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Uniruledness of stable base loci of adjoint linear systems via Mori theory

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Abstract We explain how to deduce from recent results in the Minimal Model Program a general uniruledness theorem for base loci of adjoint divisors. As a special case, we recover previous results by Takayama.

Keywords Big and pseudoeffective adjoints divisors · Stable, non-ample and non-nef base locus · Rational curves

Mathematics Subject Classification (2000) 14J40

1 Introduction

Let X be a normal projective variety defined over $\mathbb{C}$ (or any algebraically closed field of characteristic 0) and let D be an $\mathbb{R}$-divisor on X (where $\mathbb{R}$-divisor will mean $\mathbb{R}$-Cartier $\mathbb{R}$-divisor unless otherwise specified). The (real) *stable base locus* of D is defined as

$$\mathbf{B}(D) := \bigcap \{\mathrm{Supp} E \mid E \text{ effective } \mathbb{R}\text{-divisor}, \ E \sim_{\mathbb{R}} D\}, \tag{1.1}$$

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where $E \sim_{\mathbb{R}} D$ means that E is $\mathbb{R}$-linearly equivalent to D, i.e. $E - D$ is an $\mathbb{R}$-linear combination of principal divisors $\operatorname{div}(f)$, $f \in \mathbb{C}(X)^*$. This definition coincides with the usual notion of stable base locus when D is a $\mathbb{Q}$-divisor (see for instance [1, Lemma 3.5.3]).

The *augmented base locus* (or *non-ample locus*) of D is then defined by

$$\mathbf{B}_+(D) := \bigcap_{m>0} \mathbf{B}\left(D - \tfrac{1}{m}A\right) \tag{1.2}$$

and the *restricted base locus* of D by

$$\mathbf{B}_-(D) := \bigcup_{m>0} \mathbf{B}\left(D + \tfrac{1}{m}A\right) \tag{1.3}$$

where A is an ample divisor, the definition being independent of A (see [5,6,10] and [2,3] for the analytic counterpart). We thus have the inclusions

$$\mathbf{B}_-(D) \subset \mathbf{B}(D) \subset \mathbf{B}_+(D),$$

and

$$\mathbf{B}_+(D) \neq X \Longleftrightarrow D \text{ big}, \ \ \mathbf{B}_+(D) = \emptyset \Longleftrightarrow D \text{ ample},$$
$$\mathbf{B}_-(D) \neq X \Longleftrightarrow D \text{ pseudoeffective}, \ \ \mathbf{B}_-(D) = \emptyset \Longleftrightarrow D \text{ nef}.$$

While $\mathbf{B}_+(D)$ is Zariski closed by definition, $\mathbf{B}_-(D)$ is *a priori* only a countable union of Zariski closed sets, which might not be Zariski closed in general even though no specific example seems to be known at the moment.

On the other hand, the *non-nef locus* (or *numerical base locus*) $\operatorname{NNef}(D)$ of an $\mathbb{R}$-divisor D [2,3,11], is defined in terms of the asymptotic or numerical vanishing orders attached to D (cf. Definition 2.5 below). We always have

$$\operatorname{NNef}(D) \subset \mathbf{B}_-(D)$$

and equality was shown to hold when X is smooth in [11, Lemma V.1.9] (see also [5, Proposition 2.8]), but seems to be unknown when X is an arbitrary normal variety (for a very recent progress on this problem see [4]).

The goal of the present paper is to investigate the uniruledness properties of the above loci in the case of adjoint divisors. In Sect. 2, we collect and prove some basic properties of this loci which will be essential in the proof of our main result. We then explain in Sect. 3 how to obtain the following general result using known parts of the Minimal Model Program [1,7].

Theorem A *Let X be a normal $\mathbb{Q}$-factorial projective variety, and let Δ be an effective $\mathbb{R}$-Weil divisor such that the pair (X, Δ) has klt singularities.*

(i) We have $\operatorname{NNef}(K_X + \Delta) = \mathbf{B}_-(K_X + \Delta)$*, and each of its irreducible components is uniruled.*

(ii) If $K_X + \Delta$ is furthermore big, then

$$\operatorname{NNef}(K_X + \Delta) = \mathbf{B}_-(K_X + \Delta) = \mathbf{B}(K_X + \Delta),$$

while every irreducible component of $\mathbf{B}_+(K_X + \Delta)$ *is uniruled as well.*

As already noticed in [14], the above uniruledness results both fail in the more general case when (X, Δ) has log canonical singularities, even in the log smooth case (i.e. when X is smooth and Δ has simple normal crossing support).

The special case of Theorem A where X is smooth and either K_X or Δ vanishes was obtained by Takayama in [14] by a completely different (and more direct) method, which combined his extension result for log pluricanonical forms (see [13, Theorem 4.5]) with the characterization of uniruled varieties in terms of the non-pseudoeffectivity of the canonical class [3,9].

As a corollary of Theorem A we obtain the following result, extending [14].

Corollary A *Let X be a smooth projective variety, and let D a pseudoeffective $\mathbb{R}$-divisor on X such that either $-K_X$ or $D - K_X$ is nef.*

(i) Every irreducible component of the restricted base locus $\mathbf{B}_-(D)$ is uniruled.
(ii) If D is furthermore big, then every irreducible component of the stable base locus $\mathbf{B}(D)$ or of the non-ample locus $\mathbf{B}_+(D)$ is uniruled.

2 Base loci

By convention, divisor (resp. $\mathbb{Q}$-divisor, $\mathbb{R}$-divisor) will mean Cartier divisor (resp. $\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier) unless otherwise specified. In this paper, a *pair* (X, Δ) is a normal projective variety together with an effective $\mathbb{R}$-Weil divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A birational morphism $f : Y \to X$ is a *log resolution* of the pair (X, Δ) if Y is smooth, f is projective, the exceptional locus $\operatorname{Exc}(f)$ of f is a divisor and $\operatorname{Exc}(f) + f_*^{-1}\Delta$ has simple normal crossing support. A pair (X, Δ) is *klt* if for every (equivalently for some) log resolution $f : Y \to X$, the divisor Γ defined by the following equality

$$K_Y + \Gamma = f^*(K_X + \Delta)$$

has coefficients strictly less than one.

2.1 Augmented base loci

We collect in this section some preliminary results regarding augmented base loci. It will be convenient to use the following terminology.

Definition 2.1 (*Kodaira decompositions*) Let X be a normal projective variety and D be a big $\mathbb{R}$-divisor on X. A *Kodaira decomposition* of D is a decomposition $D \sim_{\mathbb{R}} A + E$ with A an ample $\mathbb{Q}$-divisor and E an effective $\mathbb{R}$-divisor.

By [5, Remark 1.3], for a big $\mathbb{R}$-divisor D, the augmented base locus (as defined in the introduction) satisfies

$$\mathbf{B}_+(D) = \bigcap_{D \sim_{\mathbb{R}} A+E} \operatorname{Supp} E, \tag{2.1}$$

where the intersection runs over all Kodaira decompositions of D. The following result shows that one obtains the same locus by allowing Kodaira decompositions on arbitrary birational models.

Lemma 2.2 *Let X be a normal projective variety, and let D be a big $\mathbb{R}$-divisor on X. Then its augmented base locus satisfies*

$$\mathbf{B}_+(D) = \bigcap_{\pi^* D \sim_{\mathbb{R}} A'+E'} \pi(\operatorname{Supp} E'),$$

where π runs over all birational morphisms $X' \to X$ with X' normal and projective, and $\pi^* D \sim_{\mathbb{R}} A' + E'$ runs over all Kodaira decompositions of $\pi^* D$ on X'.

Proof In view of (2.1), it is clear that

$$\mathbf{B}_+(D) \supset \bigcap_{\pi^* D \sim_{\mathbb{R}} A' + E'} \pi(\operatorname{Supp} E').$$

Conversely, consider a birational morphism $\pi : X' \to X$ and a Kodaira decomposition

$$\pi^* D \sim_{\mathbb{R}} A' + E'$$

on X', and let $x \in X \setminus \pi(\operatorname{Supp} E')$. We are going to produce a Kodaira decomposition $D \sim_{\mathbb{R}} A + E$ on X such that $x \notin \operatorname{Supp} E$.

Since $E' = \pi^* D - A'$ is both effective and π-antiample, its support must contain every curve contracted by π, i.e. $\operatorname{Exc}(\pi) \subset \operatorname{Supp} E'$. Since $x \notin \pi(\operatorname{Supp} E')$ it follows that there is a unique preimage x' of x by π and that $x' \notin \operatorname{Supp} E'$. Now pick an ample $\mathbb{Q}$-divisor A on X such that $A' - \pi^* A$ is ample on X' (any small enough multiple of a given ample divisor will do). We then have $\mathbf{B}(A' - \pi^* A) = \emptyset$, which means that there exists an effective $\mathbb{R}$-divisor F' on X' with

$$F' \sim_{\mathbb{R}} A' - \pi^* A$$

and such that $x' \notin \operatorname{Supp} F'$. If we set $E := \pi_*(G' + F')$, then x does not belong to $\operatorname{Supp} E$, and $G' \sim_{\mathbb{R}} \pi^*(D - A)$ implies that $E \sim_{\mathbb{R}} D - A$, which concludes the proof. □

The next result describes the behavior of augmented base loci under birational transforms.

Proposition 2.3 *Let $\pi : X \to Y$ be a birational morphism between normal projective varieties. For any big $\mathbb{R}$-divisor D on Y and any effective π-exceptional $\mathbb{R}$-divisor F on X we then have*

$$\mathbf{B}_+(\pi^* D + F) = \pi^{-1}(\mathbf{B}_+(D)) \cup \operatorname{Exc}(\pi).$$

Proof Let $x \in X \setminus \mathbf{B}_+(\pi^* D + F)$, so that there exists a Kodaira decomposition

$$\pi^* D + F \sim_{\mathbb{R}} A + E$$

with $x \notin \operatorname{Supp} E$. Then $G := E - F$ is π-antiample and $\pi_* G = \pi_* E$ is effective since F is π-exceptional, thus the "negativity lemma" ([8, Lemma 3.39] or rather its version for $\mathbb{R}$-divisors [12, 1.1]) shows that G is effective. Since G is also π-antiample, it contains $\operatorname{Exc}(\pi)$ in its support, which shows that $x \notin \operatorname{Exc}(\pi)$. We also get a Kodaira decomposition

$$\pi^* D \sim_{\mathbb{R}} A + G$$

such that $\pi(x) \notin \pi(\operatorname{Supp} G)$, hence $\pi(x) \notin \mathbf{B}_+(D)$ by Lemma 2.2. We have thus shown that

$$\pi^{-1}(\mathbf{B}_+(D)) \cup \operatorname{Exc}(\pi) \subset \mathbf{B}_+(\pi^* D + F).$$

In order to prove the reverse inclusion, we first consider the special case where D is an ample $\mathbb{Q}$-divisor on Y and $F = 0$. Our goal is then to show that $\mathbf{B}_+(\pi^* D) \subset \operatorname{Exc}(\pi)$. Pick $x \notin \operatorname{Exc}(\pi)$ and choose a hyperplane section H of X such that $x \notin H$. Since π is an isomorphism above $\pi(x)$, it follows that $\pi(x)$ does not belong to the zero locus of the ideal sheaf $\mathcal{I} := \pi_* \mathcal{O}_X(-H)$. If we choose k sufficiently large and divisible, then $\mathcal{O}_Y(kD) \otimes \mathcal{I}$ is globally generated since D is an ample $\mathbb{Q}$-divisor. So we get the existence of a section in $H^0(Y, \mathcal{O}_Y(kD) \otimes \mathcal{I})$ that does not vanish at $\pi(x)$, hence a section $s \in H^0(X, k\pi^* D - H)$ with $s(x) \neq 0$, which indeed shows that $x \notin \mathbf{B}_+(\pi^* D)$.

We now treat the general case. We thus pick $x \in X \setminus \mathrm{Exc}(\pi)$ such that $\pi(x) \notin \mathbf{B}_+(D)$, and we have to show that $x \notin \mathbf{B}_+(\pi^*D + F)$. Since $\pi(x) \notin \mathbf{B}_+(D)$, there exists a Kodaira decomposition

$$D = A + E,$$

with $\pi(x) \notin \mathrm{Supp}E$. By the special case treated above we have $\mathbf{B}_+(\pi^*A) \subset \mathrm{Exc}(\pi)$, so that there exists a Kodaira decomposition

$$\pi^*A = B + G,$$

with B ample and $x \notin \mathrm{Supp}G$. Putting all together yields a Kodaira decomposition

$$\pi^*D + F = B + (G + \pi^*E + F),$$

with $x \notin \mathrm{Supp}(G + \pi^*E + F)$, which concludes the proof. □

2.2 Restricted base locus vs. non-nef locus

We use [5,11] as general references for what follows. Let D be a big $\mathbb{R}$-divisor on the normal projective variety X. Given a divisorial valuation v on X, the *asymptotic vanishing order* of D along v is defined as

$$v(\|D\|) := \inf\{v(E) \mid E \text{ effective } \mathbb{R}\text{-divisor}, \ E \sim_{\mathbb{R}} D\}.$$

By [5, Lemma 3.3], this is also equal to

$$\inf\{v(E) \mid E \text{ effective } \mathbb{R}\text{-divisor}, \ E \equiv D\}.$$

In particular, it only depends on the numerical equivalence class of D; the function it defines on the open convex cone

$$\mathrm{Big}(X) \subset N^1(X)$$

of big classes is homogeneous and convex, hence continuous and sub-additive. When D is a pseudoeffective $\mathbb{R}$-divisor, one sets as in [11, Definition III.1.6] (see also [2, §3] for the case of $(1, 1)$-classes)

$$v(\|D\|) := \lim_{\varepsilon \to 0} v(\|D + \varepsilon A\|), \tag{2.2}$$

where A is any ample divisor, the definition being easily seen to be independent of the choice of A. The corresponding function on the pseudoeffective cone

$$\mathrm{Psef}(X) = \overline{\mathrm{Big}(X)} \subset N^1(X)$$

is now lower semicontinuous, but *not* continuous up to the boundary of the pseudoeffective cone in general (cf. [11, p.135, Example 2.8]). A pseudoeffective $\mathbb{R}$-divisor D is nef iff $v(\|D\|) = 0$ for every divisorial valuation v.

We will use the following result, which is a consequence of [11, Lemma III.1.7].

Lemma 2.4 *Let $\pi : X' \to X$ be a birational morphism and let D be a pseudoeffective $\mathbb{R}$-divisor on X. Then we have*

$$v(\|\pi^*D\|) = v(\|D\|)$$

for every divisorial valuation v.

Definition 2.5 Let X be a normal projective variety. Let D be a pseudoeffective $\mathbb{R}$-divisor on X. The *non-nef locus* of D (called its *numerical base locus* in [11]) is defined as

$$\mathrm{NNef}(D) := \bigcup \{c_X(v) \mid v(\|D\|) > 0\},$$

where $c_X(v)$ denotes the center on X of a given divisorial valuation v, viewed as a subvariety of X.

The non-nef locus is always contained in the restricted base locus:

Lemma 2.6 *Let X be a normal projective variety. For every pseudoeffective $\mathbb{R}$-divisor D we have*

$$\mathrm{NNef}(D) \subset \mathbf{B}_-(D).$$

Proof Let $x \notin \mathbf{B}_-(D)$, and let v be any divisorial valuation such that $x \in c_X(v)$. Given an ample divisor A we have $x \notin \mathbf{B}(D + \varepsilon A)$ for each $\varepsilon > 0$, thus there exists an effective $\mathbb{R}$-divisor $E_\varepsilon \sim_{\mathbb{R}} D + \varepsilon A$ such that $x \notin \mathrm{Supp} E_\varepsilon$, and we infer that

$$v(\|D + \varepsilon A\|) \leqslant v(E_\varepsilon) = 0.$$

Letting $\varepsilon \to 0$ yields $v(\|D\|) = 0$, hence $x \notin \mathrm{NNef}(D)$. □

When X is *smooth* it was shown in [11, Lemma V.1.9] (see also [5, Proposition 2.8]) that equality holds, i.e.

$$\mathrm{NNef}(D) = \mathbf{B}_-(D) \tag{2.3}$$

for every pseudoeffective $\mathbb{R}$-divisor D. This shows in particular that $\mathrm{NNef}(D)$ is an at most countable union of Zariski closed subsets of X. The latter property carries on to the case where X is an arbitrary normal variety, since choosing a resolution of singularities $\pi : X' \to X$ yields

$$\mathrm{NNef}(D) = \pi(\mathrm{NNef}(\pi^* D))$$

by Lemma 2.4. It is however natural to ask:

Conjecture 2.7 [1] *For every normal projective variety X and every pseudoeffective $\mathbb{R}$-divisor D on X, we have* $\mathrm{NNef}(D) = \mathbf{B}_-(D)$.

Using [1] we prove:

Proposition 2.8 *Let (X, Δ) be a klt pair with $K_X + \Delta$ pseudoeffective. Then we have* $\mathrm{NNef}(K_X + \Delta) = \mathbf{B}_-(K_X + \Delta)$, *which furthermore coincides with* $\mathbf{B}(K_X + \Delta)$ *when $K_X + \Delta$ is big.*

Proof By Lemma 2.6 we only need to show that

$$\mathrm{NNef}(K_X + \Delta) \supset \mathbf{B}_-(K_X + \Delta). \tag{2.4}$$

We claim that it is enough to prove the result when $K_X + \Delta$ is big. Indeed, by the very definition of $\mathbf{B}_-$, given an irreducible component V of $\mathbf{B}_-(K_X + \Delta)$, there exists an ample $\mathbb{Q}$-divisor A such that V is a component of $\mathbf{B}(K_X + \Delta + 2A)$. Since

$$\mathbf{B}(K_X + \Delta + 2A) \subset \mathbf{B}_-(K_X + \Delta + A) \subset \mathbf{B}_-(K_X + \Delta),$$

[1] After the posting of this article on the arXiv, Cacciola and Di Biagio [4] proved the conjecture for surfaces and, in dimension ≥3, in the klt case.

we deduce that V is an irreducible component of $\mathbf{B}_-(K_X+\Delta+A)$. Notice moreover that $\mathrm{NNef}(K_X+\Delta+A)\subset\mathrm{NNef}(K_X+\Delta)$. Upon changing A in its $\mathbb{Q}$-linear equivalence class we may assume that $(X,\Delta+A)$ is klt, and we are indeed reduced to the case where $K_X+\Delta$ is big.

By [1, Theorem 1.2] $K_X+\Delta$ admits an ample model, which means that there exist birational morphisms $\pi: Y\to X$ and $\pi': Y\to X'$ such that

$$\pi^*(K_X+\Delta)=\pi'^*H+F,$$

where H is ample on X' and F is effective and π'-exceptional, and Y may be assumed to be smooth. By the negativity lemma [8, Lemma 3.39], every effective $\mathbb{R}$-divisor E on Y such that $E\equiv\pi'^*H+F$ satisfies $E\geqslant F$, and it easily follows that

$$v(\|K_X+\Delta\|)=v(\|\pi^*(K_X+\Delta)\|)=v(F)$$

for every divisorial valuation v, so that

$$\mathrm{NNef}(K_X+\Delta)=\pi(\mathrm{Supp}F).$$

On the other hand we have

$$\mathbf{B}(K_X+\Delta)=\pi(\mathbf{B}(\pi^*(K_X+\Delta)))=\pi(\mathrm{Supp}F),$$

and the result follows. □

3 Theorem A and its corollary

3.1 Proof of Theorem A

Let X be a normal, projective and $\mathbb{Q}$-factorial variety and let Δ be an effective $\mathbb{R}$-Weil divisor on X such that (X,Δ) is klt. The equality $\mathrm{NNef}(K_X+\Delta)=\mathbf{B}_-(K_X+\Delta)$ (respectively $\mathrm{NNef}(K_X+\Delta)=\mathbf{B}_-(K_X+\Delta)=\mathbf{B}(K_X+\Delta)$ when $K_X+\Delta$ is big) has been proved in 2.8. Therefore we concentrate on the uniruledness of the irreducible components of these loci. If $K_X+\Delta$ is not pseudoeffective, then by [3] X is uniruled: in fact, considering a log resolution $f: Y\to X$ of (X,Δ), and an effective divisor Γ such that

$$K_Y+\Gamma=f^*(K_X+\Delta)+E,$$

with E effective f-exceptional, we have that $K_Y+\Gamma$ is not pseudoeffective, since E is f-exceptional and $f^*(K_X+\Delta)$ not pseudoeffective. As Γ is effective, K_Y is not pseudoeffective either, thus Y is uniruled and $X=\mathrm{NNef}(K_X+\Delta)=\mathbf{B}_-(K_X+\Delta)$ is uniruled too.

We now assume that $K_X+\Delta$ is pseudoeffective, and let V be an irreducible component of $\mathbf{B}_-(K_X+\Delta)$. By [5, Lemma 1.14] we have

$$\mathbf{B}_-(K_X+\Delta)=\bigcup_{A\text{ ample}}\mathbf{B}_+(K_X+\Delta+A),$$

thus there exists an ample $\mathbb{Q}$-divisor A such that V is a component of $\mathbf{B}_+(K_X+\Delta+A)$. Since A is ample, we may furthermore assume that $(X,\Delta+A)$ is klt. This reduces us to the following situation: assume that (X,Δ) is klt, $K_X+\Delta$ is big and let V be an irreducible component of $\mathbf{B}_+(K_X+\Delta)$. We are then to show that V is uniruled.

Consider a commutative diagram of birational maps

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & Z. & \end{tikzcd} \tag{3.1}$$

with $-(K_X + \Delta)$ π-ample, and either π is a divisorial contraction and π' is the identity, or π is a small contraction and ψ is its flip. Since $-(K_X + \Delta)$ is π-ample, we have $\mathrm{Exc}(\pi) \subset \mathbf{B}_+(K_X + \Delta)$. Indeed, $K_X + \Delta$ has negative degree along any curve C contracted by π, and therefore $C \subset \mathbf{B}_+(K_X + \Delta)$. If V is contained in $\mathrm{Exc}(\pi)$ it must therefore be one of its irreducible components, and it follows that V is uniruled by [7, Theorem 1]. Otherwise we may consider its strict transform V' on X', since ψ is in both cases an isomorphism away from $\mathrm{Exc}(\pi)$. If we denote by Δ' the strict transform of Δ on X' then (X', Δ') is klt and $K_{X'} + \Delta'$ is big. We claim that

$$V' \text{ is a component of } \mathbf{B}_+(K_{X'} + \Delta'). \tag{3.2}$$

Indeed consider a resolution of the indeterminancies of ψ

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ X \arrow[rr, dashed, "\psi"] & & X' \end{tikzcd} \tag{3.3}$$

which may be chosen such that μ (resp. μ') is an isomorphism above the generic point of V (resp. V'). Let F be the μ'-exceptional divisor on Y defined by the following equation:

$$\mu^*(K_X + \Delta) = \mu'^*(K_{X'} + \Delta') + F.$$

Notice that $-F$ is nef over X' (since it is nef over Z), thus $F \geqslant 0$ by the Negativity Lemma. The claim now follows by Proposition 2.3.

Since X is $\mathbb{Q}$-factorial, by [1, Corollary 1.4.2] there exists a finite composition of maps

$$\begin{aligned} &\psi_0 : X := X_0 \dashrightarrow X_1, \ldots, \psi_i : X_i \dashrightarrow X_{i+1}, \ldots, \psi_r : X_r \dashrightarrow X_{r+1} =: X_{\min} \\ &\quad \Delta_0 := \Delta,\ \Delta_i := (\psi_i)_* \Delta_{i-1},\ i = 1, \ldots, r, \end{aligned}$$

with

$$\begin{tikzcd} X_i \arrow[rr, dashed, "\psi_i"] \arrow[dr, "\pi_i"'] & & X_{i+1} \arrow[dl, "\pi_i'"] \\ & Z_i. & \end{tikzcd} \tag{3.4}$$

as in (3.4) such that $K_{X_{\min}} + \Delta_{\min}$ is nef at the final stage. We have two cases. If, for some $i = 0, \ldots, r$, the strict transform of V in X_i is contained in $\mathrm{Exc}(\pi_i)$, then it is uniruled by [7, Theorem 1]. Otherwise, by (3.2), the strict transform $V_{\min}$ of V inside $X_{\min}$ is a component of $\mathbf{B}_+(K_{X_{\min}} + \Delta_{\min})$. By the base point free theorem there exists a further birational morphism $\rho : X_{\min} \to W$ such that $K_{X_{\min}} + \Delta_{\min} = \rho^* A$ with A ample on W, and Proposition 2.3 shows that $\mathbf{B}_+(K_{X_{\min}} + \Delta_{\min}) = \mathrm{Exc}(\rho)$. Hence $V_{\min}$ is a component of $\mathrm{Exc}(\rho)$. We then conclude that $V_{\min}$ is uniruled as desired, by a final application of [7, Theorem 2].

3.2 Theorem A implies Corollary A

We argue as we did at the beginning of the proof of Proposition 2.8. As in the proof of Theorem A, we then have the flexibility to assume that D is big, upon adding to it a small multiple of an ample divisor.

Assume first that $-K_X$ is nef. We then have

$$\varepsilon D = K_X + (\varepsilon D - K_X)$$

and, for $\varepsilon > 0$ small enough, $\varepsilon D - K_X$ is numerically equivalent to a divisor Δ such that (X, Δ) is klt. Indeed we can write $D \equiv A + E$ where A is ample and E is effective, hence

$$\varepsilon D - K_X \equiv \varepsilon E + \varepsilon A - K_X,$$

where $\varepsilon A - K_X$ is ample and $(X, \varepsilon E)$ is klt for ε small enough. Since both $\mathbf{B}_-(D)$ and $\mathbf{B}_+(D)$ are invariant under scaling D we thus get the result by Theorem A applied to (X, Δ).

Now assume instead that $D - K_X =: N$ is nef. We can then write

$$\frac{1}{1-\varepsilon} D = K_X + N + \frac{\varepsilon}{1-\varepsilon} D$$

and $N + \frac{\varepsilon}{1-\varepsilon} D$ is numerically equivalent to a divisor Δ such that (X, Δ) is klt for $\varepsilon > 0$ small enough just as before, and Theorem A again implies the desired result after scaling D.

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