REMARKS ON MINIMAL MODELS OF DEGENERATIONS

S.BOUCKSOM

1. Birational uniqueness of minimal models

Theorem 1.1. *Let $X \to C$, $X' \to C$ be two flat projective morphisms to a germ of a smooth curve $0 \in C$, and let $\phi : X \dashrightarrow X'$ be a birational map over C inducing an isomorphism over $C \setminus \{0\}$.*

(i) *Assume that both pairs $(X, X_{0,\mathrm{red}})$ and $(X', X'_{0,\mathrm{red}})$ are lc, with relatively nef log canonical class. Then these pairs are crepant birational.*

(ii) *If we further assume that (X, X_0) is dlt (and hence X_0 is reduced), then ϕ is surjective in codimension one (i.e. ϕ^{-1} doesn't contract any divisor).*
In particular, if X_0 is irreducible, then ϕ is an isomorphism in codimension one.

The first assertion is well-known, see [KM98, 3.52], [dFKX12, Definition 15], [NX13, Theorem 2.2.6]. The minor observation we make is (ii), which is an elaboration of a remark of C. Favre.

Proof. Let Y be the normalization of the graph of ϕ, with its two projections $\pi : Y \to X$, $\pi' : Y \to X'$. Let K_Y be a given canonical Weil divisor on Y, and denote by K_X and $K_{X'}$ the corresponding canonical Weil divisors on X and X', respectively.

Let us first recall the proof of (i), which is an easy consequence of the negativity lemma. Since $(X, X_{0,\mathrm{red}})$ is lc, we have an equality of $\mathbb{Q}$-Weil divisors

$$K_Y + \pi_*^{-1} X_{0,\mathrm{red}} + E = \pi^*(K_X + X_{0,\mathrm{red}}) + F$$

where E is the reduced sum of all π-exceptional prime divisors and F is an effective π-exceptional $\mathbb{Q}$-Weil divisor.

Since ϕ is an isomorphism away from X_0, E is supported over X_0. We thus have

$$\pi_*^{-1} X_{0,\mathrm{red}} + E = Y_{0,\mathrm{red}},$$

the reduced special fiber of $Y \to C$, and hence

$$K_Y + Y_{0,\mathrm{red}} = \pi^*(K_X + X_{0,\mathrm{red}}) + F.$$

By symmetry, we also have $K_Y + Y_{0,\mathrm{red}} = \pi'^*(K_{X'} + X'_{0,\mathrm{red}}) + F'$, with F' an effective π'-exceptional $\mathbb{Q}$-Weil divisor.

Now introduce the $\mathbb{Q}$-Cartier divisor

$$D := F - F' = \pi'^*(K_{X'} + X'_{0,\mathrm{red}}) - \pi^*(K_X + X_{0,\mathrm{red}}).$$

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Our goal is to show that $D = 0$. On the one hand, since F is π-exceptional, we have $\pi_* D = -\pi_* F' \leq 0$. But D is also π-nef since $\pi'^*(K_{X'} + X'_{0,\mathrm{red}})$ is nef. By the negativity lemma, we obtain $D \leq 0$. By symmetry we get as desired $D = 0$.

(ii) Let E be an irreducible component of X'_0. Our goal is to show that E doesn't get contracted on X.

Since (X', X'_0) has log discrepancy 0 along the corresponding divisorial valuation ord_E, so does (X, X_0) by (i). As the latter pair is dlt, it follows that X_0 has simple normal crossings at the (scheme theoretic) center ξ of ord_E on X, and that ord_E is necessarily monomial with respect to the local equations at ξ of the irreducible components $E_1, ..., E_r$ of X_0 passing through ξ. It is thus enough to prove that $r = 1$. But we have

$$1 = \mathrm{ord}_E(X'_0) = \mathrm{ord}_E(Y'_0) = \mathrm{ord}_E(X_0) = \sum_{i=1}^{r} \mathrm{ord}_E(E_i)\,\mathrm{ord}_{E_i}(X_0),$$

hence the claim since $\mathrm{ord}_E(E_i)$ and $\mathrm{ord}_{E_i}(X_0)$ are positive integers. □

Remark 1.2. If X and X' are $\mathbb{Q}$-factorial with (X, X_0) and (X', X'_0) dlt, (ii) shows that ϕ is an isomorphism in codimension one, and the argument of [Kaw08] then yields that ϕ factors into a sequence of flops.

2. Uniqueness of polarized limits

The following result implies in particular the separatedness of the moduli functor of polarized varities with log terminal singularities and nef canonical class, a fact which is undoubtedly well-known to experts.

Theorem 2.1. *Let $0 \in C$ be a germ of smooth curve and $(X, L) \to C$, $(X', L') \to C$ be two polarized families (i.e. a flat projective morphism with connected fibers together with a relatively ample line bundle). Assume that:*

- (i) *X is $\mathbb{Q}$-Gorenstein, K_X is relatively nef, and all fibers of X are log terminal;*
- (ii) *X' is $\mathbb{Q}$-Gorenstein, $K_{X'}$ is relatively nef, and all fibers of X' are log terminal, except perhaps the central fiber X'_0, which is allowed to be slc.*

Then any isomorphism of the the two polarized families over $C \setminus \{0\}$ automatically extends to an isomorphism over C.

The result is false as soon as X_0 is reducible and, say, X is $\mathbb{Q}$-factorial. Indeed, it is then possible to perturb L by a 'small' non-numerically trivial $\mathbb{Q}$-divisor supported on X_0.

Remark 2.2. Note that Theorem 2.1 does not directly follow from [Ale06], since polarizations are only defined up to linear equivalence.

Remark 2.3. Odaka proved in [Oda12] that every polarized variety (X, L) such that X is klt with nef canonical class is K-stable. The result is in fact only stated for K_X numerically trivial, but the proof also covers the nef case. Theorem 1.1 therefore fits with the conjectural result of [OT13], which states that the moduli functor of K-stable polarized varieties should be separated.

We first recall the following well-known fact, of common use in the MMP.

Lemma 2.4. *Let X, X' be normal varieties over a base S, and $\phi : X \dashrightarrow X'$ be birational map over S which is an isomorphism in codimension one. Let H be a relatively ample $\mathbb{Q}$-divisor on X whose strict transform H' is $\mathbb{Q}$-Cartier and relatively ample on X'. Then ϕ is an isomorphism.*

Proof. Let Y be a desingularization of the graph of ϕ, with its two projections $\pi : Y \to X$ and $\pi' : Y \to X'$. Then $D := \pi^* H - \pi'^* H'$ is exceptional and π-nef, hence $D \le 0$ by the negativity lemma. By symmetry we get $D = 0$, which yields an isomorphism of algebras of sections

$$R(X/S, H) \simeq R(Y/S, \pi^* H) = R(Y/S, \pi'^* H') \simeq R(X'/S, H'),$$

and hence

$$X = \operatorname{Proj}_S R(X/S, H) \simeq \operatorname{Proj}_S R(X'/S, H') = X'.$$

□

Proof of Theorem 2.1. By inversion of adjunction [Kol97, Theorem 7.5], the assumption that X_0 is log terminal implies (in fact, is equivalent to) (X, X_0) being dlt. Similarly, X_0' slc implies that (X', X_0') is lc.

By (ii) in Theorem 1.1, we thus get that the birational map $\phi : X \dashrightarrow X'$ is an isomorphism in codimension one.

According to Lemma 2.4, it remains to see that $\phi_* L$ is $\mathbb{Q}$-Cartier and relatively ample. As a $\mathbb{Q}$-Weil divisor class, $\phi_* L$ is $\mathbb{Q}$-linearly equivalent to L' away from X_0' by assumption, so that $\phi_* L$ is $\mathbb{Q}$-linearly equivalent as a Weil divisor class to $L' + D$, with D a $\mathbb{Q}$-Weil divisor supported on X_0'. But since X_0' is assumed to be irreducible, it follows that D is a rational multiple of X_0', which shows as desired that $\phi_* L$ is $\mathbb{Q}$-Cartier and relatively ample.

□

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CNRS-Université Pierre et Marie Curie, Institut de Mathématiques de Jussieu, F-75251 Paris Cedex 05, France
E-mail address: boucksom@math.jussieu.fr