal measures μ_k is weakly Bernstein–Markov and the convergence $\mu_k\to\mu_{\mathrm{eq}}(K,\phi)$ follows by Theorem A.

Recursively extremal configurations

Finally, we will consider a recursive way of constructing configurations with certain extremal properties. Even if the precise construction seems to be new, it should be emphasized that it is inspired by the elegant algorithmic construction of determinantal random point processes in [HKPV].

Fix a weighted compact set (K,ϕ) and a probability measure μ on K. A configuration $P=(x_1, ..., x_N)$ will be said to be *recursively extremal* for (μ,ϕ) if it arises in the following way. Denote by $\mathcal{H}_N$ the corresponding Hilbert space $H^0(X,L)$ of dimension N. Take a pair (x_N, s_N) maximizing the pointwise norm $|s(x)|^2_\phi$ over all points x in the set K and sections s in the unit-sphere of $\mathcal{H}_N$. Next, replace $\mathcal{H}_N$ by the Hilbert space $\mathcal{H}_{N-1}$ of dimension $N-1$ obtained as the orthogonal complement of s_N in $\mathcal{H}_N$ and repeat the procedure to get a new pair (x_{N-1}, s_{N-1}), where now $s_{N-1}\in\mathcal{H}_{N-1}$. Continuing in this way gives a configuration $P:=(x_1, ..., x_N)$ after N steps.

Note that x_N may be equivalently obtained as a point maximizing the Bergman distortion function $\varrho(x)$ of $\mathcal{H}_N$ and so on. The main advantage of recursively extremal configurations over Fekete configurations is thus that they are obtained by maximizing functions defined on X and not on the space X^N of increasing dimension. This advantage should make them useful in numerical interpolation problems. We show that a sequence of recursively extremal configurations P_k is, in fact, *asymptotically Fekete* in the sense that (0.10) holds. As a direct consequence P_k is equilibrium distributed.

COROLLARY D. *Let μ be a Bernstein–Markov measure for the weighted set (K,ϕ) and $(P_k)_{k=1}^\infty$ be a sequence of configurations which are* recursively extremal *for $(\mu, k\phi)$. Then*

$$\lim_{k\to\infty} \delta_{P_k} = \mu_{\mathrm{eq}}(K,\phi)$$

in the weak topology of measures.

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1. Regular sets and Bernstein–Markov measures

Recall that L denotes a given big line bundle over a complex compact manifold X. The existence of a such a big line bundle on X is equivalent to X being *Moishezon*, i.e. bimeromorphic to a projective manifold, and X is then projective if and only if it is Kähler.

1.1. Pluripolar subsets and regularity

The goal of this section is to recall some preliminary results from [BB1] and to quickly explain how to adapt further results on equilibrium weights, that are standard in the classical situation, to our big line bundle setting. We refer to Klimek's book [K] and Demailly's survey [Dem] for details.

First recall that a subset A of X is said to be (*locally*) *pluripolar* if it is locally contained in the polar set of a local psh function. For a big line bundle L this is equivalent to the following global notion of pluripolarity (as was shown by Josefson in the $\mathbb{C}^n$-setting).

PROPOSITION 1.1. *If* $A\subset X$ *is* (*locally*) *pluripolar*, *then there exists a psh weight* ϕ *on* L *such that* $A\subset\{x:\phi(x)=-\infty\}$.

Proof. Since L is big, we can find a proper modification $\mu: X'\to X$ and an effective divisor E with $\mathbb{Q}$-coefficients such that μ^*L-E is ample (so that X' is in particular Kähler). By Guedj–Zeriahi's extension of Josefson's result to the Kähler situation [GZ], there exists a closed positive $(1,1)$-current T on X' which is cohomologous to μ^*L-E and whose polar set contains $\mu^{-1}(A)$. We can then find a psh weight ϕ on L such that $dd^c\phi=\mu_*(T+[E])$, and the polar set of ϕ contains A as desired. □

By definition, a *weighted compact subset* (K,ϕ) consists of a non-pluripolar compact set $K\subset X$ together with a continuous Hermitian metric $e^{-\phi}$ on $L|_K$. By the Tietze–Urysohn extension theorem, ϕ extends to a continuous weight on L over all of X. Now if E is an *arbitrary* subset of X and ϕ is a continuous weight on L (over all of X), we define the associated extremal function by

$$\phi_E := \sup\{\psi \text{ psh weight on } L : \psi \leqslant \phi \text{ on } E\}.$$

It is obvious that $\phi_E \equiv \infty$ on $X \setminus E$, when E is pluripolar, and a standard argument relying on Choquet's lemma shows conversely that ϕ_E^* is a psh weight when E is non-pluripolar (compare the proof of [GZ, Theorem 5.2]).

Using Proposition 1.1 one proves the following two useful facts exactly as in the classical setting (cf. for instance [K, p. 194]).

PROPOSITION 1.2. *Let ϕ be a continuous weight and let $E, A \subset X$ be two subsets with A pluripolar. Then we have $\phi_{E\cup A}^* = \phi_E^*$.*

COROLLARY 1.3. *If E is the increasing union of subsets E_j, then $\phi_{E_j}^*$ decreases pointwise to ϕ_E^* as $j \to \infty$.*

Adapting a classical notion to our setting, we introduce the following concept.

Definition 1.4. If E is a non-pluripolar subset of X and ϕ is a continuous weight, we say that (E, ϕ) is *regular* (or that E is regular with respect to ϕ) if and only if ϕ_E is upper semi-continuous.

As opposed to the classical case (cf. [Dem, Theorem 15.6]), we are unable to prove that ϕ_E is a priori lower semi-continuous when L is not ample, hence our definition (note that ϕ_E^* has a non-empty polar set in general in the big case, see [BB1, Remark 1.14] for a short discussion on this issue).

Note that ϕ_E is usc if and only if $\phi_E^* \leqslant \phi$ on E, that is if and only if the set of psh weights ψ such that $\psi \leqslant \phi$ on E admits a largest element.

Examples of non-regular sets are obtained by adding to a given subset a pluripolar one, in view of Proposition 1.2. Conversely, since ϕ is in particular usc, we see that X, or in fact any open subset of X, is regular with respect to ϕ. In order to get a less trivial class of regular sets recall that a compact subset $K \subset X$ is said to be *locally regular* at $x \in K$ if there exists an open neighborhood U of x such that for every non-decreasing uniformly bounded sequence u_j of psh functions on U such that $u_j \leqslant 0$ on $K \cap U$ their usc upper envelope also satisfies

$$\Big(\sup_j u_j\Big)^* \leqslant 0 \quad \text{on } K \cap U.$$

It is easy to adapt the argument of [N, Proposition 6.1] to prove the following result.

PROPOSITION 1.5. *Let $K \subset X$ be a non-pluripolar compact subset. Then K is locally regular* (*i.e. locally regular at each point of ∂K*) *if and only if* (K, ϕ) *is regular for* every *continuous weight ϕ on L.*

As a way to test local regularity we have the so-called accessibility criterion.

PROPOSITION 1.6. *If $K \subset X$ is a compact subset of X and there exists a real-analytic arc $\gamma: [0,1] \to X$ such that $\gamma(]0,1])$ is contained in the topological interior K^0, then K is locally regular at $\gamma(0)$.*

This follows from the fact that any subharmonic function u defined around $[0,1] \subset \mathbb{C}$ satisfies

$$u(0) = \limsup_{\substack{z \to 0 \\ z \in]0,1]}} u(z).$$

COROLLARY 1.7. *Let Ω and M be a smoothly bounded domain and a real-analytic n-dimensional totally real compact submanifold of X, respectively. Then $\overline{\Omega}$ and M are locally regular.*

Proof. The first assertion follows from the accessibility criterion just as in [K, Corollary 5.3.13] and the second from the fact that $\mathbb{R}^n$ is locally regular in $\mathbb{C}^n$. □

Remark 1.8. It seems to be unknown whether 'real-analytic' can be relaxed to C^∞ in Corollary 1.7.

1.2. Bernstein–Markov and determining measures

Recall from the introduction that given a weighted compact subset (K, ϕ) we say that a probability measure μ on K is *Bernstein–Markov* for (K, ϕ) if and only if the distortion between the $L^\infty(K, k\phi)$-norm and the $L^2(\mu, k\phi)$-norm on $H^0(kL)$ has sub-exponential growth as $k \to \infty$, that is:

For each $\varepsilon > 0$ there exists $C > 0$ such that

$$\sup_K |s|^2_{k\phi} \leqslant C e^{\varepsilon k} \int_K |s|^2_{k\phi} \, d\mu. \tag{1.1}$$

for each k and each section $s \in H^0(kL)$.

Following [S] we are going to obtain a characterization of the following stronger property.

Definition 1.9. Let (K, ϕ) be a weighted compact subset and let μ be a positive measure on K. Then μ will be said to be *Bernstein–Markov with respect to psh weights* for (K, ϕ) if and only if for each $\varepsilon > 0$ there exists $C > 0$ such that

$$\sup_K e^{p(\psi - \phi)} \leqslant C e^{\varepsilon p} \int_K e^{p(\psi - \phi)} \, d\mu \tag{1.2}$$

for all $p \geqslant 1$ and all psh weights ψ on L.

Remark 1.10. One virtue of Definition 1.9 is that it obviously makes sense in the more general situation of θ-psh functions with respect to a smooth $(1,1)$-form θ as considered for example in [GZ], [BEGZ] and [BBGZ].

It is immediate to see that μ is Bernstein–Markov for (K,ϕ) if it is Bernstein–Markov with respect to psh weights, since (1.2) implies (1.1) with $\psi:=(\log|s|)/k$ and $p:=2k$.

Question 1.11. Let (K,ϕ) be a *regular* weighted compact set. Is it true that any Bernstein–Markov measure μ for (K,ϕ) with respect to sections is necessarily Bernstein–Markov with respect to psh weights, i.e. that (1.1) implies (1.2)?

We will only consider *non-pluripolar* measures μ, i.e. measures putting no mass on pluripolar subsets. Note that the equilibrium measure $\mu_{\text{eq}}(K,\phi)$ is non-pluripolar, since it is defined as the non-pluripolar Monge–Ampère measure of ϕ_K^* (cf. [BB1]).

Following essentially [S] we shall say that μ is *determining* for (K,ϕ) if and only if the following equivalent properties hold (compare [S, Theorem A]).

PROPOSITION 1.12. *Let (K,ϕ) be a weighted compact subset and let μ be a non-pluripolar probability measure on K. Then the following properties are equivalent, and imply that (K,ϕ) is regular*:

(i) *each Borel subset $E\subset K$ such that $\mu(K\setminus E)=0$ satisfies $\phi_E=\phi_K$;*

(ii) *for each psh weight ψ we have*

$$\psi\leqslant\phi\ \mu\text{-a.e.}\quad\Longrightarrow\quad\psi\leqslant\phi\ \text{on } K, \tag{1.3}$$

i.e.

$$\sup_K(\psi-\phi)=\log\|e^{\psi-\phi}\|_{L^\infty(\mu)}.$$

Proof. Assume that (i) holds and let ψ be a psh weight such that $\psi\leqslant\phi$ μ-a.e. Consider the Borel subset

$$E:=\{x\in K:\psi(x)\leqslant\phi(x)\}\subset K.$$

We then have $\mu(K\setminus E)=0$ by assumption, and hence $\phi_E=\phi_K$. On the other hand, we have $\psi\leqslant\phi$ on E by definition of E. Hence $\psi\leqslant\phi_E=\phi_K$ on X, and we infer that $\psi\leqslant\phi$ on K. We have thus shown that (i) $\Rightarrow$ (ii).

Assume conversely that (ii) holds. The set $\{x:\phi_K(x)<\phi_K^*(x)\}$ is negligible, and hence pluripolar by [BT]. Therefore it has μ-measure 0, since μ is non-pluripolar by assumption. We thus have $\phi_K^*=\phi_K\leqslant\phi$ μ-a.e., and (ii) implies that $\phi_K^*\leqslant\phi$ everywhere on K, which means that (K,ϕ) is regular.

Now let ψ be a psh weight on X and assume that $\psi\leqslant\phi$ on $E\subset K$ with $\mu(K\setminus E)=0$. Then we have in particular $\psi\leqslant\phi$ μ-a.e., hence $\psi\leqslant\phi$ on K, and it follows that $\phi_E=\phi_K$ as desired. □

Proposition 1.13. *Let (K,ϕ) be a weighted compact subset. Then the equilibrium measure $\mu_{\mathrm{eq}}(K,\phi)$ is determining for (K,ϕ) if and only if (K,ϕ) is regular.*

Proof. Suppose that (K,ϕ) is regular. The domination principle, itself an easy consequence of the so-called comparison principle, states that given two psh weights ψ and ψ' on L such that ψ has minimal singularities we have

$$\psi' \leqslant \psi \text{ a.e. for } \mathrm{MA}(\psi) \quad \Longrightarrow \quad \psi' \leqslant \psi \text{ on } X$$

(cf. [BEGZ, Corollary 2.5] for a proof in our context). Applying this to $\psi := \phi_K^*$ immediately yields the result since we have $\phi_K^* \leqslant \phi$ on K by the regularity assumption. The converse follows from Proposition 1.12. □

The next result provides a new proof of [S] while extending it to our context.

Theorem 1.14. *Let (K,ϕ) be a weighted compact subset and μ be a non-pluripolar probability measure on K. Then the following properties are equivalent*:

(i) *μ is determining for (K,ϕ);*

(ii) *μ is Bernstein–Markov with respect to psh weights for (K,ϕ).*

Proposition 1.13 combined with Theorem 1.14 shows that the equilibrium measure of a regular weighted set (K,ϕ) is Bernstein–Markov, which generalizes the result in [NZ].

Proof. We introduce the functionals

$$F_p(\psi) := \frac{1}{p} \log \int_X e^{p(\psi-\phi)}\, d\mu = \log \|e^{\psi-\phi}\|_{L^p(\mu)}$$

for $p>0$ and

$$F(\psi) := \sup_K (\psi-\phi),$$

both defined on the set $\mathcal{P}(X,L)$ of all psh weights ψ on L. For each ψ, $pF_p(\psi)$ is a convex function of p by convexity of the exponential (Hölder's inequality), and we have $pF_p(\psi)\to 0$ as $p\to 0_+$ by dominated convergence since $p(\psi-\phi)\to 0$ μ-a.e. (μ puts no mass on the polar set $\{x:\psi(x)=-\infty\}$). As a consequence, $F_p(\psi)$ is a non-decreasing function of p, and we have

$$\lim_{p\to\infty} F_p(\psi) = \log \|e^{\psi-\phi}\|_{L^\infty(\mu)}$$

by a basic fact from integration theory. We can therefore reformulate (i) and (ii) as follows:

(i′) $F_p \to F$ pointwise on $\mathcal{P}(X,L)$;

(ii′) $F_p - F$ is bounded on $\mathcal{P}(X,L)$, uniformly for $p \geqslant 1$, and $F_p \to F$ uniformly on $\mathcal{P}(X,L)$ as $p\to\infty$.

This clearly shows that (ii) $\Rightarrow$ (i). Let us show conversely that (i) $\Rightarrow$ (ii). By Hartogs' lemma F is *upper semi-continuous* on $\mathcal{P}(X, L)$. On the other hand, Lemma 1.15 below says that F_p is *continuous* on $\mathcal{P}(X, L)$ for each $p>0$, so that $F-F_p$ is usc on $\mathcal{P}(X, L)$. Now the main point is that $F-F_p$ is invariant by translation (by a constant), and thus descends to a usc function on

$$\mathcal{P}(X, L)/\mathbb{R} \simeq \mathcal{T}(X, L),$$

the space of all closed positive $(1,1)$-currents lying in the cohomology class $c_1(L)$, which is *compact* (in the weak topology of currents).

By monotonicity we have $0 \leqslant F-F_p \leqslant F-F_1$ when $p \geqslant 1$. But $F-F_1$ is usc on a compact set, hence is bounded from above, and it follows that $F-F_p$ is at any rate bounded on $\mathcal{P}(X, L)$ uniformly for $p \geqslant 1$. By the above discussion, we thus see that (i) $\Rightarrow$ (ii) amounts to the fact that F_p converges to F uniformly as soon as pointwise convergence holds, which is a consequence of Dini's lemma since $F-F_p$ is usc and non-increasing on $\mathcal{T}(X, L)$ as a function of p. □

LEMMA 1.15. *The functional $F_p: \mathcal{P}(X, L) \to \mathbb{R}$ is continuous for each $p>0$.*

The proof relies on more or less standard arguments.

Proof. Let $\psi_k \to \psi$ be a (weakly) convergent sequence in $\mathcal{P}(X, L)$ and set

$$u_k := p(\psi_k - \phi) \quad \text{and} \quad u := p(\psi - \phi),$$

all of which are θ-psh functions for $\theta := p\, dd^c \phi$ (the language of quasi-psh functions is more convenient for what follows).

Let us first recall the following general consequences of Hartogs' lemma (compare [GZ, Proposition 2.6]). If $(\varphi_k)_{k=1}^{\infty}$ is a sequence of θ-psh functions which is uniformly bounded above and if φ_k converges Lebesgue-a.e. to a θ-psh function φ, then $\varphi_k \to \varphi$ in $L^1(X)$ and $\varphi = \limsup_{k\to\infty} \varphi_k$ quasi-everywhere (q.e. for short), i.e. outside a pluripolar set (using that negligible sets are pluripolar by [BT]).

As $u_k \to u$ in $L^1(X)$ and e^{u_k} is uniformly bounded, we may assume upon extracting a subsequence that $u_k \to u$ Lebesgue-a.e. and $\int_X e^{u_k}\, d\mu \to l$ for some $l \in \mathbb{R}$. We have to show that $l = \int_X e^u\, d\mu$.

Since the functions e^{u_k} stay in a weakly compact subset of the Hilbert space $L^2(\mu)$, the closed convex subsets

$$C_k := \overline{\operatorname{Conv}\{e^{u_j} : j \geqslant k\}} \subset L^2(\mu)$$

are weakly compact in $L^2(\mu)$, and it follows that there exists v lying in the intersection of the decreasing sequence of compact sets C_k. For each k we may thus find finite convex combinations

$$v_k := \sum_{j\in I_k} t_{k,j} e^{u_j}$$

with $I_k \subset [k,\infty[$ such that $v_k \to v$ strongly in $L^2(\mu)$. Note that

$$\int_X v_k\,d\mu \to l \quad \text{as} \quad \int_X e^{u_k}\,d\mu \to l,$$

and hence $\int_X v\,d\mu = l$.

Observe, on the other hand, that the θ-psh functions $w_k := \log v_k$ converge to u Lebesgue-a.e., since we have arranged that $u_j \to u$ Lebesgue-a.e. As $(w_k)_{k=1}^\infty$ is also uniformly bounded above, it follows from the general consequences of Hartogs' lemma recalled above that $w_k \to u$ weakly and $\limsup_{k\to\infty} w_k = u$ q.e., and hence μ-a.e., since μ puts no mass on pluripolar sets. But $v_k \to v$ in $L^2(\mu)$ implies that a subsequence of $(v_k)_{k=1}^\infty$ converges to v μ-a.e. and we conclude that $v = e^u$ μ-a.e. This implies as desired that

$$l = \int_X v\,d\mu = \int_X e^u\,d\mu. \qquad \square$$

Corollary 1.16. *If (K,ϕ) is a* regular *weighted compact subset then*

$$\psi \longmapsto \sup_K (\psi - \phi)$$

is continuous on $\mathcal{P}(X,L)$.

Compare [ZZ, Lemma 27] for a related result in the $\mathbb{C}$-case.

Proof. By Proposition 1.13 the equilibrium measure $\mu := \mu_{\mathrm{eq}}(K,\phi)$ is determining for (K,ϕ) since (K,ϕ) is regular. By Lemma 1.15, the functionals $\log \|e^{\psi-\phi}\|_{L^p(\mu)}$ are continuous, and they converge uniformly to $\sup_K(\psi-\phi)$ by Theorem 1.14. $\square$

2. Volumes of balls

2.1. Convexity properties

Let (K,ϕ) be a weighted compact subset and let μ be a non-pluripolar probability measure on K. Let $S=(s_1,...,s_N)$ be a basis of $H^0(L)$ and let vol be the corresponding Lebesgue measure. As is well known, the Gram determinant satisfies

$$-\log\det(\langle s_i, s_j\rangle_{L^2(\mu,\phi)})_{i,j} = \log \operatorname{vol} \mathcal{B}^2(\mu,\phi) - \log \frac{\pi^N}{N!}, \tag{2.1}$$

where $\pi^N/N!$ is of course the Euclidian volume of the unit ball in $\mathbb{C}^N$.

Now let $\det S$ be the image of $s_1\wedge...\wedge s_N$ under the natural map

$$\bigwedge^N H^0(X,L) \longrightarrow H^0(X^N, L^{\boxtimes N}),$$

that is, the global section on X^N locally defined by

$$(\det S)(x_1,...,x_N) := \det(s_i(x_j))_{i,j}.$$

By [BB1, Lemma 5.3], we have the following result.

LEMMA 2.1. *The L^2-norm of $\det S$ with respect to the weight and measure induced by ϕ and μ satisfies*

$$\|\det S\|^2_{L^2(\mu,\phi)} = N!\det(\langle s_i,s_j\rangle_{L^2(\mu,\phi)})_{i,j}.$$

On the other hand, a straightforward computation yields the following identity.

LEMMA 2.2. *If $P\in X^N$ is a configuration of points, then*

$$\|\det S\|^2_{L^2(\delta_P,\phi)} = \frac{N!}{N^N}|\det S|^2_\phi(P).$$

Combining these results, we record the following consequence.

PROPOSITION 2.3. *We have*

$$\log\operatorname{vol}\mathcal{B}^2(\mu,\phi) = -\log\|\det S\|^2_{L^2(\mu,\phi)} + N\log\pi. \tag{2.2}$$

If $\mu=\delta_P$, then

$$\log\operatorname{vol}\mathcal{B}^2(\delta_P,\phi) = -\log|\det S|^2_\phi(P) + \log\frac{\pi^N}{N!} + N\log N. \tag{2.3}$$

Note that the last formula reads

$$\log\frac{\operatorname{vol}\mathcal{B}^2(\delta_P,\phi)}{\operatorname{vol}\mathcal{B}^2(\nu,\psi)} = -\log|\det S|^2_\phi(P) + N\log N \tag{2.4}$$

when S is $L^2(\nu,\psi)$-orthonormal. The volume of balls satisfies the following convexity properties.

PROPOSITION 2.4. *Let (K,ϕ) be a weighted compact subset and μ be a probability measure on K. The functional* $\log\operatorname{vol}\mathcal{B}^2(\mu,\phi)$ *is* convex *in its μ-variable and* concave *in its ϕ-variable.*

Proof. The function log det is concave on positive definite Hermitian matrices. Since the Gram matrix $(\langle s_i,s_j\rangle_{L^2(\mu,\phi)})_{i,j}$ depends linearly on μ, formula (2.1) implies that

$$\mu \longmapsto \log\operatorname{vol}\mathcal{B}^2(\mu,\phi)$$

is convex. On the other hand, concavity in ϕ follows from equation (2.2) and Hölder's inequality. □

2.2. Directional derivatives

Proposition 2.5. *The $\mathcal{L}$-functional has directional derivatives given by*

$$\frac{\partial}{\partial\phi}\log\operatorname{vol}\mathcal{B}^2(\mu,\phi)=\langle N\beta(\mu,\phi),\cdot\,\rangle$$

and

$$\frac{\partial}{\partial\mu}\log\operatorname{vol}\mathcal{B}^2(\mu,\phi)=-\langle\,\cdot\,,\varrho(\mu,\phi)\rangle.$$

Proof. This is similar to [BB1, Lemma 5.1], itself a variant of [D1, Lemma 2]. By (2.1) we have to show that, given two paths ϕ_t and μ_t, we have

$$\frac{d}{dt}\bigg|_{t=0}\log\det\biggl(\int_X s_i\bar{s}_j e^{-2\phi_t}\,d\mu\biggr)_{i,j}=-2\int_X\biggl(\frac{d}{dt}\bigg|_{t=0}\phi_t\biggr)\varrho(\mu,\phi_0)\,d\mu$$

and

$$\frac{d}{dt}\bigg|_{t=0}\log\det\biggl(\int_X s_i\bar{s}_j e^{-2\phi}\,d\mu_t\biggr)_{i,j}=\int_X\varrho(\mu_0,\phi)\biggl(\frac{d}{dt}\bigg|_{t=0}d\mu_t\biggr).$$

The only thing to remark is that the variations are independent of the choice of the basis S by (2.1), so that one may assume that $S=(s_j)_{j=1}^{\infty}$ is $L^2(\mu,\phi)$-orthonormal. The result then follows from a straightforward computation. □

Remark 2.6. If (μ,ϕ) is a weighted subset, the condition

$$\beta(\mu,\phi)=\mu$$

holds by definition if and only if

$$\varrho(\mu,\phi)=N\quad\mu\text{-a.e.}$$

According to Proposition 2.5, this is the case if and only if ϕ is a critical point of the convex functional

$$\langle\mu,\cdot\,\rangle-\frac{1}{N}\log\operatorname{vol}\mathcal{B}^2(\mu,\cdot\,).$$

On the other hand, this condition is related to Donaldson's notion of *μ-balanced metric* (cf. [D2, §2.2]). Indeed ϕ is μ-balanced in Donaldson's sense if and only if $\varrho(\mu,\phi)=N$ holds *everywhere* on X.

Proposition 2.7. *For any configuration $P\in X^N$, the pair (δ_P,ϕ) satisfies*

$$\beta(\delta_P,\phi)=\delta_P.$$

The proof is immediate from the definition. On the other hand, following [BBLW], we introduce the following concept.

Definition 2.8. If (K,ϕ) is a weighted compact subset, we say that a probability measure μ on K is a (K,ϕ)-*optimal measure* if and only if it minimizes $\log\operatorname{vol}\mathcal{B}^2(\cdot,\phi)$ over the compact convex set $\mathcal{P}_K$ of all probability measures on K.

As in [Bos] we obtain the following characterizations.

PROPOSITION 2.9. *A probability measure μ on K is (K,ϕ)-optimal if and only if the $L^2(\mu,\phi)$-$L^\infty(K,\phi)$ distortion takes the least possible value $N^{1/2}$, i.e. if and only if*

$$\sup_K \varrho(\mu,\phi)=N.$$

In particular we then have

$$\beta(\mu,\phi)=\mu.$$

Proof. By convexity of $\mu\mapsto\log\operatorname{vol}\mathcal{B}^2(\mu,\phi)$, the minimum on $\mathcal{P}_K$ is achieved at μ if and only if

$$\left\langle \frac{\partial}{\partial\mu}\log\operatorname{vol}\mathcal{B}^2(\phi,\mu),\nu-\mu\right\rangle\geqslant 0$$

for all $\nu\in\mathcal{P}_K$, i.e. if and only if

$$\langle\varrho(\phi,\mu),\nu\rangle\leqslant N$$

for all probability measures ν on K, which is in turn equivalent to

$$\sup_K \varrho(\phi,\mu)\leqslant N$$

and implies that $\varrho(\mu,\phi)=N$ μ-a.e. since $\langle\varrho(\mu,\phi),\mu\rangle=N$. □

We note that the optimal value satisfies

$$\min_{\mu\in\mathcal{P}_K}\log\operatorname{vol}\mathcal{B}^2(\mu,\phi)\geqslant\log\operatorname{vol}\mathcal{B}^\infty(K,\phi),$$

but equality does *not* hold as soon as $N\geqslant 2$ since it would imply that $\mathcal{B}^\infty(K,\phi)=\mathcal{B}^2(\mu,\phi)$ for some measure $\mu\in\mathcal{P}_K$ and thus that $1=\sup_K\varrho(\mu,\phi)\geqslant N$.

Next, we have the following basic result.

PROPOSITION 2.10. *Let P be a Fekete configuration for the weighted set (K,ϕ). Then the $L^\infty(K,\phi)$-$L^1(\delta_P,\phi)$ distortion is at most equal to N.*

Proof. Fix a configuration $P=(x_1,...,x_N)$ and let $e_i\in H^0(L\otimes L^*_{x_i})$ be defined by

$$e_i(x):=\det S(x_1,...,x_{i-1},x,x_i,...,x_N)\otimes\det S(x_1,...,x_i\,...,x_N)^{-1}$$

(the Lagrange interpolation 'polynomials'). If P is a Fekete configuration for (K,ϕ) then we clearly have $\sup_K|e_i|_\phi=1$. The result follows since any $s\in H^0(L)$ may be written as $s=\sum_{i=1}^N s(x_i)\otimes e_i$. □

2.3. Energy at equilibrium

As in the introduction, we now suppose given a reference weighted compact subset (K_0, ϕ_0) and a probability measure μ_0 on K_0 which is Bernstein–Markov for (K_0, ϕ_0). We normalize the Haar measure vol on $H^0(kL)$ by the condition

$$\operatorname{vol} \mathcal{B}^2(K_0, k\phi_0) = 1$$

and we consider the corresponding $\mathcal{L}$-functionals, defined by

$$\mathcal{L}_k(\mu, \phi) = \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^2(\mu, k\phi)$$

and

$$\mathcal{L}_k(K, \phi) = \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^\infty(K, k\phi).$$

We will use the following results [BB1, Theorems A and B].

THEOREM 2.11. *If* (K, ϕ) *is a given compact weighted subset, then*

$$\lim_{k\to\infty} \mathcal{L}_k(K, \phi) = \mathcal{E}_{\mathrm{eq}}(K, \phi).$$

THEOREM 2.12. *The map* $\phi \mapsto \mathcal{E}_{\mathrm{eq}}(K, \phi)$, *defined on the affine space of continuous weights over* K, *is concave and Gâteaux differentiable, with directional derivatives given by integration against the equilibrium measure*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{\mathrm{eq}}(\phi + tv) = \langle v, \mu_{\mathrm{eq}}(K, \phi)\rangle.$$

This differentiability property of the energy at equilibrium really is the key to the proof of Theorem C. Even though $\mathcal{E}_{\mathrm{eq}}(K, \phi)$ is by definition the composition of the projection operator $P_K: \phi \mapsto \phi_K^*$ on the convex set of psh weights with the Monge–Ampère energy $\mathcal{E}$, whose derivative at ϕ_K^* is equal to $\mu_{\mathrm{eq}}(K, \phi)$, this result is not a mere application of the chain rule, since P_K is definitely *not* differentiable in general.

3. Proof of the main results

3.1. Proof of Theorem C

Let $v \in C^0(X)$, and set

$$f_k(t) := \mathcal{L}_k(\mu_k, \phi + tv) \quad \text{and} \quad g(t) := \mathcal{E}_{\mathrm{eq}}(K, \phi + tv).$$

Theorem 2.11 combined with $\mathcal{L}_k(\mu,\phi)\geqslant\mathcal{L}_k(K,\phi)$ shows that $g(t)$ is an asymptotic lower bound for $f_k(t)$ as $k\to\infty$, that is

$$\liminf_{k\to\infty} f_k(t)\geqslant g(t),$$

and the assumption means that this asymptotic lower bound is achieved for $t=0$, that is

$$\lim_{k\to\infty} f_k(0)=g(0).$$

Now f_k is concave for each k by Proposition 2.4, and we have

$$f_k'(0)=\langle\beta(\mu_k,k\phi),v\rangle$$

by Proposition 2.5. On the other hand, g is differentiable with

$$g'(0)=\langle\mu_{\mathrm{eq}}(K,\phi),v\rangle$$

by Theorem 2.12. The elementary lemma below thus shows that

$$\lim_{k\to\infty}\langle\beta(\mu_k,k\phi),v\rangle=\langle\mu_{\mathrm{eq}}(K,\phi),v\rangle$$

for each continuous function v, and the proof of Theorem C is complete.

LEMMA 3.1. *Let f_k by a sequence of* concave *functions on $\mathbb{R}$ and let g be a function on $\mathbb{R}$ such that*

- $\liminf_{k\to\infty} f_k\geqslant g$;
- $\lim_{k\to\infty} f_k(0)=g(0)$.

If the f_k and g are differentiable at 0, *then* $\lim_{k\to\infty} f_k'(0)=g'(0)$.

Proof. Since f_k is concave, we have

$$f_k(0)+f_k'(0)t\geqslant f_k(t)$$

for all t, and hence

$$\liminf_{k\to\infty} t f_k'(0)\geqslant g(t)-g(0).$$

The result now follows by first letting $t>0$ and then $t<0$ tend to 0. □

The same lemma underlies the proof of Yuan's equidistribution theorem given in [BB1], and was in fact inspired by the variational principle in the original equidistribution result (in the strictly psh case) by Szpiro, Ullmo and Zhang [SUZ].

3.2. Proof of Theorem B

As noted in the introduction, the condition on the sequence of probability measures μ_k in Theorem C is equivalent to

$$\log \frac{\operatorname{vol} \mathcal{B}^2(\mu_k, k\phi)}{\operatorname{vol} \mathcal{B}^\infty(K, k\phi)} = o(kN_k). \tag{3.1}$$

This condition can be understood as a weak Bernstein–Markov condition for the sequence $(\mu_k)_{k=1}^\infty$, in view of the following easy result.

Lemma 3.2. *For any probability measure μ on K,*

$$0 \leqslant \log \frac{\operatorname{vol} \mathcal{B}^2(\mu, \phi)}{\operatorname{vol} \mathcal{B}^\infty(K, \phi)} \leqslant N \log \sup_K \varrho(\mu, \phi).$$

The proof is immediate if we recall that $\sup_K \varrho(\mu, k\phi)^{1/2}$ is the distortion between the two norms and vol is homogeneous of degree $2N_k = \dim_{\mathbb{R}} H^0(kL)$.

Since a given measure μ is Bernstein–Markov for (K, ϕ) if and only if

$$\log \sup_K \varrho(\mu, k\phi) = o(k),$$

we now see that Theorem B directly follows from Theorem C.

3.3. Proof of Theorem A

Let $P_k \in K^{N_k}$ be a Fekete configuration for $(K, k\phi)$. Since $\beta(\delta_{P_k}, k\phi_k) = \delta_{P_k}$ by Proposition 2.7, Theorem C will imply Theorem A if we can show that

$$\lim_{k\to\infty} \mathcal{L}_k(\delta_{P_k}, k\phi) = \mathcal{E}_{\mathrm{eq}}(K, \phi). \tag{3.2}$$

Now let S_k be an $L^2(\mu_0, k\phi_0)$-orthonormal basis of $H^0(kL)$. The metric $|\det S_k|$ does not depend on the specific choice of the orthonormal basis S_k, simply because $|\det U| = 1$ for any unitary matrix U. We recall the following definition from [BB1], which is a generalization of Leja and Zaharjuta's notion of *transfinite diameter*.

Definition 3.3. Let (K, ϕ) be a weighted compact subset. Its k-*diameter* (with respect to (μ_0, ϕ_0)) is defined by

$$\mathcal{D}_k(K, \phi) := -\frac{1}{kN_k} \log \|\det S_k\|_{L^\infty(K, k\phi)} = \inf_{P \in K^{N_k}} \frac{1}{kN_k} \log |\det S_k(P_k)|^{-1}_{k\phi}.$$

A Fekete configuration $P_k \in K^{N_k}$ for $(K, k\phi)$ is thus a point $P_k \in K^{N_k}$ where the infimum defining $\mathcal{D}_k(K, \phi)$ is achieved. The following result was proved in [BB1].

Theorem 3.4. *If (K,ϕ) is a weighted compact subset, then*

$$\lim_{k\to\infty}\mathcal{D}_k(K,\phi)=\mathcal{E}_{\mathrm{eq}}(K,\phi).$$

We set $\mu_k:=\delta_{P_k}$. Since P_k is a Fekete configuration for $(K,k\phi)$, we have

$$-\frac{1}{kN_k}\log|\det S_k|_{k\phi}(P_k)=\mathcal{D}_k(K,\phi)$$

by definition, and formula (2.4) thus yields

$$\frac{1}{kN_k}\log\frac{\mathrm{vol}\,\mathcal{B}^2(\mu_k,k\phi)}{\mathrm{vol}\,\mathcal{B}^2(\mu_0,k\phi_0)}=\mathcal{D}_k(K,\phi)+\frac{1}{2k}\log N_k.$$

This implies that

$$\mathcal{L}_k(\mu_k,k\phi)=\frac{1}{2kN_k}\log\frac{\mathrm{vol}\,\mathcal{B}^2(\mu_k,k\phi)}{\mathrm{vol}\,\mathcal{B}^\infty(K_0,k\phi_0)}$$

converges to $\mathcal{E}_{\mathrm{eq}}(K,\phi)$ as desired, as

$$\log N_k=O(\log k)$$

on the one hand, and

$$\log\frac{\mathrm{vol}\,\mathcal{B}^2(\mu_0,k\phi_0)}{\mathrm{vol}\,\mathcal{B}^\infty(K_0,k\phi_0)}=o(kN_k)$$

by Lemma 3.2 above, since μ_0 is Bernstein–Markov for (K_0,ϕ_0). The proof of Theorem A is thus complete.

3.4. Proof of Corollary C

Step 1. For each (p,q)-optimal configuration $P_k\in K^{N_k}$ the $L^p(\mu)$-$L^\infty(\delta_{P_k})$ distortion is at most equal to N_k. Indeed pick a Fekete configuration $Q_k\in K^{N_k}$. For each $s\in H^0(kL)$ we then have, using Proposition 2.10,

$$\|s\|_{L^p(\mu,k\phi)}\leqslant\|s\|_{L^\infty(K,k\phi)}\leqslant N_k\|s\|_{L^1(\delta_{Q_k},k\phi)}\leqslant N_k\|s\|_{L^q(\delta_{Q_k},k\phi)},$$

and the result follows since the $L^p(\mu)$-$L^q(\delta_{P_k})$ distortion is at most equal to that of $L^p(\mu)$-$L^q(\delta_{Q_k})$.

Step 2. For each (p,q)-optimal configuration $P_k\in K^{N_k}$ the $L^\infty(X)$-$L^\infty(\delta_{P_k})$ distortion has subexponential growth. Indeed the BM-property of μ implies, by Step 1, that

$$\|s\|_{L^\infty(K,k\phi)}\leqslant Ce^{\varepsilon k}\|s\|_{L^p(\mu,k\phi)}\leqslant CN_ke^{\varepsilon k}\|s\|_{L^\infty(\delta_{P_k},k\phi)}.$$

Step 3. Every sequence $P_k \in K^{N_k}$ such that the $L^\infty(X)$-$L^\infty(\delta_{P_k})$ distortion has subexponential growth is equilibrium distributed. Indeed denote by C_k the $L^\infty(X)$-$L^\infty(\delta_{P_k})$ distortion. Applying (0.12) successively to each variable of the section $\det S_k$ successively (as in [BB1, p. 378]) and using the fact that $\det S_k$ is anti-symmetric yields

$$\|\det S_k\|_{L^\infty(K^{N_k},k\phi)} \leqslant C_k^{N_k} |\det S_k|_{k\phi}(P_k).$$

Since we are assuming that $C_k = O(e^{\varepsilon k})$ for each $\varepsilon > 0$, it follows that the sequence $(P_k)_{k=1}^\infty$ is asymptotically Fekete for (K,ϕ), i.e. the measures $\mu_k = \delta_{P_k}$ satisfy the growth conditions in Theorem C, proving the convergence $\delta_{P_k} \to \mu_{\mathrm{eq}}(K,\phi)$.

3.5. Proof of Corollary D

The sections $s_1, ..., s_N$ appearing in the construction of the recursively extremal configuration $P = (x_1, ..., x_N)$ constitute an orthononormal basis S in $H^0(L)$. Moreover, by definition, x_j maximizes the Bergman distortion function $\varrho^{\mathcal{H}_j}(x)$ of the sub-Hilbert space $\mathcal{H}_j$ and

(i) $\varrho^{\mathcal{H}_j}(x_j) = |s_j(x_j)|_\phi^2$;

(ii) $s_i(x_j) = 0$ for $i < j$.

Indeed, (i) is a direct consequence of the extremal definition (0.3) of the Bergman distortion function $\varrho^{\mathcal{H}_j}$ of the space $\mathcal{H}_j$. Then (ii) follows from (i) by expanding $\varrho^{\mathcal{H}_j}$ in terms of the orthonormal base $s_1, ..., s_j$ of $\mathcal{H}_j$ (using formula (0.4)) and evaluating at x_j.

Now, by (ii) above, we have that the matrix $(s_i(x_j))$ is triangular and thus

$$(\det S)(P) := \det(s_i(x_j)) = s_1(x_1) \dots s_N(x_N).$$

Hence, (i) gives that

$$|(\det S)(P)|_\phi^2 = \varrho^{\mathcal{H}_1}(x_1) \dots \varrho^{\mathcal{H}_N}(x_N).$$

But since x_i maximizes $\varrho^{\mathcal{H}_i}(x)$, where $\int_X \varrho^{\mathcal{H}_i}(x)\, d\mu = \dim \mathcal{H}_i = i$, it follows that $\varrho^{\mathcal{H}_i}(x) \geqslant i$. Thus, $|(\det S)(P)|_\phi^2 / N! \geqslant 1$ and replacing P by P_k then gives that P_k is asymptotically Fekete, i.e. (0.10) holds. The corollary now follows from Theorem C.

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