SINGULARITIES OF PLURISUBHARMONIC FUNCTIONS AND MULTIPLIER IDEALS

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Contents

Introduction

These notes are based on second year Master course I delivered in Orsay in 2015–2017.

1. Subharmonic functions

We use [DemAG] as a reference.

1.1. **The Green-Riesz representation formula.** Denote by

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

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the usual (negative definite) Laplacian on $\mathbb{R}^d$, and recall that a fundamental solution of Δ is given by the *Newton kernel*

$$N_d(x) = \begin{cases} \frac{1}{2}|x| \text{ for } d=1; \\ \frac{1}{2\pi}\log|x| \text{ for } d=2; \\ \frac{1}{(2-d)|S^{d-1}|}|x|^{2-d} \text{ for } d\geq 3, \end{cases} \tag{1.1}$$

where

$$|S^{d-1}| = d|B^d| = d\pi^{d/2}/(d/2)!$$

is the area of the unit sphere. Observe that $N_r(x)$ only depends on $r=|x|$, with

$$\frac{\partial}{\partial r}N_d = \frac{1}{|S^{d-1}|r^{d-1}}.$$

Given a smoothly bounded domain $\Omega \Subset \mathbb{R}^d$ and $f \in C^\infty(\overline{\Omega})$, $v \in C^\infty(\partial\Omega)$, , the Dirichlet problem

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = v \end{cases} \tag{1.2}$$

admits a unique solution $u \in C^\infty(\overline{\Omega})$ (uniqueness being a simple consequence of the maximum principle). The resulting isomorphism

$$C^\infty(\overline{\Omega}) \oplus C^\infty(\partial\Omega) \simeq C^\infty(\overline{\Omega})$$

gives rise to two kernels, the *Green kernel*

$$G_\Omega : \overline{\Omega} \times \overline{\Omega} \to [-\infty, 0]$$

and the *Poisson kernel*

$$P_\Omega : \overline{\Omega} \times \partial\Omega \to \mathbb{R}$$

with the property that

$$u(x) = \int_\Omega G_\Omega(x,y)\Delta u(y) + \int_{\partial\Omega} P_\Omega(x,y)u(y)d\sigma(y) \tag{1.3}$$

for all $u \in C^\infty(\overline{\Omega})$, with $d\sigma(y)$ the Lebesgue measure $\partial\Omega$. The Green kernel G_Ω is the unique solution of

$$\begin{cases} \Delta_x G_\Omega(x,y) = \delta_y \\ G_\Omega(\cdot, y)|_{\partial\Omega} \equiv 0. \end{cases} \tag{1.4}$$

It is symmetric in (x,y), smooth outside the diagonal of $\overline{\Omega} \times \overline{\Omega}$, and satisfies

$$G_\Omega(x,y) = N_d(x-y) \bmod C^\infty$$

on $\Omega \times \Omega$. The Poisson kernel P_Ω is smooth, and $P_\Omega(x,y)d\sigma(y)$ is a probability measure on $\partial\Omega$ for all $x \in \partial\Omega$.

Example 1.1. The Poisson kernel of the open ball $B(0,r)$ is given by

$$P_r(x,y) = \frac{1}{|S^{d-1}|r}\frac{r^2-|x|^2}{|x-y|^d}.$$

The unique solution of (1.4) on $B(0,r)$ with $y=0$ is given by

$$G_r(x,0) = N_d(x) - N_d(re_1) = \frac{1}{|S^{d-1}|}\int_r^{|x|} \frac{dt}{t^{d-1}}.$$

Injecting this in (1.3) and translating at a given $a \in \mathbb{R}^d$, we get the fundamnetal *Gauss-Jensen formula.*

Corollary 1.2. *For each $u \in C^\infty(\bar{B}(a,r))$, the difference between $u(a)$ and the mean value $⨍_{S(a,r)} u$ on the sphere $S(a,r)$ is expressed as*

$$⨍_{S(a,r)} u - u(a) = \frac{1}{|S^{d-1}|}\int_0^r \frac{dt}{t^{d-1}} \int_{B(a,t)} \Delta u. \tag{1.5}$$

1.2. **Subharmonic functions.** In what follows, $\Omega \subset \mathbb{R}^d$ denotes any open subset.

Definition 1.3. A function $u : \Omega \to [-\infty, +\infty)$ is *subharmonic* if:

(a) u is usc, not identically $-\infty$ on any connected component of Ω;
(b) for each ball $\bar{B}(a,r) \subset \Omega$, the mean value inequality $u(a) \le ⨍_{S(a,r)} u$ holds.

Note that u is locally bounded above by (a), so that $\int_{S(a,r)} u\, d\sigma$ makes sense in $[-\infty, +\infty)$. Since

$$\int_{B(a,r)} u(x)dx = \int_0^r dt \int_{S(a,t)} u(y)d\sigma(y), \tag{1.6}$$

(b) implies $u(a) \le ⨍_{B(a,r)} u$. Using this, it is easy to see that u is locally integrable.

Example 1.4. The Gauss-Jensen formula (1.5) directly shows that a smooth function u is subharmonic if and only if $\Delta u \ge 0$.

Proposition 1.5. *Subharmonic functions satisfy the following properties.*

(i) *For any family (u_α) of subharmonic functions on Ω, locally bounded above, the usc regularization $(\sup_\alpha u_\alpha)^\star$ is subharmonic.*
(ii) *If $u_1, \dots, u_r$ are subharmonic on Ω, then $\chi(u_1, \dots, u_r)$ is subharmonic for any convex function χ on $\mathbb{R}^r$ that is non-decreasing in each variable.*
(iii) *If u is subharmonic on Ω and ν is a probability measure supported on a compact set $K \subset \mathbb{R}^d$, then $u \star \nu$ is subharmonic on any open set U such that $U + K \subset \Omega$.*

Pick a nonnegative radial function $\rho \in C_c^\infty(\mathbb{R}^d)$, compactly supported in the unit ball and such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$, i.e. $\rho(x) = \tilde{\rho}(|x|)$ with $|S^{d-1}| \int_0^1 \tilde{\rho}(r) r^{d-1} dr = 1$, and consider the associated approximation of unity

$$\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\varepsilon^{-1}x) \tag{1.7}$$

We then have the following key monotone regularization result.

Theorem 1.6. *Let u be subharmonic function on Ω. Then $u \star \rho_\varepsilon$ is smooth and subharmonic on $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$, and decreases to u as $\varepsilon \to 0$.*

Proof. Since ρ_ε is an approximation of unity, $u \star \rho_\varepsilon$ is smooth on Ω_ε, $u \star \rho_\varepsilon \to u$ in L^1_{loc}, and $\limsup_{\varepsilon \to 0} u \star \rho_\varepsilon \le u$, u being usc. Noting that

$$(u \star \rho_\varepsilon)(a) = |S^{d-1}| \int_0^1 \tilde{\rho}(r) r^{d-1} dr \fint_{S(a,\varepsilon r)} u \, d\sigma, \tag{1.8}$$

the mean value inequality yields $u \le u \star \rho_\varepsilon$. By Proposition 1.5, $u \star \rho_\varepsilon$ is subharmonic, and it therefore remains to show that $\varepsilon \mapsto u \star \rho_\varepsilon$ is non-decreasing. When u is smooth, it follows from (1.8) and the Gauss-Jensen formula (Corollary 1.2), which shows that $\fint_{S(a,r)} u$ is a non-decreasing function of r. In the general case, $u \star \rho_\delta$ is smooth and subharmonic on Ω_δ for any $\delta > 0$. By the smooth case, $\varepsilon \mapsto (u \star \rho_\varepsilon) \star \rho_\delta = (u \star \rho_\delta) \star \rho_\varepsilon$ is thus non-decreasing, and hence so is $\varepsilon \mapsto u \star \rho_\varepsilon$ in the limit as $\delta \to 0$. □

Corollary 1.7. *If u is subharmonic, then $\Delta u \ge 0$ in the sense of distributions. Conversely, if $v \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\Delta v \ge 0$, then there exists a unique subharmonic function u on Ω with $u = v$ a.e.*

Using the monotone regularization procedure of Theorem 1.6, one easily extends the Green-Riesz representation formula as follows.

Corollary 1.8. *Any subharmonic function u defined in a neighborhood of the closure of a smooth bounded domain $\Omega \Subset \mathbb{R}^d$ is integrable on $\partial\Omega$, and*

$$u(x) = \int_\Omega G_\Omega(x,y)(\Delta u)(y) dy + \int_{\partial\Omega} P_\Omega(x,y) u(y) d\sigma(y).$$

for all $x \in \Omega$.

By the maximum principle, we have $G_\Omega \le 0$, which yields the generalized submean value inequality

$$u(x) \le \int_{\partial\Omega} P_\Omega(x,y) u(y) d\sigma(y)$$

for $x \in \Omega$. In particular, a subharmonic function u defined on a neighborhood of $\bar{B}(0,1)$ satisfies

$$u(x) \le \frac{1}{|S^{d-1}|} \int_{S(0,1)} \frac{1 - |x|^2}{|x-y|^d} u(y) d\sigma(y),$$

for all $x \in B(0,1)$. After translation and scaling, this implies the following basic *Harnack inequality.*

Corollary 1.9. *For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, d)$ such that*

$$\sup_{B(a,(1-\varepsilon)r)} u \le C \fint_{S(a,r)} u.$$

for any non-positive subharmonic function $u \le 0$ defined on a neighborhood of a ball $\bar{B}(a,r)$.

1.3. **Regularity properties.** As we next show, every subharmonic function belongs to the Sobolev space $W^{1,1+\varepsilon}$ for $0 < \varepsilon \ll 1$.

Proposition 1.10. *Let u be a subharmonic function on an open subset $\Omega \subset \mathbb{R}^d$. Then $u \in L^p_{\mathrm{loc}}$ for $1 \le p < \frac{d}{d-2}$ and $\nabla u \in L^p_{\mathrm{loc}}$ for $1 \le p < \frac{d}{d-1}$.*

When $d = 2$, e^{-cu} is even locally integrable near any given $a \in \Omega$ for all $c > 0$ small enough. More precisely, this holds as soon as $c^{-1} > \frac{1}{\pi}\Delta u(\{a\})$.

Proof. A direct check shows that the Newton kernel N_d satisfies these properties. Let $\chi \in C^\infty_c(\Omega)$ be a cut-off function with $\chi \equiv 1$ on a neighborhood U of a given $a \in \Omega$, and consider the compactly supported positive measure $\mu := \chi\Delta u$. On U, the function $v - N_d \star \mu$ is harmonic, and hence smooth, and it is thus enough to show the result for $N_d \star \mu$. For the L^p conditions, this follows directly from Lemma 1.11 below, by homogeneity of $\chi(t) = t^p$. Assume now that $d = 2$, and pick $c > 0$ with $c^{-1} > \frac{1}{\pi}\Delta u(\{a\})$. We may then choose $\operatorname{supp}\chi$ such that the total mass m of μ satisfies $(cm)^{-1} > \frac{1}{\pi}$. Applying Lemma 1.11 with $\chi(t) = e^{-cmt}$ and $\nu = m^{-1}\mu$, we get as desired that

$$\exp\left(-(cm)N_d \star (m^{-1}\mu)\right) = \exp(-cN_d \star \mu)$$

is integrable. ☐

Lemma 1.11. *Let $\chi : [0, +\infty) \to [0, +\infty)$ be a convex, nondecreasing function, f a measurable function on $\mathbb{R}^d$, ν a compactly supported probability measure on $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} \chi(|f \star \nu|)\, dx \le \int_{\mathbb{R}^d} \chi(|f|)\, dx.$$

Proof. Since χ is non-decreasing and convex, Jensen's inequality yields

$$\chi\left(|f \star \nu|(x)\right) \le \chi\left(\int |f(x-y)| d\nu(y)\right) \le \int \chi\left(|f(x-y)|\right) d\nu(y),$$

and hence

$$\int \chi(|f \star \nu|(x)) dx \le \int d\nu(y) \int \chi(|f(x-y)|) dx = \int \chi(|f|)(x) dx.$$

☐

2. Plurisubharmonic functions

2.1. **First properties.** Let $\Omega \subset \mathbb{C}^n$ be an open subset.

Definition 2.1. A function $u : \Omega \to [-\infty, +\infty)$ is *plurisubharmonic* (*psh* for short) if u is usc, not identically $-\infty$ on any connected component of Ω, and the restriction of u to each complex line is subharmonic, i.e.

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + \xi e^{i\theta}) d\theta$$

for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ such that $\{a + z\xi \mid z \in \bar{\mathbb{D}}\} \subset \Omega$.

Proposition 2.2. *Plurisubharmonic functions satisfy the following properties.*

(i) *If u is psh, then it is also subharmonic as a function of $2n$ real variables.*

(ii) *If u is smooth, then u is psh iff its complex Hessian $\left(\partial^2 u/\partial z_j \partial \bar{z}_k\right)_{1\le j,k\le n}$ is pointwise semipositive. Equivalently, $i\partial\overline{\partial} u \ge 0$.*
(iii) *For any family (u_α) of psh functions on Ω, locally bounded above, the usc regularization $(\sup_\alpha u_\alpha)^\star$ is psh.*
(iv) *If $u_1, \dots, u_r$ are psh on Ω, then $\chi(u_1, \dots, u_r)$ is psh for any convex function χ on $\mathbb{R}^r$ that is non-decreasing in each variable.*
(v) *If u is psh on Ω and ν is a probability measure supported on a compact set $K \subset \mathbb{R}^d$, then $u \star \nu$ is subharmonic on any open set U such that $U + K \subset \Omega$.*
(vi) *If u is psh and ρ_ε is a radially symmetric approximation of unity as in (1.7), then $u \star \rho_\varepsilon$ is smooth, psh, and decreases pointwise to u as $\varepsilon \searrow 0$.*
(vi) *if u is psh on Ω and $f : \Omega' \to \Omega$ is holomorphic on an open subset of $\mathbb{C}^p$, then $u \circ f$ is psh, or $u \circ f \equiv -\infty$ on some component of Ω'.*

The above properties follow from those already established for subharmonic functions, except for the key new property (vi). The latter is true in the smooth case by (ii), and the general case follows by regularization. As a result, psh functions make sense on any complex manifold.

2.2. **Convexity properties.** The class of psh functions contains that of convex functions, as follows:

Lemma 2.3. *Let $\omega \subset \mathbb{R}^n$ be a convex open subset. A function f on ω is then convex if and only if $f(x+iy)$ is a psh function of $z = x + iy \in \omega + i\mathbb{R}^n$.*

Proof. This follows from the Hessian criterion when f is smooth, and the general case is obtained by regularizing f. □

As a consequence, we get the following generalized Hadamard three-circle theorem:

Corollary 2.4. *Let u be any psh function on an open subset Ω of $\mathbb{C}^n$ and $a \in \Omega$. Let also $\rho_r(z) = r^{-2n}\rho(r^{-1}z)$ be a radial approximation of unity. The following quantities are then convex as functions of $\log r$:*

(i) $\sup_{B(a,r)} u$;
(ii) $⨍_{S(a,r)} u$;
(iii) $⨍_{B(a,r)} u$;
(iv) $(u \star \rho_r)(a)$.

Proof. After scaling, we assume for simplicity that $\bar{B}(a,1)$ is contained in Ω, so that $f(t) = \sup_{B(a,e^t)} u$ and $g(t) = ⨍_{S(a,e^t)} u$ are both defined on $\mathbb{R}_-$. By (1.6) and (1.8), $⨍_{B(a,e^t)} u$ and $(u \star \rho_{e^t})(a)$ are both of the form

$$\int_{\mathbb{R}_+} g(t-s)\,d\mu(s) \tag{2.1}$$

with μ a probability measure on $\mathbb{R}_+$, and it is thus enough to show that f and g are convex. For $\xi \in \bar{B}(0,1)$, $z \mapsto u(a + e^z\xi)$ is a subharmonic function on the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, and hence so are

$$\sup_{B(a,e^z)} u = \sup_{\xi \in \bar{B}(0,1)} u(a + e^z \xi)$$

and

$$\fint_{S(a,e^z)} u = \frac{1}{|S^{2n-1}|} \int_{S^{2n-1}} u(a + e^z \xi) d\sigma(\xi).$$

Being independent of $\operatorname{Im} z$, they are therefore convex functions of $\operatorname{Re} z$. $\square$

When u is continuous, the various means of u considered in Corollary 2.4 all converge to $u(a)$ as $r \to 0$. As observed in [BK07, Lemma 5], they remain equivalent as $r \to 0$ when u is merely locally bounded:

Lemma 2.5. *Assume further that u is locally bounded in Corollary 2.4. Then the difference between any two of the quantities (i)—(iv), as well as those obtained by replacing r with by cr for some $c > 0$, all tend to 0 as $r \to 0$, locally uniformly with respect to $a \in \Omega$:*

This fact will later allow us to easily compare the regularization by convolution in various coordinate charts, which is the key point in the Blocki–Kolodziej extension of the Richberg regularization procedure, cf. §3.2.

Proof. Using the above notation, we first claim that there exists $C > 0$ such that

$$0 \le f(t) - g(t) \le C\left(f(t) - f(t-1)\right). \tag{2.2}$$

for all $t \in \mathbb{R}_-$. Indeed, $u - f(t)$ is a non-positive subharmonic function on $B(a, e^t)$, and the Harnack inequality of Corollary 1.9 therefore shows that

$$g(t) - f(t) = \fint_{S(a,e^t)} (u - f(t))$$

$$\ge C \sup_{B(a,e^{t-1})} (u - f(t)) = C(f(t-1) - f(t)),$$

which proves the claim. By convexity of f, we have for all $t \in \mathbb{R}_-$ and $s > 0$

$$\frac{f(t) - f(t-s)}{s} \le \frac{f(0) - f(t-s)}{-t+s}.$$

Since u is locally bounded, we infer

$$0 \le f(t) - f(t-\delta) \le C\frac{s}{-t+s} \tag{2.3}$$

for some constant $C > 0$, which can be chosen locally uniformly with respect a. Combined with (2.2), this shows that

$$0 \le f(t) - g(t) \le \frac{C}{-t+1},$$

and hence

$$f(t) = g(t) + o(1) = f(t-s) + o(1)$$

as $t \to -\infty$. We have seen that $⨍_{B(a,e^t)} u$ and $(u \star \rho_{e^t})(a)$ are both of the form $\int_{\mathbb{R}_+} g(t-s)d\mu(s)$ with μ a probability measure on $\mathbb{R}_+$. The above estimates yield

$$0 \le f(t) - \int_{\mathbb{R}_+} g(t-s)d\mu(s) \le C \int_{\mathbb{R}_+} \alpha_t(s)d\mu(s).$$

with $\alpha_t(s) := \frac{s}{-t+s} + \frac{1}{-t+s+1}$. Since $0 \le \alpha_t(s) \le 2$ and $\lim_{t\to-\infty} \alpha_t(s) = 0$ for all $s \in \mathbb{R}_+$, dominated convergence yields $f(t) - \int_{\mathbb{R}_+} g(t-s)d\mu(s) \to 0$ for $t \to -\infty$, and we are done. □

2.3. **Lelong numbers.** In what follows, u still denotes a psh function on an open subset $\Omega \subset \mathbb{C}^n$. Pick $a \in \Omega$ and $r > 0$ with $\bar{B}(a,r) \subset \Omega$. By convexity of $f(t) := \sup_{B(a,e^t)} u$ on $(-\infty, \log r]$,

$$\frac{f(t) - f(\log r)}{t - \log r} \ge 0$$

is a non-decreasing function of t, and $f(t)/t$ therefore admits a limit in $\mathbb{R}_+$ as $t \to -\infty$. In other words,

$$\nu_a(u) := \lim_{r\to 0} \frac{\sup_{B(a,r)} u}{\log r}$$

exists is $\mathbb{R}_+$, and is called the *Lelong number* of u at a. By convexity of f, we further have

$$f(t) - f(\log r) \le (t - \log r)\left(\lim_{t'\to-\infty} \frac{f(t') - f(\log r)}{t' - \log r}\right) = (t - \log r)\nu_a(u).$$

for $t \le \log r$. In other words,

$$u(z) \le \nu_a(u) \log \frac{|z-a|}{r} + \sup_{B(a,r)} u \tag{2.4}$$

for all $z \in B(a,r)$. It follows that

$$\nu_a(u) = \max\{\gamma \in \mathbb{R}_+ \mid u(z) \le \gamma \log|z-a| + O(1) \text{ near } a\}. \tag{2.5}$$

which intuitively says that $\nu_a(u)$ is the 'vanishing order' at a of e^u, and hence measures the 'size' of the singularity of u at a. Using the estimate (2.4), it is easy to establish:

Lemma 2.6. *The Lelong number $\nu_a(u)$ is usc with respect to $a \in \Omega$, and also with respect the weak (equivalently, L^1_{loc}) topology of psh functions. In particular, if (u_j) is an increasing sequence of psh functions converging a.e. to a psh function u, then $\nu_a(u_j) \searrow \nu_a(u)$.*

While a locally bounded psh function u has zero Lelong numbers, the converse is far from true.

Example 2.7. For each convex, non-decreasing function $\chi : \mathbb{R}_- \to \mathbb{R}$, $u(z) = \chi(\log|z|)$ is subharmonic on the unit disc $\mathbb{D} \subset \mathbb{C}$, with

$$\nu_0(u) = \chi'(-\infty) = \lim_{t\to-\infty} \frac{\chi(t)}{t}.$$

If $\chi(t) = -\log(-t)$ or $(-t)^\alpha$ with $0 < \alpha < 1$, then $u(0) = -\infty$ but $\nu_0(u) = 0$.

Recall from Corollary 2.4 that $\fint_{S(a,r)} u$ is a convex function of $\log r$.

Lemma 2.8. *For each psh function u on Ω and $a \in \Omega$, we have*

$$\nu_a(u) = \lim_{r \to 0} \frac{\fint_{S(a,r)} u}{\log r}.$$

Proof. The Harnack inequality of Corollary 1.9 shows that

$$0 \le \frac{\fint_{S(a,r)} u - \sup_{B(a,r)} u}{\log r} \le C \frac{\sup_{B(a,r/2)} u - \sup_{B(a,r)} u}{\log r} \tag{2.6}$$

for $0 < r \ll 1$. The right-hand side clearly tends to 0, hence the result. □

Remark 2.9. Arguing as in the proof of Lemma 2.5, one similarly checks that

$$\nu_a(u) = \lim_{r \to 0} \frac{\fint_{B(a,r)} u}{\log r} = \lim_{r \to 0} \frac{(u \star \rho_r)(a)}{\log r}.$$

Corollary 2.10. *For any two psh functions u, v we have*

$$\nu_a(\max\{u, v\}) = \min\{\nu_a(u), \nu_a(v)\}$$

and

$$\nu_a(u + v) = \nu_a(u) + \nu_a(v).$$

The next result provides an alternative characterization of $\nu_a(u)$ as a limit density. Set $\omega := \frac{i}{2}\partial\overline{\partial}|z|^2$, which is the Kähler form associated to the Euclidian metric on $\mathbb{C}^n$. The Lebesgue measure of $\mathbb{C}^n$ is then equal to $\omega^n/n!$. Since u is psh, $T := \frac{i}{\pi}\partial\overline{\partial}u$ is a closed positive $(1,1)$-current, and $\sigma_T := T \wedge \frac{\omega^{n-1}}{(n-1)!}$ is thus a positive measure on Ω, called the *trace measure* of T. The choice of normalization for T comes from the following special case:

Example 2.11. When $u = \log|z_1|$, the trace measure $\sigma_T = \sigma_H$ coincides with the Lebesgue measure on the hyperplane $H = (z_1 = 0)$. For $n = 1$, this amounts to the well-known formula $\frac{i}{\pi}\partial\overline{\partial}\log|z| = \delta_0$. Pulling-back by the projection $(z_1, \dots, z_n) \mapsto z_1$ shows that $\frac{i}{\pi}\partial\overline{\partial}\log|z_1|$ is the integration current δ_H on H (a special case of the Lelong-Poincaré formula), and $\sigma_H = \delta_H \wedge \frac{\omega^{n-1}}{(n-1)!}$ is thus the Lebesgue measure on H.

We are now in a position to decribe $\nu_a(u)$ as a ratio of trace measures:

Proposition 2.12. *If u is a psh function and $T = \frac{i}{\pi}\partial\overline{\partial}u$, then*

$$\frac{\sigma_T\left(B(a,r)\right)}{\sigma_H\left(B(0,r)\right)} \searrow \nu_a(u)$$

as $r \searrow 0$.

Proof. Set $f(r) := \sigma_T\left(B(a,r)\right)/\sigma_H\left(B(0,r)\right)$. Since $\omega^n/n!$ is the Lebesgue measure on $\mathbb{C}^n$, the Gauss-Jensen formula of Corollary 1.2 yields

$$\fint_{S(a,r)} u - u(a) = \frac{1}{|S^{2n-1}|} \int_0^r \frac{dt}{t^{2n-1}} \int_{B(a,t)} \Delta u \frac{\omega^n}{n!}.$$

Using

$$f(r) = \frac{(n-1)!}{(\pi r^2)^{n-1}} \int_{B(a,r)} T \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{\pi^{n-1} r^{2n-2}} \int_{B(a,r)} T \wedge \omega^{n-1},$$

$$i\partial\overline{\partial} u \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \Delta u \frac{\omega^n}{n!},$$

and $|S^{2n-1}| = 2n|B^{2n}| = 2\pi^n/(n-1)!$, we get

$$\fint_{S(a,r)} u - u(a) = \int_0^r f(t) \frac{dt}{t}. \tag{2.7}$$

Since $\fint_{S(a,r)} u$ is a convex function of $\log r$, it follows that f is non-decreasing, with

$$\lim_{r\to 0} f(r) = \lim_{r \to 0} \frac{\fint_{S(a,r)} u}{\log r},$$

and we conclude by Lemma 2.8. □

Remark 2.13. When $n = 1$, Proposition 2.12 says that $\nu_a(u) = \frac{1}{2\pi} \Delta u(\{a\})$, and Proposition 1.10 can thus be rewritten as

$$\nu_a(u) < 1 \Longrightarrow e^{-u} \in L^2_{\mathrm{loc}} \text{ near } a.$$

The converse is also true, and follows immediately from (10.1). By a nontrivial result originally due to Skoda, the above implication remains valid for all n, cf. Corollary 9.18.

3. Regularization of quasi-psh functions

3.1. **Quasi-psh functions and singular metrics.** As mentioned above, psh functions make sense on any complex manifold X.

Definition 3.1. A usc function $\varphi : X \to [-\infty, +\infty)$ on a complex manifold X is *quasi-psh* if it is locally of the form $\varphi = u + f$ where u is psh and f is smooth.

It will be convenient to introduce the real operator $dd^c := \frac{i}{\pi}\partial\overline{\partial}$ (the role of the normalization factor π will become apparent below). By the Poincaré lemma, any closed, real $(1,1)$-form θ on X is locally of the form $\theta = dd^c f$ for a smooth real-valued function f, called a *local potential* of θ. In particular, θ is a Kähler form if and only if its local potentials are strictly psh.

Definition 3.2. A quasi-psh function φ is *θ-psh* if $\theta + dd^c\varphi \geq 0$ in the sense of currents; equivalently, $f + \varphi$ is (locally) psh for any choice of local potentials f for θ.

Let now L be a Hermitian holomorphic line bundle on X. The *curvature form* of L is the closed, real $(1,1)$-form θ locally defined by $\theta := -dd^c \log|\tau|$ for any local trivializing (holomorphic) section τ of L. This $(1,1)$-form is indeed globally well-defined, as any other local trivializing section is of the form $\tau' = u\tau$ with $u \in \mathcal{O}_X^\star$, and $dd^c \log|\tau'| = dd^c \log|\tau|$ since $\log|u|$ pluriharmonic.

This definition of the curvature form θ is simply a concrete description of the curvature form of the Chern connection of L, normalized so that θ represents the first Chern class of L in de Rham cohomology, i.e. the image in $H^2(X,\mathbb{R})$ of $c_1(L)\in H^2(X,\mathbb{Z})$.

Any other Hermitian metric on L is of the form $|\cdot|'=|\cdot|e^{-\varphi}$ with $\varphi\in C^\infty(X)$, and the curvature forms are related by $\theta'=\theta+dd^c\varphi$. The data of a θ-psh function φ can thus be interpreted as that of a singular metric $|\cdot|e^{-\varphi}$ on L with semipositive curvature current $\theta+dd^c\varphi$.

3.2. **Local regularization of quasi-psh functions.** Local regularization of quasi-psh functions is easily achieved by convolution with a smoothing kernel. As above, choose a radially symmetric, nonnegative function $\rho\in C_c^\infty(\mathbb{C}^n)$ such that $\int_{\mathbb{C}^n}\rho=1$, and consider the corresponding approximation of unity (ρ_j), defined by

$$\rho_j(z):=j^{2n}\rho(jz).$$

Theorem 3.3. *Let X be a complex manifold equipped with a reference positive $(1,1)$-form ω. Let θ be a closed, real $(1,1)$-form, and φ be a θ-psh function on X. Pick local holomorphic coordinates z centered at a given point $a\in X$, and denote by $\varphi\star_z\rho_j$ the (locally defined) smooth function obtained by convolution in these coordinates. Then the following holds:*

(i) *$\varphi\star_z\rho_j+c_j$ decreases pointwise to φ for some sequence $c_j\to 0_+$;*
(ii) *$\theta+dd^c(\varphi\star_z\rho_j)\ge-\varepsilon_j\omega$ for some sequence $\varepsilon_j\to 0_+$;*
(iii) *if φ is further locally bounded and z' is any other local coordinate system centered at a, then $\varphi\star_z\rho_j-\varphi\star_{z'}\rho_j$ tends to 0 locally uniformly.*

Proof. Let f be a local potential of θ near a. Since $u:=f+\varphi$ is psh, Proposition 2.2 shows that $u\star_z\rho_j$ is psh and decreases pointwise to u. As f is smooth, $f\star_z\rho_j\to f$ in C^∞ topology. In particular, $f\star_z\rho_j\to f$ locally uniformly, and

$$-\varepsilon_j\omega\le dd^c(f\star_z\rho_j-f)\le\varepsilon_j\omega$$

with $\varepsilon_j\to 0_+$, which proves (i) and (ii). Choose another local coordinate system z' centered at a, and denote by $B(a,r)$ and $B'(a,r)$ the corresponding Euclidian balls. Since z and z' are locally Lipschitz equivalent, we have

$$B(a,C^{-1}\varepsilon)\subset B'(a,\varepsilon)\subset B(a,C\varepsilon)$$

for some locally uniform constant $C>0$, and hence

$$\sup_{B(a,C^{-1}\varepsilon)}u\le\sup_{B'(a,\varepsilon)}u\le\sup_{B(a,C\varepsilon)}u.$$

By Lemma 2.5, we have

$$(u\star_{z'}\rho_\varepsilon)(a)=\sup_{B'(a,\varepsilon)}u+o(1)$$

and

$$(u\star_z\rho_\varepsilon)(a)=\sup_{B(a,C^{-1}\varepsilon)}u+o(1)=\sup_{B(a,C\varepsilon)}u+o(1)$$

locally uniformly with respect to a. As a result, $u\star_z\rho_\varepsilon-u\star_{z'}\rho_\varepsilon\to 0$ locally uniformly. By continuity of f, we also have $f\star_z\rho_\varepsilon-f\star_{z'}\rho_\varepsilon\to 0$ locally uniformly, hence (iii). $\square$

3.3. **Global regularization of bounded quasi-psh functions.** A classical result of Richberg [Ric68] (see also [DemAG, §5.E]) states that a continuous θ-psh φ on a complex manifold X can be written as the locally uniform limit of a sequence of smooth functions $\varphi_j \in C^\infty(X)$ such that the negative part of $\theta + dd^c\varphi_j$ tends to 0 on compact sets.

The following extension to the case where φ is merely locally bounded was initially obtained as a consequence of deep regularization results due to Demailly [Dem92], before Blocki and Kolodziej realized in [BK07] that it can in fact be proved along the lines of the Richberg argument. The importance of passing from the continuous to the locally bounded will be illustrated by Theorem 3.8 below.

Theorem 3.4. *Let X be a complex manifold, equipped with a reference positive $(1,1)$-form ω. Given a* locally bounded *quasi-psh function φ on X, there exists a sequence $\varphi_j \in C^\infty(X)$ with the following properties:*

(i) *$\varphi_j \searrow \varphi$ pointwise on X;*
(ii) *for each closed, real $(1,1)$-form θ such that $\theta + dd^c\varphi \geq 0$ and each relatively compact open $U \Subset X$, there exists $\varepsilon_j \to 0+$ such that $\theta + dd^c\varphi_j \geq -\varepsilon_j\omega$ on U.*

The proof presented below combines [DemAG, §5.E] and [BK07]. Note first that the local boundedness assumption cannot be dropped:

Example 3.5. Let $E \subset X$ be the exceptional divisor of the blowup of a point in a compact Kähler surface, and recall that the cohomology class $[E] \in H^2(X)$ satisfies $[E]^2 = -1$. Pick a Hermitian metric $|\cdot|$ on the line bundle $\mathcal{O}(E)$, with curvature form θ, and let $s \in H^0(X, \mathcal{O}(E))$ be the canonical section, so that $\varphi := \log|s|$ is θ-psh. Assume by contradiction that φ can be written as the weak limit of smooth functions $\varphi_j \in C^\infty(X)$ such that $\theta + dd^c\varphi_j \geq -\varepsilon_j\omega$ for some Kähler form ω on X and $\varepsilon_j \to 0$. Then

$$0 \leq \int_X (\theta + dd^c\varphi_j + \varepsilon_j\omega)^2 = ([E] + \varepsilon_j[\omega])^2 \to [E]^2 = -1,$$

a contradiction.

The Richberg patching technique relies on the use of regularized max functions.

Definition 3.6. Given a non-negative function $\rho \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\rho \subset [0,1]$, $\int_{\mathbb{R}} \rho(t)dt = 1$ and an r-tuple of positive numbers $\delta = (\delta_1, \dots, \delta_r)$, the *regularized max function* $\max_\delta\{x_1, \dots x_r\}$ is defined as the convolution product

$$\max_\delta\{x_1, \dots, x_r\} = \int_{t\in[0,1]^r} \max\{x_1 + \delta_1 t_1, \dots, x_r + \delta_r t_r\}\rho(t_1)\dots\rho(t_r)dt_1\dots dt_r.$$

The following properties are trivially satisfied:

(i) $\max_\delta$ is smooth and convex on $\mathbb{R}^r$, non-decreasing in each variable, and
$$\max_\delta\{x_1 + c, \dots, x_r + c\} = \max_\delta\{x_1, \dots, x_r\} + c$$
for all $x_1, \dots, x_r, c \in \mathbb{R}$.
(ii) If $x_i + \delta_i \leq \max_{j\neq i}(x_j - \delta_j)$, then
$$\max_\delta\{x_1, \dots, x_r\} = \max_\delta\{x_1, \dots, \widehat{x_i}, \dots, x_r\}.$$

(iii) $\max\{x_1,\dots,x_r\} \le \max_\delta\{x_1,\dots,x_p\} \le \max\{x_1+\delta_1,\dots,x_r+\delta_r\}$.

By (i) and Lemma 3.7 below, $\max_\delta(\varphi_1,\dots,\varphi_r)$ is thus θ-psh for any choice of θ-psh functions φ_i.

Lemma 3.7. *Suppose that $\chi : \mathbb{R}^r \to \mathbb{R}$ is convex, non-decreasing in each variable and satisfies $\chi(t_1+c,\dots,t_r+c) = \chi(t_1,\dots,t_r)+c$ for all $c \in \mathbb{R}$. If $\varphi_1,\dots,\varphi_r$ are θ-psh functions, then so is $\chi(\varphi_1,\dots,\varphi_r)$.*

Proof. Let f be a local potential of θ. Since $f+\varphi_j$ is psh, Proposition 2.2 implies that

$$\chi(f+\varphi_1,\dots,f+\varphi_r) = f+\chi(\varphi_1,\dots,\varphi_r)$$

is psh, and the result follows. □

Proof of Theorem 3.4. Choose a locally finite cover of X by coordinate charts (V_α), and open subsets $U'_\alpha \Subset U_\alpha \Subset V_\alpha$ with (U'_α) still covering K. For each α, the local regularization result of Theorem 3.3 yields a sequence of smooth functions $\varphi_{\alpha,j}$ defined on a neighborhood of $\overline{U}_\alpha$ such that

(i) $\varphi_{\alpha,j} \searrow \varphi$;
(i) for every closed real $(1,1)$-form θ with $\theta+dd^c\varphi \ge 0$, we have $\theta+dd^c\varphi_{\alpha,j} \ge -\varepsilon_{\alpha,j}\omega$ with $\varepsilon_{\alpha,j} \to 0_+$;
(ii) $\sup_{\overline{U}_\alpha\cap\overline{U}_\beta} |\varphi_{\alpha,j}-\varphi_{\beta,j}| \le \delta_{\alpha,j}$ with $\delta_{\alpha,j} \searrow 0_+$.

For each α, choose a cut-off function $\chi_\alpha \in C_c^\infty(U_\alpha)$ with $0 \le \chi_\alpha \le 1$ and $\chi_\alpha \equiv 1$ on a neighborhood of $\overline{U}'_\alpha$, and set for each $z \in X$

$$\varphi_j(z) := \max_{\delta_{\alpha,j}}\{\varphi_{\alpha,j}(z)+2\delta_{\alpha,j}\chi_\alpha(z)\}$$

with α ranging over all indices such that $z \in U_\alpha$. The monotonicity properties of the regularized max then imply that

$$\varphi(z) \le \varphi_{j+1}(z) \le \varphi_j(z) \le \max\{\varphi_{\alpha,j}(z)+2\delta_{\alpha,j}\chi_\alpha(z)+\delta_{\alpha,j} \mid z \in U_\alpha\},$$

and hence $\varphi_j(z) \searrow \varphi(z)$. Now pick $z_0 \in X$ and $\alpha_0 \in A$ with $z_0 \in U'_{\alpha_0}$, and set

$$V := U'_{\alpha_0} \setminus \left(\bigcup_{z_0\notin\overline{U}_\alpha} \overline{U}_\alpha \cup \bigcup_{z_0\in\partial U_\alpha} \operatorname{supp}\chi_\alpha \right).$$

Using the basic properties of regularized max functions, a simple check shows that for all $z \in V$ we have

$$\varphi_j(z) = \max_{\delta_{\alpha,j}}\{\varphi_{\alpha,j}(z)+2\delta_{\alpha,j}\chi_\alpha(z)\},$$

where α now ranges over the fixed set of indices such that $z_0 \in U_\alpha$. As a first consequence, φ_j is smooth on X. For each compact $K \subset X$, denote by $A_K \subset A$ the (finite) set indices such that V_α intersects K, and choose $C_K > 0$ such that $dd^c\chi_\alpha \ge -C_K\omega$ near K for all $\alpha \in A_K$. If $\theta+dd^c\varphi \ge 0$ for a given θ, Lemma 3.7 yields

$$\theta+dd^c\varphi_j \ge -(\min_{\alpha\in A_K}\varepsilon_{\alpha,j}+2C_K\min_{\alpha\in A_K}\delta_{\alpha,j})\omega$$

near K, and we are done. □

3.4. **Global regularization of unbounded quasi-psh functions.** We now show how to get rid of the local boundedness condition in Theorem 3.4, under adequate extra assumptions. This is only briefly mentioned in [BK07], and we provide some details.

Recall that a complex manifold X is *strongly pseudoconvex* if it admits a smooth, strictly psh exhaustion function $\tau : X \to \mathbb{R}$. If X is Stein, it admits a closed embedding in some $\mathbb{C}^N$, and the restriction to X of $|z|^2$ yields a strictly psh exhaustion function. The Levi problem asked for the converse, and was positively answered by Grauert in 1958. A proof using L^2-estimates for $\overline{\partial}$ will be provided below.

Theorem 3.8. *Let φ be a quasi-psh function on a complex manifold X, and assume given finitely many closed, real $(1,1)$-forms θ_α such that $\theta_\alpha + dd^c\varphi \geq 0$ for all α. Suppose either that X is strongly pseudoconvex, or that $\theta_\alpha > 0$ for all α. Then we can find a sequence $\varphi_j \in C^\infty(X)$ with the following properties:*

(i) *φ_j converges pointwise to φ;*
(ii) *for each relatively compact open subset $U \Subset X$, there exists $j_U \gg 1$ such that the sequence (φ_j) becomes decreasing with $\theta_\alpha + dd^c\varphi_j > 0$ for $j \geq j_U$.*

Proof. Suppose first that $\theta_\alpha > 0$ for all α. By Lemma 3.7, the locally bounded functions $\psi_j := \max\{\varphi, -j\}$ satisfy $\theta_\alpha + dd^c\psi_j \geq 0$ for all α, j. Note that ψ_j is generally not continuous, which is why the improvement from continuous to locally bounded functions in Richberg's theorem is so crucial.

Choose next an exhaustion of X by compact subsets $K_j \subset \mathring{K}_{j+1}$. We are going to construct by induction on $j \geq 1$ a sequence $\varphi_j \in C^\infty(X)$ such that

(i) on K_{j+1}, $\varphi_j > \psi_j$ and $\theta_\alpha + dd^c\varphi_j > 0$ for all α;
(ii) on K_j, $\varphi_j \leq \varphi_{j-1}$ and $\|\varphi_j - \psi_j\|_{L^1(K_j)} \leq 1/j$.

Since $\psi_j \to \varphi$ in L^1_{loc}, (φ_j) will then satisfy the conditions of the theorem. Start with any smooth function $\psi_0 > \max\{\varphi, 0\}$, and assume that $\psi_0, \dots, \psi_{k-1}$ have been constructed. Let $\psi_{j,k} \in C^\infty(X)$ be a sequence obtained by applying Theorem 3.4 to the locally bounded quasi-psh function ψ_j, i.e. $\psi_{j,k} \searrow \psi_j$ and

$$\theta_\alpha + dd^c\psi_{j,k} > -\varepsilon_k\theta_\alpha \tag{3.1}$$

on K_{j+1} for some sequence $\varepsilon_k \searrow 0$ depending on j (but not on α), by positivity of θ_α. On K_j, we have by induction

$$\lim_k \psi_{j,k} = \psi_j \leq \psi_{j-1} < \varphi_{j-1},$$

and hence $\psi_{j,k} \leq \varphi_{j-1} - \eta_j$ on K_j for some constant $\eta_j > 0$ and all $k \gg 1$, by Dini's lemma. Set $\delta_k := 1 - (1+\varepsilon_k)^{-1}$ and

$$\varphi_{j,k} := (1-\delta_k)\psi_{j,k} + \delta_k(1 + \sup_{K_{j+1}} \psi_j).$$

On K_{j+1}, we have

$$\varphi_{j,k} \geq (1-\delta_k)\psi_j + \delta_k(1 + \sup_{K_{j+1}} \psi_j) > \psi_j,$$

and $\theta_\alpha + dd^c\varphi_{j,k} > 0$ by (3.1). Since $\psi_{j,k} \le \varphi_{j-1} - \eta_j$ on K_j, we have $\varphi_{j,k} < \varphi_{j-1}$ on K_j for $k \gg 1$, and also $\|\varphi_{j,k} - \psi_j\|_{L^1(K_j)} \le 1/j$ since $\varphi_{j,k} \to \psi_j$ in L^1_{loc}. Setting $\varphi_j := \varphi_{j,k}$ with $k \gg 1$ thus completes the inductive step.

Assume finally that X is strongly pseudoconvex, with θ_α not necessarily positive anymore. Given a smooth, strictly psh exhaustion function $\tau : X \to \mathbb{R}$, we can find a positive continuous function $f : X \to \mathbb{R}_+$ such that $f\, dd^c\tau + \theta_\alpha > 0$ for all α. Choosing a smooth, convex, non-decreasing function $\chi : \mathbb{R} \to \mathbb{R}$ with fast enough growth to guarantee that $\chi''(c) \ge \sup_{\{\tau = c\}} f$ for all $c \in \mathbb{R}$, we get

$$dd^c\chi(\tau) = \chi''(\tau) dd^c\tau + \chi'(\tau) d\tau \wedge d^c\tau \ge f\, dd^c\tau.$$

For each α, $\tilde{\theta}_\alpha := \theta_\alpha + dd^c\chi(\tau)$ is thus positive with $\tilde{\theta}_\alpha + dd^c[\varphi - \chi(\tau)] \ge 0$. The first case treated above yields a sequence $\tilde{\varphi}_i \in C^\infty(X)$ converging pointwise to $\varphi - \chi(\tau)$, and which becomes decreasing with $\tilde{\theta}_\alpha + dd^c\varphi'_j > 0$ for $j \gg 1$ on any given relatively compact open subset. Setting $\varphi_j := \tilde{\varphi}_j + \chi(\tau)$ yields the desired sequence for φ. □

4. Basic analysis of first order differential operators

4.1. **Differential operators.** We use [Dem02, §2] as a reference for what follows. Let M be a smooth manifold and E, F be smooth, complex vector bundles on M. A *linear differential operator of order (at most) δ* is a $\mathbb{C}$-linear map $A : C^\infty(M, E) \to C^\infty(M, F)$ such that for any coordinate chart in which E, F are trivialized, we have

$$A = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \le \delta} a_\alpha D^\alpha \tag{4.1}$$

with $a_\alpha \in \mathrm{Hom}(E, F)$. More invariantly, the space of differential operator of order δ can be written as

$$\mathcal{D}^\delta(M; E, F) = C^\infty(M, \mathrm{Hom}(J^\delta E, F))$$

with $J^\delta E = J^\delta_M \otimes E$ the vector bundle of δ-jets of sections of E. The jet exact sequence

$$0 \to S^\delta T^\star_M \to J^d_M \to J^{\delta-1}_M \to 0$$

gives rise to an isomorphism

$$\mathcal{D}^\delta(M; E, F)/\mathcal{D}^{\delta-1}(M; E, F) \simeq C^\infty(M, S^\delta T_M \otimes \mathrm{Hom}(E, F)).$$

The image $\sigma_A \in C^\infty(M, S^\delta T_M \otimes \mathrm{Hom}(E, F))$ of $A \in \mathcal{D}^\delta(M; E, F)$ is called the *principal symbol* of A, and is more concretely described as the $\mathrm{Hom}(E, F)$-valued homogeneous form of degree δ on $T^\star_M$

$$\sigma_A(\xi) = \sum_{|\alpha| = \delta} a_\alpha \xi^\alpha,$$

when A is locally written in the form (4.1).

Proposition 4.1. *Let $A : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator of order δ.*

(i) *If* $A' : C^\infty(M,F) \to C^\infty(M,G)$ *is a differential operator of order* δ'*, then* $A'A$ *is a differential operator of order* $\delta' + \delta$*, and for each* $\xi \in T_M^\star$ *we have*

$$\sigma_{A'A}(\xi) = \sigma_{A'}(\xi)\sigma_A(\xi).$$

(ii) *For* $\delta = 1$*, we may view* σ_A *is an* $\mathrm{Hom}(E,F)$*-valued vector field, and we have the commutator relation*

$$[A,f]u := A(fu) - fAu = \sigma_A(df)u = (\sigma_A \cdot f)u \tag{4.2}$$

for each $f \in C^\infty(M)$ *and* $u \in C^\infty(M,E)$*.*

In particular, the space of order 1 differential operators $E \to F$ with a given (degree 1) symbol is an affine space modeled on the space of bundle homomorphisms.

Example 4.2. A connection ∇ on E induces a differential operator of order 1

$$D_\nabla : C^\infty(M, \Lambda^\bullet T_M^\star \otimes E) \to C^\infty(M, \Lambda^\bullet T_M^\star \otimes E),$$

with principal symbol $\xi \wedge \bullet$.

4.2. **Adjoint.** Now assume given a smooth volume form dV on M and Hermitian scalar products on E and F. If $A : C^\infty(M,E) \to C^\infty(M,F)$ is a differential operator of order δ, its *formal adjoint* $A^\star : C^\infty(M,F) \to C^\infty(M,E)$ is uniquely determined by the requirement

$$\langle Au, v\rangle_{L^2} = \langle u, A^\star v\rangle_{L^2} \tag{4.3}$$

for all $u \in C^\infty(M,E)$, $v \in C^\infty(M,F)$ such that u (or v) has compact support.

Lemma 4.3. *Any differential operator* $A \in \mathcal{D}^\delta(M;E,F)$ *admits a formal adjoint* $A^\star \in \mathcal{D}^\delta(M;F,E)$*. Further, its principal symbol is given by*

$$\sigma_{A^\star} = (-1)^\delta \sigma_A^\star.$$

Further, given a section $v \in C^\infty(M,F)$*, the value of* $A^\star v$ *at a given point only depends on the* δ*-jet of* dV *and of the metrics of* E *and* F*.*

Proof. By linearity and a partition of unity, we may assume that E, F are trivial and $A = aD^\alpha$ with $a \in \mathrm{Hom}(E,F)$. Denote by h_E, h_F the matrices representing the scalar product of E, F and write $dV = f dx$ with $f > 0$. Then

$$\langle Au, v\rangle_{L^2} = \int \langle aD^\alpha u, v\rangle_F dV = \int \langle D^\alpha u, a^\star v\rangle_E dV$$

$$= \int D^\alpha({}^t u) h_E \overline{a^\star v} f dx = (-1)^d \int {}^t u\, D^\alpha\left(h_E \overline{a^\star v} f\right) dx,$$

and hence

$$A^\star v = (-1)^\delta\, {}^t h_E^{-1} D^\alpha\left({}^t h_E a^\star v f\right) f^{-1} = (-1)^\delta a^\star D^\alpha v + \text{l.o.t.}$$

□

4.3. **Friedrich's lemma for complete metrics.** A differential operator $A \in \mathcal{D}^\delta(M;E,F)$ acts in a natural way on distributional sections, and (4.3) remains valid when u is a distribution and $v \in C^\infty_c$. In particular, A uniquely extends to a densely defined, closed operator (i.e. with closed graph) $A : L^2(M,E) \dashrightarrow L^2(M,F)$, with $u \in \operatorname{Dom} A$ iff $Au \in L^2$. However, in general, (4.3) does not hold for all $u \in \operatorname{Dom} A$, $v \in \operatorname{Dom} A^\star$ (i.e. $\operatorname{Dom} A^\star$ does not coincide with the domain of the Hilbert space adjoint of A).

Example 4.4. Let $I \subset \mathbb{R}$ be a bounded open interval endowed with the Euclidian metric, and $A = d/dx$. Then $A^\star = -A$, $\operatorname{Dom} A = \operatorname{Dom} A^\star$ is the Sobolev space $W^{1,2}(I) \subset C^0(\bar{I})$, and $\langle Au, v\rangle = \langle u, A^\star v\rangle + [uv]_{\partial I}$ for $u, v \in W^{1,2}$.

Theorem 4.5. *If the Riemannian metric g of M is* complete*, then any $u \in L^2(M,E)$ can be written as the L^2-limit of a sequence of test sections $u_j \in C^\infty_c(M,E)$ with the following property: for each first order linear differential operator $A \in \mathcal{D}^1(M;E,F)$ with bounded symbol σ_A, $Au \in L^2(M,F)$ implies $Au_j \to Au$ in L^2.*

The regularizing sequence u_j will be obtained by truncation and local convolution.

Lemma 4.6. *Let (M,g) be a Riemannian manifold, possibly with boundary, fix $p \in [1,\infty]$, and consider the following conditions:*

(i) *M admits a smooth exhaustion function $\psi : M \to \mathbb{R}$ with $d\psi \in L^p$;*
(ii) *there exists an exhaustion by compact sets $K_j \subset \mathring{K}_{j+1} \Subset M$ and a sequence of cut-off functions $\chi_j \in C^\infty_c(\mathring{K}_{j+1})$ with $0 \le \chi_j \le 1$, $\chi_j \equiv 1$ on a neighborhood of K_j, such that $\|d\chi_j\|_{L^p} \to 0$;*
(iii) *there exists K_j, χ_j as in (ii) with $d\chi_j$ bounded in L^p.*

Then (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii); when $p > 1$, we conversely have (iii)$\Longrightarrow$(i). For $p = \infty$, these conditions are also equivalent to the completeness of g.

The case of $M = (0,1]$ shows that (iii)$\Longrightarrow$(i) fails in general for $p = 1$. A Riemannian manifold (M,g) satisfying the above conditions with $p = 2$ is classically called *parabolic*, cf. [Gla83] and the references therein for more information.

Proof. Suppose given ψ as in (i), and pick a smooth function cut-off function $\theta \in C^\infty_c(\mathbb{R})$ with $0 \le \theta \le 1$, $\theta \equiv 1$ near 0. Then $\chi_j(x) = \theta(j^{-1}\psi(x))$ satisfies (ii). (ii)$\Longrightarrow$(iii) is trivial. Let χ_j as in (iii) and $p > 1$. Then

$$\psi := \sum_{j\ge1} j^{-1}(1-\chi_j)$$

is a smooth exhaustion function, since we have $\psi \ge S_j$ outside K_{j+1} with $S_j = \sum_{k=1}^j k^{-1} \to +\infty$. Since the supports of the gradients $d\chi_j$ are disjoint, we have

$$\int_M |d\psi|^p dV \le \sum_{j\ge1} j^{-p} \int_M |d\chi_j|^p < +\infty,$$

which proves (i).

Fix a basepoint $x_0 \in M$. By the Hopf-Rinow Lemma, g is complete iff $d(\bullet, x_0)$ is an exhaustion function. If ψ is a smooth exhaustion function with bounded gradient, then $\psi(x) \le \psi(x_0) + C d(x, x_0)$, and $d(\bullet, x_0)$ is thus exhaustive. Conversely, if the Lipschitz continuous function $d(\bullet, x_0)$ is exhaustive, then a regularization argument shows the existence of a smooth exhaustion function with bounded gradient. ☐

Proof of Theorem 4.5. By Lemma 4.6, we may find an exhausting sequence of cut-off functions $\chi_j \in C_c^\infty(M)$ with $|d\chi_j|_g \le 1$. For each $u \in L^2(M, E)$, we have $\chi_j u \to u$ in L^2. Further, (4.2) yields $A(\chi_j u) = \chi_j A u + \sigma_A(d\chi_j) u$ with

$$|\sigma_A(d\chi_j)u| \le C|u|$$

since $|d\chi_j|_g \le 1$ and σ_A is bounded. By dominated convergence, we thus have $\sigma_A(d\chi_j)u \to 0$ in L^2, and hence $A(\chi_j u) \to Au$ in L^2 whenever the latter is in L^2.

We are thus reduced to the case where u has compact support. Using a partition of unity, we can further assume that u is compactly supported in a coordinate chart. We then conclude using Lemma 4.7 below, usually known as Friedrich's lemma. ☐

Lemma 4.7. *Pick $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho(x)dx = 1$, and set $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\varepsilon^{-1}x)$. Let also $A = \sum_\alpha a_\alpha D^\alpha \in \mathcal{D}^1(\mathbb{R}^d)$ be a first order linear differential operator with $|da_\alpha| \le C$. For each $f \in L^2(\mathbb{R}^d)$, we then have $A(f \star \rho_\varepsilon) - (Af) \star \rho_\varepsilon \to 0$ in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$.*

Proof. The result is clearly true if f is a test function, since $f \star \rho_\varepsilon \to f$ in C^∞. By density of test funtions in L^2, it is thus enough to prove the existence of $C > 0$ such that

$$\|A(f \star \rho_\varepsilon) - (Af) \star \rho_\varepsilon\|_{L^2} \le C\|f\|_{L^2}$$

for all $f \in L^2$. To do this, we may assume by linearity that $A = a\partial_j$ where $a \in C^\infty$ has bounded gradient. Write

$$(a\partial_j f) \star \rho_\varepsilon = (af) \star \partial_j \rho_\varepsilon - ((\partial_j a)f) \star \rho_\varepsilon.$$

Since $L^2 \star L^1 \subset L^2$, the second term satisfies

$$\|\left((\partial_j a)f\right) \star \rho_\varepsilon\|_{L^2} \le \|(\partial_j a)f\|_{L^2}\|\rho_\varepsilon\|_{L^1} \le C\|f\|_{L^2}.$$

Next,

$$\left((af) \star \partial_j \rho_\varepsilon\right)(x) - a(x)(\partial_j f \star \rho_\varepsilon)(x) = \int (a(x-y) - a(x)) f(x-y)(\partial_j \rho_\varepsilon)(y) dy$$

yields

$$|(af) \star \partial_j \rho_\varepsilon - a(\partial_j f \star \rho_\varepsilon)| \le C \left| f \star (x\partial_j \rho_\varepsilon)\right|,$$

whose L^2 norm is bounded above by a uniform multiple of $\|f\|_{L^2}$ since $\|x\partial_j\rho_\varepsilon\|_{L^1} = \|x\partial_j\rho\|_{L^1}$ is bounded. ☐

4.4. **Removable singularities for first order PDEs.**

Proposition 4.8. *Let $A \in \mathcal{D}^1(M; E, F)$ be a first order differential operator, $u \in L^2_{\mathrm{loc}}(M, E)$, $v \in L^1_{\mathrm{loc}}(M, F)$, and assume that $Au = v$ holds outside a closed submanifold $N \subset M$ of codimension at least 2. Then $Au = v$ on M.*

Lemma 4.9. *If M is a compact Riemannian manifold with boundary and $N \subset M$ is a closed submanifold of codimension at least 2, then $M \setminus N$ is parabolic, i.e. it satisfies the equivalent condition of Lemma 4.6 with $p = 2$.*

Proof. Using a partition of unity, we are reduced to a local statement. We may then choose local coordinates $x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}^d$ such that $N = \{x' = 0\}$. Pick any smooth function $\chi : \mathbb{R} \to \mathbb{R}_+$ with $\chi = 0$ near 0, $\chi \equiv 1$ on $[1, +\infty)$, and set

$$\chi_j(x) := \chi\left(j\|x'\|\right).$$

Then $\int |d\chi_j|^2 dV \le Cj^2 \operatorname{Vol}(\|x'\| \le j^{-1}) \le Cj^{2-n}$ is bounded since $n \ge 2$ by assumption. □

Proof of Proposition 4.8. Let χ_j be a sequence of cut-off functions as in (ii) of Lemma 4.6, and write $A(\chi_j u) = \chi_j v + \sigma_A(d\chi_j)u$. We have $\chi_j u \to u$ and $\chi_j v \to v$ in L^1_{loc}, which implies $A(\chi_j u) \to Au$ in the sense of distributions. Further, $\|\sigma_A(d\chi_j)\|_{L^2} \to 0$ since σ_A is bounded and $d\chi_j \to 0$ in L^2. Since $u \in L^2_{\mathrm{loc}}$, it follows that $\sigma_A(d\chi_j)u \to 0$ in L^1_{loc}, and we are done. □

More generally, we prove:

Proposition 4.10. *Let $p \in (1, m]$ with conjugate exponent $q \in [\frac{m}{d-1}, \infty)$. Let $A \in \mathcal{D}^1(M; E, F)$ be a first order differential operator, $u \in L^q_{\mathrm{loc}}(M, E)$, $v \in L^1_{\mathrm{loc}}(M, F)$, and assume that $Au = v$ as distributions outside a closed subset $F \subset M$ of finite $(m-p)$-Hausdorff measure. Then $Au = v$ on M.*

The Heaviside function shows that result fails in general for $p = 1$. As before, Proposition 4.10 will be a direct consequence of the following result.

Lemma 4.11. *Let M be a compact Riemannian manifold with boundary, and let $F \subset M$ be a closed subset of finite $(m-p)$-dimensional Hausdorff measure for some $p \in [1, m]$. Then $M \setminus F$ admits an exhausting sequence of cut-off functions with gradient bounded in L^p, as in (iii) of Lemma 4.6.*

Proof. Following [EG92, p.154, Theorem 3], we are going to show that any compact set $K \subset M \setminus F$, we can find a cut-off function $\chi \in C^\infty_c(M)$ with $0 \le \chi \le 1$, $\chi \equiv 1$ on a neighborhood of K, and $\|\chi\|_{L^p} \le C$ for a uniform constant $C > 0$.

Since F is compact and has finite $\mathcal{H}_{m-p}$-measure, there exists a constant $C > 0$ such that F can be covered by finitely many open balls $B(x_i, r_i)$ of radius $r_i \ll 1$ such that $\sum_i r_i^{m-p} \le C$.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be the piecewise affine function defined by $\theta(t) = 0$ for $t \le 1$, $\theta(t) = t - 1$ for $1 \le t \le 2$, and $\chi(t) = 1$ for $t \ge 2$, and consider the Lipschitz continuous function

$$\chi_i(x) = \theta\left(r_i^{-1} d(x, x_i)\right).$$

Then $\chi := \max_i \chi_i$ is also Lipschitz continuous, it has compact support in $M \setminus F$, and $\chi \equiv 1$ on a neighborhood of K. Further, we have

$$|d\chi| \le \max_i |d\chi_i| \le \max_i r_i^{-1} \mathbf{1}_{\{x \mid r_i \le d(x,x_i) \le 2r_i\}}$$

a.e. on M, hence

$$\int_M |d\chi|^p dV \le C \sum_i r_i^{-p} \operatorname{Vol} B(x_i, 2r_i) \le C' \sum_i r_i^{m-p} \le C''$$

with $C'' > 0$ independent of K. This proves the claim, after regularizing χ near the boundary of $\{\chi = 1\}$. $\square$

5. Fundamental identities of Kähler geometry

5.1. **Some Hermitian algebra.** Let V be a finite dimensional complex vector space, and denote by

$$\Lambda = \bigoplus_{p,q \in \mathbb{N}} \Lambda^{p,q} V^\star$$

the corresponding bigraded exterior algebra. The graded commutator of $A, B \in \operatorname{End}(\Lambda)$ is defined by

$$[A, B] = AB - (-1)^{|A||B|} BA$$

for A, B of pure degree $|A|, |B|$. It satisfies the graded Jacobi identity

$$[[A, B], C] = [A, [B, C]] - (-1)^{|A||B|} [B, [A, C]], \tag{5.1}$$

which expresses that $A \mapsto \operatorname{ad}(A) := [A, \bullet]$ is a graded Lie algebra homomorphism. For each $\eta \in \Lambda$, we also denote by η the associated endomorphism $\eta \wedge \bullet \in \operatorname{End}(\Lambda)$. Note that $[\eta, \eta'] = 0$ for any two $\eta, \eta' \in \Lambda$.

Now suppose that V is equipped with a Hermitian scalar product, and denote by $\omega \in \Lambda^{1,1}$ the corresponding positive $(1,1)$-form. Given an orthonormal basis (ξ_i) of $\Lambda^{1,0} \simeq V^\star$, $(\xi_I \wedge \bar{\xi}_J)_{|I|=p, |J|=q}$ is an orthonormal basis of $\Lambda^{p,q}$, and

$$\omega = i \sum_j \xi_j \wedge \bar{\xi}_j. \tag{5.2}$$

For $\eta \in \Lambda$, denote by $\eta^\star \in \operatorname{End}(\Lambda)$ the adjoint of $\eta \wedge \bullet$, which has bidegree $(-p, -q)$ when $\eta \in \Lambda^{p,q} V^\star$.

We next discuss in more detail $[\eta, \omega^\star] \in \operatorname{End}(\Lambda)$ for $\eta \in \Lambda^{p,q} V^\star$ of bidegree $(p, q) = (1, 0)$ or $(1, 1)$. The next result is the 'linear algebra' part of the Kähler commutation identities.

Lemma 5.1. *For each $\xi \in \Lambda^{1,0} \simeq V^\star$, the following identities hold:*

(i) $[\xi, \omega^\star] = i\bar{\xi}^\star$;
(ii) $[\xi^\star, \omega] = i\bar{\xi}$;
(iii) $[\omega, \bar{\xi}^\star] = i\xi$;
(iv) $[\omega^\star, \bar{\xi}] = i\xi^\star$.

Proof. A direct check shows that $\xi^\star$ is given by contraction against the dual vector $\xi^\# \in V$. As a result, $\xi^\star$ is an anti-derivation of Λ, i.e.

$$\xi^\star(\alpha \wedge \beta) = \xi^\star(\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \xi^\star(\beta), \tag{5.3}$$

or, equivalently, $[\xi^\star, \alpha] = \xi^\star(\alpha)$ for all $\alpha \in \Lambda$. We get (ii) thanks to $\xi^\star(\omega) = i\bar{\xi}$, which is easily seen using (5.2). Applying adjunction and conjugation yields the remaining identities. □

Now fix a real $(1,1)$-form $\theta \in \Lambda^{1,1}$. The operator $[\theta, \omega^\star]$ will play a crucial role in the L^2 estimates for $\overline{\partial}$, and we therefore study it in some detail.

Proposition 5.2. [Dem82, Lemme 3.2] *Assume that $\theta \in \Lambda^{1,1}$ is positive, and fix an integer $q \geq 1$.*

(i) *The endomorphism $[\theta, \omega^\star] = \theta\omega^\star$ of $\Lambda^{n,q}$ is positive definite. More precisely, if we pick $\varepsilon > 0$ then $\theta \geq \varepsilon\omega$ implies $[\theta, \omega^\star] \geq q\varepsilon \,\mathrm{id}$ on $\Lambda^{n,q}V^\star$;*
(ii) *for each $u \in \Lambda^{n,q}$, $\langle[\theta, \omega^\star]^{-1}u, u\rangle dV_\omega$ is non-increasing with respect to ω and θ;*
(iii) *when $u \in \Lambda^{n,1}$, $\langle[\theta, \omega^\star]^{-1}u, u\rangle dV_\omega$ is independent of ω, and is given by*

$$\langle[\theta, \omega^\star]^{-1}u, u\rangle dV_\omega = i^{n^2}\{u, \bar{u}\}_\theta,$$

where $\{u, \bar{u}\}_\theta$ denotes the the image of $iu \otimes \bar{u}$ under the contraction morphism

$$\Lambda^{n,1}V^\star \otimes \Lambda^{1,n}V^\star \xrightarrow{\simeq} \Lambda^{1,1} \otimes \Lambda^{n,n}V^\star \xrightarrow{\mathrm{tr}_\theta} \Lambda^{n,n}V^\star.$$

Proof of Proposition 5.2. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of θ with respect to ω, and pick an ω-orthonormal basis (ξ_j) such that $\theta = i\sum_j \lambda_j \xi_j \wedge \bar{\xi}_j$. Setting $\Omega := \xi_1 \wedge \cdots \wedge \xi_n \in \Lambda^{n,0}V^\star$, a direct check shows that

$$\theta\omega^\star(\Omega \wedge \bar{\xi}_J) = (\sum_{j \in J} \lambda_j)\Omega \wedge \bar{\xi}_J, \tag{5.4}$$

which implies (i). Pick $u \in \Lambda^{n,1}V^\star$, and write $u = \Omega \wedge v$ with $v = \sum_j v_j \bar{\xi}_j$ in $\Lambda^{0,1}$. By (5.4), we have

$$\langle[\theta, \omega^\star]^{-1}u, u\rangle = \sum_j \lambda_j^{-1}|v_j|^2 = \mathrm{tr}_\theta(iv \wedge \bar{v}),$$

and (iii) follows since

$$dV_\omega = \bigwedge_j i\xi_j \wedge \bar{\xi}_j = i^{n^2}\Omega \wedge \bar{\Omega}.$$

Monotonicity with respect to θ in (ii) is clear, in view of (5.4). Observe that the quadratic form

$$q_\omega(u) := \langle[\theta, \omega^\star]^{-1}u, u\rangle_\omega$$

satisfies

$$q_\omega(u) = \sup_{0 \neq v \in \Lambda^{n,q}V^\star} \frac{|\langle u, v\rangle_\omega|^2}{\langle\theta\omega^\star v, v\rangle_\omega}. \tag{5.5}$$

Now let $\omega' \ge \omega$ be another positive $(1,1)$-form with eigenvalues $\mu_i \ge 1$ with respect to ω, and let (ζ_j) be an ω-orthonormal basis of $\Lambda^{1,0}V$ such that $\omega' = \sum_j \mu_j \zeta_j \wedge \bar\zeta_j$ (which is however not necessarily diagonalizing θ). Then $dV'_\omega = \mu_1 \dots \mu_n dV_\omega$, and our goal is thus to show that

$$q_{\omega'}(u)\mu_1 \dots \mu_n \le q_\omega(u). \tag{5.6}$$

Setting $\mu = \operatorname{diag}(\mu_j)$, we have $\omega'^\star = \mu^{1/2}\omega^\star\mu^{-1/2}$, and (5.5) yields

$$q_{\omega'}(u) = \sup_{0\ne v\in\Lambda^{n,q}V^\star} \frac{|\langle u,v\rangle_{\omega'}|^2}{\langle \theta\omega'^\star v, v\rangle_{\omega'}} = \sup_{0\ne v\in\Lambda^{n,q}V^\star} \frac{|\langle u,v\rangle_{\omega}|^2}{\langle \theta\mu^{1/2}\omega^\star\mu^{1/2} v, v\rangle_{\omega}}.$$

Using $\mu_j \ge 1$, one easily sees that

$$\langle \theta\mu^{1/2}\omega^\star\mu^{1/2}v,v\rangle_\omega \ge \mu_1\dots\mu_n\langle\theta\omega^\star v,v\rangle_\omega,$$

which yields as desired (5.6). $\square$

5.2. **Kähler identities.** In what follows, (X,ω) is a Kähler manifold, and E is a Hermitian holomorphic vector bundle on X, with Chern connection $\partial_E + \overline{\partial}_E$. We start with the existence of local normal forms for ω and E.

Lemma 5.3. *Near each point of X, we may find:*

(i) *holomorphic coordinates $(z_0,\dots,z_n)$ such that*

$$\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\rangle_\omega = \delta_{jk} + O(|z|^2).$$

(ii) *a holomorphic basis (e_j) of E such that $\langle e_j, e_k\rangle_E = \delta_{jk} + O(|z|^2)$.*

Conversely, note that a Hermitian metric ω admitting local holomorphic coordinates as in (i) is closed, and hence Kähler.

Proof. Locally, we have $\omega = dd^c\varphi$ for a smooth (strictly psh) function φ. Choose holomorphic coordinates w such that $\omega = \sum_j i dw_j \wedge d\bar w_j$ at 0. The Taylor expansion of φ at 0 then writes

$$\varphi = |w|^2 + \operatorname{Re}\left(\sum_j P_j(w)\bar w_j\right) + \operatorname{Re} P(w) + O(|w|^4)$$

where each $P_j(w)$ is a holomorphic quadratic polynomial, and $P(w)$ is a holomorphic polynomial of degree at most 3. Setting $z_j = w_j - \frac{1}{2}P_j(w)$ yields (i). The proof of (ii) is similar and left to the reader. $\square$

Theorem 5.4. *Assume that (X,ω) is Kähler, let E be a Hermitian holomorphic vector bundle on X, with Chern connection $\partial_E + \overline{\partial}_E$. Then*

$$\begin{aligned}[\overline{\partial}_E^\star, \omega] &= i\partial_E\\ [\partial_E^\star,\omega] &= -i\overline{\partial}_E\\ [\omega^\star, \overline{\partial}_E] &= -i\partial_E^\star\\ [\omega^\star,\partial_E] &= i\overline{\partial}_E^\star.\end{aligned}$$

Proof. Since the adjoint of a first order operator only depends on the 1-jets of the metrics involves, the normal forms of Lemma 5.3 reduces us to the case of $\mathbb{C}^n$ with its Euclidian metric and E the trivial line bundle. In that case, the first order operators involved have constant coefficients. It is thus enough to prove that their symbols coincide, which is the content of Lemma 5.1. □

The key consequence for us is the following Bochner-type formula, which goes back to the work of Kodaira and Nakano.

Corollary 5.5 (Bochner–Kodaira–Nakano identity)**.** *With the previous notation, the Laplacians of ∂_E and $\overline{\partial}_E$ satisfy*

$$\Delta_{\overline{\partial}_E} = \Delta_{\partial_E} + [\theta_E, \omega^\star]$$

with $\theta_E \in C^\infty(X, \Lambda^{1,1}T^\star_X \otimes \mathrm{Hom}(E, E))$ the curvature tensor.

Proof. The above Kähler identities combined with the Jacobi identity (5.1) yield

$$\Delta_{\overline{\partial}_E} = [\overline{\partial}^\star_E, \overline{\partial}_E] = i[[\partial_E, \omega^\star], \overline{\partial}_E]$$

$$= i[\partial_E, [\omega^\star, \overline{\partial}_E]] - i[\omega^\star, [\partial_E, \overline{\partial}_E]]$$

$$= [\partial_E, \partial^\star_E] + i[[\partial_E, \overline{\partial}_E], \omega^\star] = \Delta_{\partial_E} + i[\theta_E, \omega^\star].$$

□

As a first application, we obtain the following classical cohomology vanishing result.

Theorem 5.6 (Kodaira vanishing)**.** *If L is a Hermitian holomorphic line bundle with positive curvature form $\theta_L > 0$, then*

$$H^{n,q}(X, L) = H^q(X, K_X + L) = 0$$

for each $q \geq 1$.

Proof. By Hodge theory, it is enough to show that for every L-valued (n, q)-form u with $\Delta_{\overline{\partial}_L} u = 0$ vanishes. The Bochner–Kodaira–Nakano equality yields

$$0 = \|\partial_L u\|^2 + \|\partial^\star_L u\|^2 + \langle [\theta_L, \omega^\star]u, u\rangle_{L^2}.$$

By Proposition 5.2, the operator $[\theta, \omega^\star]$ is positive definite on (n, q)-forms, hence the result. □

6. L^2-estimates for the $\overline{\partial}$-equation

The results in this section go back to the fundamental work of Hörmander, with successive improvements and generalizations by Andreotti–Vesentini, Skoda, Demailly.

6.1. **The smooth, complete case.**

Theorem 6.1. *Let (X, ω) be a Kähler manifold, and let be L a Hermitian holomorphic line bundle on X with positive curvature form $\theta > 0$. Assume also that X admits a complete Kähler metric (not necessarily equal to ω or θ). If*

$$v \in L^2_{\mathrm{loc}}(X, \Lambda^{n,q} T_X^\star \otimes L)$$

satisfies $\overline{\partial} v = 0$ and

$$\int_X \langle [\theta, \omega^\star]^{-1} v, v \rangle dV_\omega < +\infty,$$

then $v = \overline{\partial} u$ for some $u \in L^2(X, \Lambda^{n,q-1} T_X^\star \otimes L)$ such that

$$\|u\|^2_{L^2} = \int_X |u|^2 dV_\omega \le \int_X \langle [\theta, \omega^\star]^{-1} v, v \rangle dV_\omega.$$

If v is smooth, then u can be chosen smooth.

Recall that the endomorphism $[\theta, \omega^\star] = \theta\omega^\star$ of $\Lambda^{n,q} T_X^\star$ is positive definite, by Proposition 5.2.

Remark 6.2. For $q = 1$, the result does in fact not involve ω, since

$$\langle [\theta, \omega^\star]^{-1} v, v \rangle dV = i^{n^2} \{v, \bar{v}\}_\theta$$

and

$$|u|^2 dV = i^{n^2} u \wedge \bar{u}.$$

As the next result shows, the assumption on X is satisfied by a fairly general class of Kähler manifolds.

Lemma 6.3. *Assume that X is* weakly pseudoconvex*, i.e. admits a smooth, psh exhaustion function. Then X admits a complete Kähler metric.*

In particular, any Stein manifold admits a complete Kähler metric.

Proof. Let $\varphi : X \to \mathbb{R}$ be a smooth, psh exhaustion function. Since φ is bounded below, we may assume that $\varphi \ge 0$. Then

$$i\partial\overline{\partial}(\varphi^2) = 2\varphi\, i\partial\overline{\partial}\varphi + 2i\partial\varphi \wedge \overline{\partial}\varphi$$

shows the gradient of φ is bounded with respect to the Kähler metric $\omega + i\partial\overline{\partial}(\varphi^2)$, which is thus complete. □

The key tool in the proof of Theorem 6.1 the following consequence of the Bochner–Kodaira–Nakano identity.

Lemma 6.4. *If ω is complete, then each $u \in L^2(X, \Lambda^{n,q} T_X^\star \otimes L)$ with $\overline{\partial} u, \overline{\partial}^\star u \in L^2$ satisfies*

$$\|\overline{\partial} u\|^2_{L^2} + \|\overline{\partial}^\star u\|^2_{L^2} \ge \int_X \langle [\theta, \omega^\star] u, u \rangle dV.$$

Proof. The symbol of

$$\overline{\partial} : C^\infty(X, \Lambda^{n,q}T_X^\star \otimes L) \to C^\infty(X, \Lambda^{n,q+1}T_X^\star \otimes L)$$

is given by $\sigma_{\overline{\partial}}(\xi) = \xi \wedge \bullet$, and is thus bounded. The symbol of its adjoint $\sigma_{\overline{\partial}^\star}(\xi) = -\sigma_{\overline{\partial}}(\xi)^\star$ is thus also bounded, and Theorem 4.5 yields a sequence $u_j \in C_c^\infty(X, \Lambda^{n,q}T_X^\star \otimes L)$ with $u_j \to u$, $\overline{\partial}u_j \to \overline{\partial}u$ and $\overline{\partial}^\star u_j \to \overline{\partial}^\star u$ in L^2 norm. After passing to a subsequence, we may assume that $u_j \to u$ a.e. and hence

$$\int_X \langle [\theta, \omega^\star]u, u\rangle dV \leq \liminf_j \int_X \langle [\theta, \omega^\star]u_j, u_j\rangle dV,$$

by Fatou's lemma. For each j, the Bochner–Kodaira–Nakano identity implies

$$\|\overline{\partial}u_j\|_{L^2}^2 + \|\overline{\partial}^\star u_j\|_{L^2}^2 \geq \int_X \langle [\theta, \omega^\star]u_j, u_j\rangle dV,$$

and we get the desired inequality in the limit. ☐

Proof of Theorem 6.1. Assume first that ω itself is complete. The existence of u clearly implies

$$|\langle w, v\rangle|^2 \leq C\|\overline{\partial}^\star w\|_{L^2}^2 \tag{6.1}$$

for all test forms $w \in C_c^\infty(X, \Lambda^{n,q}T_X^\star \otimes L)$. Assume conversely that this estimate holds. Then $\overline{\partial}^\star w \mapsto \langle w, v\rangle$ defines a continuous linear form of norm at most $C^{1/2}$ on

$$\overline{\partial}^\star C_c^\infty(X, \Lambda^{n,q}T_X^\star \otimes L) \subset L^2(X, \Lambda^{n,q-1}T_X^\star \otimes L).$$

By the Hahn-Banach theorem, this linear form admits a continuous extension to $L^2(X, \Lambda^{n,q-1}T_X^\star \otimes L)$ of norm at most $C^{1/2}$. By the Riesz representation theorem, we may thus find $u \in L^2(X, \Lambda^{n,q-1}T_X^\star \otimes L)$ with $\|u\|_{L^2} \leq C^{1/2}$ such that

$$\langle w, v\rangle = \langle \overline{\partial}^\star w, u\rangle = \langle w, \overline{\partial}u\rangle$$

for all test forms w, and hence $\overline{\partial}u = v$. In order to establish (6.1), consider the orthogonal decomposition $w = w' + w''$ in $L^2(X, \Lambda^{n,q-1}T_X^\star \otimes L)$ with $w' \in \ker\overline{\partial}$ and $w'' \in (\ker\overline{\partial})^\perp$. For each test form $\eta \in C_c^\infty(X, \Lambda^{n,q-2}T_X^\star \otimes L)$, $\overline{\partial}\eta \in \ker\overline{\partial}$ implies $\langle w'', \overline{\partial}\eta\rangle = 0$, i.e. $\overline{\partial}^\star w'' = 0$ in the sense of distributions. Since $\overline{\partial}v = 0$, the Cauchy-Schwarz inequality Lemma 6.4 yield

$$\begin{aligned}|\langle v, w\rangle|^2 = |\langle v, w'\rangle|^2 &\leq \left(\int_X \langle [\theta, \omega^\star]^{-1}v, v\rangle dV\right)\left(\int_X \langle [\theta, \omega^\star]w', w'\rangle dV\right)\\ &\leq C\|\overline{\partial}^\star w'\|^2 = C\|\overline{\partial}^\star w\|^2,\end{aligned}$$

which proves (6.1).

Next, assume that ω is arbitrary, and let ω' be a complete Kähler metric. For each $\varepsilon > 0$, $\omega_\varepsilon := \omega + \varepsilon\omega'$ is then complete as well. The monotonicity property of Proposition 5.2 implies

$$\int_X \langle [\theta, \omega_\varepsilon^\star]^{-1}v, v\rangle_{\omega_\varepsilon} dV_{\omega_\varepsilon} \leq C,$$

and the first part of the proof yields the existence of $u_\varepsilon \in L^2_{\mathrm{loc}}(X, \Lambda^{n,q-1}T^\star_X \otimes L)$ with $\overline{\partial} u_\varepsilon = v$ and $\int_X |u_\varepsilon|^2_{\omega_\varepsilon} dV_{\omega_\varepsilon} \le C$. In particular, u_ε remains bounded in L^2_{loc}, and an easy weak compactness argument yields the desired solution u.

To see the final point, note that we can always replace u by its orthogonal projection in $(\ker \overline{\partial})^\perp$. As above, this implies $\overline{\partial}^\star u = 0$ and hence $\Delta_{\overline{\partial}} u = \overline{\partial}^\star v$ in the sense of distributions. If v is smooth, then u is smooth by ellipticity of the Laplacian. □

6.2. **The singular case.**

Theorem 6.5. *Let (X, ω) be a Kähler manifold, and assume that X contains a Stein Zariski open subset. Let L be a Hermitian holomorphic line bundle on X with curvature form θ, and let also φ be a quasi-psh function such that $\theta + i\partial\overline{\partial}\varphi \ge \eta > 0$ in the sense of currents for some positive $(1,1)$-form η. If*

$$v \in L^2_{\mathrm{loc}}(X, \Lambda^{n,q}T^\star_X \otimes L)$$

satisfies $\overline{\partial} v = 0$ and

$$\int_X \langle [\eta, \omega^\star]^{-1} v, v\rangle e^{-2\varphi} dV_\omega < +\infty,$$

then $v = \overline{\partial} u$ for some $u \in L^2(X, \Lambda^{n,q-1}T^\star_X \otimes L)$ such that

$$\int_X |u|^2 e^{-2\varphi} dV_\omega \le \int_X \langle [\eta, \omega^\star]^{-1} v, v\rangle e^{-2\varphi} dV_\omega.$$

Note that θ is not assumed to be positive here. The current $\theta + dd^c\varphi$ should be interpreted as the curvature of the singular metric $|\cdot|_L e^{-\varphi}$. As in Remark 6.2, the result does in fact not involve ω when $q = 1$.

Lemma 6.6. *If u, are L^2_{loc} L-valued forms on a complex manifold X such that $\overline{\partial} u = v$ outside a closed analytic subset $A \subset X$, then $\overline{\partial} u = v$ on X.*

Proof. Arguing by induction on $\dim A$, it is enough to prove that $\overline{\partial} u = v$ on $X \setminus A_{\mathrm{sing}}$, which follows from Proposition 4.8 since A_{reg} is a closed submanifold of $X \setminus A_{\mathrm{sing}}$, of real codimension at least 2. □

Proof of Theorem 6.5. By Lemma 6.6, we may assume that X is Stein. By Theorem 3.8, we may then find an exhaustion of X by weakly pseudoconvex open subsets Ω_j such that $\varphi|_{\Omega_j}$ is the decreasing limit of a sequence $\varphi_{j,k} \in C^\infty(\Omega_j)$ with $\theta + dd^c\varphi_{j,k} \ge \eta$. By Lemma 6.3, Ω_j admits a complete Kähler metric, and Theorem 6.1 yields the existence of $u_{j,k} \in L^2(\Omega_j, \Lambda^{n,q-1}T^\star_{\Omega_j} \otimes L)$ such that $\overline{\partial} u_{j,k} = v$ on Ω_j and

$$\int_{\Omega_j} |u_{j,k}|^2 e^{-2\varphi_{j,k}} dV_\omega \le \int_{\Omega_j} \langle [\eta, \omega^\star]^{-1} v, v\rangle e^{-2\varphi_{j,k}} dV_\omega$$

$$\le C := \int_X \langle [\eta, \omega^\star]^{-1} v, v\rangle e^{-2\varphi} dV_\omega,$$

By monotonicity of $(\varphi_{j,k})_k$, we get for $k \ge l$ $\int_{\Omega_j} |u_{j,k}|^2 e^{-2\varphi_{j,l}} dV_\omega \le C$, which shows in particular that $(u_{j,k})_k$ is bounded in $L^2(\Omega_j, e^{-2\varphi_{j,l}})$. After passing to a subsequence, we thus assume that that $u_{j,k}$ converges weakly in $L^2(\Omega_j, e^{-2\varphi_{j,l}})$ to u_j, which may further be asumed to be the same for all l, by a diagonal argument. We then have $\overline{\partial} u_j = v$, and $\int_{\Omega_j} |u_j|^2 e^{-2\varphi_{j,l}} dV_\omega \le C$ for all l, hence $\int_{\Omega_j} |u_j|^2 e^{-2\varphi} dV_\omega \le C$ by monotone convergence of $\varphi_{j,l} \to \varphi$. By a diagonal argument, we may finally arrange that $u_j \to u$ weakly in $L^2(K, e^{-2\varphi})$ for each compact $K \subset X$, and we obtain the desired solution. □

Corollary 6.7. *Let (X, ω) be a Kähler manifold, and assume that X contains a Stein Zariski open subset. Let L be a Hermitian holomorphic line bundle on X with curvature form θ, and let also φ be a quasi-psh function such that $\theta + i\partial\overline{\partial}\varphi \ge \varepsilon\omega$ (resp. $\theta + i\partial\overline{\partial}\varphi + \mathrm{Ric}(\omega) \ge \varepsilon\omega > 0$) in the sense of currents for some positive $(1,1)$-form η. If $v \in L^2(X, \Lambda^{n,1} T_X^\star \otimes L)$ (resp. $v \in L^2(X, \Lambda^{0,1} T_X^\star \otimes L)$ satisfies $\overline{\partial} v = 0$, then $v = \overline{\partial} u$ for some $u \in L^2(X, \Lambda^{n,q-1} T_X^\star \otimes L)$ (resp. $L^2(X, \Lambda^{0,q-1} T_X^\star \otimes L)$) such that*

$$\int_X |u|^2 e^{-2\varphi} dV_\omega \le \frac{1}{q\varepsilon} \int_X |v|^2 e^{-2\varphi} dV_\omega.$$

Proof. The case of (n,q)-forms is a direct consequence of Proposition 5.2 and Theorem 6.5. The case of $(0,q)$-forms follows by viewing a $(0,q)$-form v with values in L as an $(n,1)$-form with values in $L - K_X$, which has curvature $\theta + \mathrm{Ric}(\omega)$. □

7. The Ohsawa–Takegoshi extension theorem

Extension theorems with L^2 estimates have a rather long history, going back to an original result of Ohsawa and Takegoshi. The following version is a slight refinement of [Ber, Theorem 2.5] (compare [Dem12, Theorem 13.6]).

Recall that for a smooth hypersurface S in a complex manifold X, we have a canonical isomorphism $K_S \simeq (K_X + S)|_S$, given by the Poincaré residue map Res_S.

Theorem 7.1. *Let X be a complex manifold containing a Stein Zariski open subset, and consider the following data:*

(i) *a smooth complex hypersurface $S \subset X$ with canonical section $s \in H^0(X, \mathcal{O}(S))$, and a smooth metric on $\mathcal{O}(S)$ with curvature form θ_S, such that $|s| \le e^{-1}$;*
(ii) *a Hermitian holomorphic line bundle L on X with curvature form θ_L, and a quasi-psh function φ such that $\theta_L + i\partial\overline{\partial}\varphi \ge 0$ and $\theta_L + i\partial\overline{\partial}\varphi \ge \theta_S$;*

Then for each $\sigma \in H^0(S, K_S + L|_S)$ such that $\int_S |\sigma|^2 e^{-2\varphi} < +\infty$, there exists $\widetilde{\sigma} \in H^0(X, K_X + S + L)$ with $\mathrm{Res}_S(\widetilde{\sigma}) = \sigma$ and

$$\int_X \frac{|\widetilde{\sigma}|^2}{|s|^2 \log^2 |s|} e^{-2\varphi} \le C \int_S |\sigma|^2 e^{-2\varphi} \tag{7.1}$$

with $C > 0$ a purely numerical constant.

The singular weight $\frac{1}{|s|^2 \log^2 |s|}$ in the left-hand integral is said to have *Poincaré growth* (by analogy with the Poincaré metric $\frac{|dz|^2}{|z|^2 \log^2 |z|}$ of the punctured disc). In particular, it is integrable, but more singular that $|s|^{-2c}$ for any $c < 1$.

Before entering the proof of Theorem 7.1, we state a 'local' version of the result.

Corollary 7.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain, φ a psh function on Ω, and $L \subset \mathbb{C}^n$ a (complex) affine subspace. Then every holomorphic function $f \in \mathcal{O}(L \cap \Omega)$ such that $|f|e^{-\varphi}$ is L^2 admits a holomorphic extension $\widetilde{f} \in \mathcal{O}(\Omega)$ with*

$$\int_\Omega |\widetilde{f}|^2 e^{-2\varphi} \le C \int_{L\cap\Omega} |f|^2 e^{-2\varphi}$$

for a constant $C > 0$ only depending on n and the diameter of Ω.

Proof. Writing L as a successive intersection of affine hyperplanes, it is enough to treat the case where $L = H$ is a hyperplane. The canonical bundles of Ω and $H \cap \Omega$ are trivial, and the metrics they inherit from the Euclidian metric on $\mathbb{C}^n$ are flat. We thus obtain Corollary 7.2 as a direct consequence of Theorem 7.1, which would even allow to add a denominator $|\ell|^2 \log |\ell|^2$ in the left-hand integral, with $\{\ell = 0\}$ an affine equation for H. □

The proof of Theorem 7.1 presented below is essentially an adaptation to the 'global setting' of Błocki's exposition [Bło13] of B.Y.Chen's proof [Chen].

7.1. **Reduction to a twisted L^2-estimate.** By the removable singularity property of L^2 holomorphic functions, it is enough to prove Theorem 7.1 when X itself is Stein. By cohomology vanishing, σ automatically admits a holomorphic extension $\widetilde{\sigma} \in H^0(X, K_X + F)$ with $F := S + L$. Arguing by compactness as in the proof of Theorem 6.5, we may further assume that $\varphi = 0$, $\theta_L > 0$ and $\theta_L > \theta_S$, and $\int_X \frac{|\widetilde{\sigma}|^2}{|s|^2 \log^2 |s|} < +\infty$ (but without any uniform estimate!).

A natural approach to prove Theorem 7.1 is to introduce a cut-off function

$$\chi_\varepsilon := \chi\left(\varepsilon^{-1}|s|^2\right)$$

with $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $t \le 1/2$ and $\chi(t) = 0$ for $t \ge 1$, and solve

$$\overline{\partial} u_\varepsilon = \overline{\partial}(\chi_\varepsilon \widetilde{\sigma}) = \varepsilon^{-1}\chi'\left(\varepsilon^{-1}|s|^2\right)\overline{\partial}\left(|s|^2\right)\widetilde{\sigma} \tag{7.2}$$

with an adequate L^2 estimate with respect to singular weight $\log |s|$. The F-valued $(n,0)$-form

$$\widetilde{\sigma}_\varepsilon := \chi_\varepsilon \widetilde{\sigma} - u_\varepsilon$$

is then also a holomorphic extension of σ, since u_ε is holomorphic near S, and hence vanishes on S (as a section of $K_X + F$) thanks to the L^2-estimate. Since

$$\lim_{\varepsilon\to 0} \int_X \frac{|\chi_\varepsilon \widetilde{\sigma}|^2}{|s|^2 \log^2 |s|} = 0$$

by integrability of the Poincaré weight, it would thus be enough to solve (7.2) with an estimate of the form

$$\int_X \frac{|u_\varepsilon|^2 |s|^{-2}}{\log^2(|s|^2 + \varepsilon)} \le \int_{\operatorname{supp} \overline{\partial} u_\varepsilon} |\widetilde{\sigma}|^2 |s|^{-2}. \tag{7.3}$$

Indeed,

$$\limsup_{\varepsilon\to 0}\int_{\operatorname{supp}\overline{\partial}u_\varepsilon}|\widetilde{\sigma}|^2|s|^{-2}\le C\int_S|\sigma|^2$$

since

$$\operatorname{supp}\overline{\partial}u_\varepsilon\subset\{\varepsilon/2\le|s|^2\le\varepsilon\},$$

and we would get the desired extension of σ as a weak L^2_{loc} limit of $\widetilde{\sigma}_\varepsilon$.

In view of (7.2), a direct application of Theorem 6.5 with weight function $\psi=\log|s|$ yields instead

$$\int_X|u_\varepsilon|^2|s|^{-2}\le C\int_{\{\varepsilon/2\le|s|^2\le\varepsilon\}}\left|\varepsilon^{-1}\overline{\partial}\left(|s|^2\right)\right|^2_\theta|\widetilde{\sigma}|^2|s|^{-2},$$

where $\left|\varepsilon^{-1}\overline{\partial}\left(|s|^2\right)\right|_\theta$ is a priori unbounded even when $|s|^2\sim\varepsilon$. The trick will be to solve (7.2) with a *twisted* L^2-estimate, replacing in particular the right-hand side with

$$\int_{\{\varepsilon/2\le|s|^2\le\varepsilon\}}e^{2\tau_\varepsilon}\left|\varepsilon^{-1}\overline{\partial}\left(|s|^2\right)\right|^2_{\theta_\varepsilon}|\widetilde{\sigma}|^2|s|^{-2},$$

where τ_ε is a smooth function such that $\theta_\varepsilon:=\theta_L+i\partial\overline{\partial}\tau_\varepsilon>0$ and

$$e^{\tau_\varepsilon}\left|\overline{\partial}\left(|s|^2\right)\right|_{\theta_\varepsilon}=O(\varepsilon)$$

when $|s|^2\sim\varepsilon$.

The general principle underlying these twisted estimates is as follows.

Lemma 7.3. *Let X be a Stein manifold, F be a Hermitian holomorphic line bundle with curvature form θ_F, and ψ be a quasi-psh function such that $\theta_F+i\partial\overline{\partial}\psi\ge\eta$ with η a Kähler form. Pick a $\overline{\partial}$-closed $v\in L^2_{\mathrm{loc}}(X,\Lambda^{n,1}T^\star_X\otimes F)$, and assume given a strictly η-psh function $\tau\in C^\infty(X)$ such that*

(i) $|\overline{\partial}\tau|^2_{\eta_\tau}\le 1/4$ *on X;*
(ii) $|\overline{\partial}\tau|^2_{\eta_\tau}\le 1/8$ *on* $\operatorname{supp}v$.

Then there exists $u\in L^2(X,\Lambda^{n,0}T^\star_X\otimes F)$ such that $\overline{\partial}u=v$ and

$$\int_X\left(1-4|\overline{\partial}\tau|^2_{\eta_\tau}\right)|u|^2e^{2(\tau-\psi)}\le 10\int_X i^{n^2}\{v,\bar v\}_{\eta_\tau}e^{2(\tau-\psi)}.$$

Proof. Arguing by regularization and weak compactness as in the proof of Theorem 6.5, we may assume that $\psi=0$. Let u be the minimal solution of $\overline{\partial}u=v$ in L^2. Since u is orthogonal to $\ker\overline{\partial}$ in $L^2(\psi)$, $ue^{2\tau}$ is orthogonal to $\ker\overline{\partial}$ in $L^2(\tau)$, and hence is the minimal solution of $\overline{\partial}(ue^{2\tau})=(v+2u\,\overline{\partial}\tau)e^{2\tau}$ in $L^2(\tau)$. By Theorem 6.1, we thus have

$$\int_X|u|^2e^{2\tau}\le\int_X i^{n^2}\left\{v+2u\,\overline{\partial}\tau,\overline{v+2u\,\overline{\partial}\tau}\right\}_{\eta_\tau}e^{2\tau}.$$

Using the Cauchy-Schwarz inequality combined with $2ab\le ta^2+t^{-1}b^2$ for $a,b\in\mathbb{R}$ and $t>0$, we infer

$$\int_X\left(1-4|\overline{\partial}\tau|^2_{\eta_\tau}\right)e^{2\tau}|u|^2\le t\int_{\operatorname{supp}v}|u|^2e^{2\tau}+(1+t^{-1})\int_X i^{n^2}\{v,\bar v\}_{\eta_\tau}e^{2\tau}.$$

Since $t:=1/4\le\left(1-4|\overline{\partial}\tau|^2_{\eta_\tau}\right)/2$ on $\operatorname{supp}v$, we get the desired result. $\square$

7.2. **Proof of Theorem 7.1.** Following the above strategy, suppose given τ_ε such that

(i) $\theta_\varepsilon := \theta_L + i\partial\overline{\partial}\tau_\varepsilon > 0$;
(ii) $4|\overline{\partial}\tau_\varepsilon|^2_{\theta_\varepsilon} \le 1$ on X;
(iii) $4|\overline{\partial}\tau_\varepsilon|^2_{\theta_\varepsilon} \le 1/2$ on $\{\varepsilon/2 \le |s|^2 \le \varepsilon\}$.

Applying Lemma 7.3 to $F = L + S$, $\psi = \log|s|$ and $\eta = \theta_L$, we get a solution u_ε of (7.2) such that

$$\int_X e^{2\tau_\varepsilon}\left(1 - 4|\overline{\partial}\tau_\varepsilon|^2_{\theta_\varepsilon}\right)|u_\varepsilon|^2|s|^{-2} \le C\int_{\{\varepsilon/2\le|s|^2\le\varepsilon\}} e^{2\tau_\varepsilon}\left|\varepsilon^{-1}\overline{\partial}\left(|s|^2\right)\right|^2_{\theta_\varepsilon}|\widetilde{\sigma}|^2|s|^{-2}.$$

We will thus be done if τ_ε further satisfies

(iv) $e^{2\tau_\varepsilon}\left(1 - 4|\overline{\partial}\tau_\varepsilon|^2_{\theta_\varepsilon}\right) \ge C^{-1}/\log^2\left(|s|^2+\varepsilon\right)$;
(v) $e^{\tau_\varepsilon}\left|\overline{\partial}\left(|s|^2\right)\right|_{\theta_\varepsilon} \le C\varepsilon$ on $\{\varepsilon/2 \le |s|^2 \le \varepsilon\}$.

This will achieved by means of the following construction.

Lemma 7.4. *Set $\rho_\varepsilon := -\log(|s|^2+\varepsilon)$. Then $\tau_\varepsilon := -\log\left(\rho_\varepsilon + \log\rho_\varepsilon\right)$ satisfies (iv), (v), and $\left|\overline{\partial}\tau_\varepsilon\right|^2_{\theta_{\tau_\varepsilon}} \le C/\log\varepsilon^{-1}$ on $\{\varepsilon/2 \le |s|^2 \le \varepsilon\}$.*

Proof. We have $\tau_\varepsilon = -\log f(\rho_\varepsilon)$ with $f(t) = t + \log t$. Outside S, we compute

$$-\overline{\partial}\rho_\varepsilon = \frac{\overline{\partial}|s|^2}{|s|^2+\varepsilon},$$

$$-\overline{\partial}\tau_\varepsilon = \frac{f'(\rho_\varepsilon)}{f(\rho_\varepsilon)}\overline{\partial}\rho_\varepsilon, \tag{7.4}$$

$$\begin{aligned}-i\partial\overline{\partial}\rho_\varepsilon = i\partial\frac{\overline{\partial}|s|^2}{|s|^2+\varepsilon} &= \frac{\varepsilon}{|s|^2}i\partial\rho_\varepsilon\wedge\overline{\partial}\rho_\varepsilon - \frac{|s|^2}{|s|^2+\varepsilon}\theta_S\\ &\ge \frac{\varepsilon}{|s|^2}\left(\frac{f(\rho_\varepsilon)}{f'(\rho_\varepsilon)}\right)^2 i\partial\tau_\varepsilon\wedge\overline{\partial}\tau_\varepsilon - \theta_L,\end{aligned}$$

since $\theta_L \ge \theta_S$ and $\theta_L \ge 0$. This yields

$$i\partial\overline{\partial}\tau_\varepsilon = -\frac{f'(\rho_\varepsilon)}{f(\rho_\varepsilon)}i\partial\overline{\partial}\rho_\varepsilon + \left(1 + \frac{f(\rho_\varepsilon)}{f'(\rho_\varepsilon)^2\rho_\varepsilon^2}\right)i\partial\tau_\varepsilon\wedge\overline{\partial}\tau_\varepsilon.$$

Since $|s| \le e^{-1}$, we have $\rho_\varepsilon \ge 2$, and hence $e^{-\tau_\varepsilon} = f(\rho_\varepsilon) \ge 2$, $f'(\rho_\varepsilon) \le 2$. We infer

$$\theta + i\partial\overline{\partial}\tau_\varepsilon \ge e^{-\tau_\varepsilon}\left(\frac{\varepsilon}{2|s|^2} + 1 + \frac{1}{4\log^2(|s|^2+\varepsilon)}\right)i\partial\tau_\varepsilon\wedge\overline{\partial}\tau_\varepsilon.$$

Finally,

$$\overline{\partial}\left(|s|^2\right) = \frac{f(\rho_\varepsilon)}{f'(\rho_\varepsilon)}(|s|^2+\varepsilon)\overline{\partial}\tau_\varepsilon,$$

and the desired estimates easily follow from this. □

8. Blowups and valuations

The goal of this section is to recall a number of basic facts on blowups, integral closures and valuations, in the setting of complex analytic spaces.

8.1. **Blowing-up along an ideal.** Assume that X is a reduced complex space. If $\mathfrak{a} \subset \mathcal{O}_X$ is a coherent ideal (sheaf) defining a complex subspace $Z \subset X$, the *blowup* $\mu : X' \to X$ of X along $\mathfrak{a}$ (or Z) is defined by setting

$$X' = \mathrm{Bl}_{\mathfrak{a}}(X) = \mathrm{Proj}_X \left(\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m \right).$$

Since X is reduced, the graded $\mathcal{O}_X$-algebra $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m$ is reduced, and X' is thus reduced as well. The morphism μ is projective, and an isomorphism over $X \setminus Z$; it is thus a proper bimeromorphic morphism, i.e. a *modification*. Note also that there is canonical isomorphism $\mathrm{Bl}_{\mathfrak{a}^m}(X) \simeq \mathrm{Bl}_{\mathfrak{a}}(X)$ for any nonzero $m \in \mathbb{N}$.

The blowup can be more concretely understood as the graph of a meromorphic map, as follows. After shrinking X near a given point, we can pick generators $(f_1, \dots, f_r)$ of $\mathfrak{a}$. This induces a surjection

$$\mathcal{O}_X[t_1, \dots, t_r] \twoheadrightarrow \bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m,$$

and hence a closed embeddding $X' \hookrightarrow X \times \mathbb{P}^{r-1}$ over X, which can be shown to coincide with the graph of the bimeromorphic map $X \dashrightarrow \mathbb{P}^{r-1}$ defined by $[f_1 : \cdots : f_r]$ in homogeneous coordinates.

Blowups are characterized by the following universal property.

Lemma 8.1. *The scheme-theoretic inverse image $E = \mu^{-1}(Z)$ is a Cartier divisor, i.e. $\mathfrak{a} \cdot \mathcal{O}_{X'}$ is locally principal ideal sheaf. Further, $-E$ is μ-very ample, and*

$$\mu_* \mathcal{O}_{X'}(-mE) = \mathfrak{a}^m$$

near any given compact subset of X for all $m \gg 1$. Finally, a proper bimeromorphic morphism $\rho : Y \to X$ with Y reduced factors through μ if and only if $\rho^{-1}(Z)$ is a Cartier divisor.

Assume now that the complex space X is normal. The *normalized blowup* $\pi : \widetilde{X} \to X$ of X along a coherent ideal $\mathfrak{a} \subset \mathcal{O}_X$ is defined as the normalization of the blowup $\mathrm{Bl}_{\mathfrak{a}}(X)$, i.e. $\pi = \mu \circ \nu$ with $\nu : \widetilde{X} \to X'$ the normalization morphism. Again, π is a projective bimeromorphic morphism, and an isomorphism over $X \setminus Z$. Normalized blowups are again characterized by a universal property:

Lemma 8.2. *The scheme-theoretic inverse image $D = \pi^{-1}(Z)$ is a Cartier divisor, and a proper bimeromorphic morphism $\rho : Y \to X$ with Y normal factors through π if and only if $\rho^{-1}(Z)$ is a Cartier divisor. Further, $-D$ is π-ample and π-globally generated.*

The meaning of this last property is that the evaluation map

$$\pi^\star \left(\pi_* \mathcal{O}_{\widetilde{X}}(-D) \right) \otimes \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(-D)$$

is surjective. Since X is normal, $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ (i.e. , π has connected fiber), by Zariski's 'main theorem'. We can thus interpret

$$\mathfrak{b} := \pi_* \mathcal{O}_{\widetilde{X}}(-D) \subset \pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$$

as an ideal sheaf, which satisfies $\mathfrak{b} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-D)$. As we shall next see, $\mathfrak{b}$ can be understood algebraically as the integral closure of $\mathfrak{a}$.

8.2. **The integral closure of an ideal.** If $I \subset A$ is an ideal in an integrally closed domain, the *integral closure* $\bar{I}$ of I is defined as the set of $f \in A$ satisfying a polynomial equation of the form

$$f^d + \sum_{j=1}^{d} a_j f^{d-j} = 0 \tag{8.1}$$

with $a_j \in I^j$. Using the usual determinant trick, one shows that $f \in A$ belongs to $\bar{I}$ iff there exists a finitely generated nonzero ideal $J \subset A$ such that $fJ \subset IJ$. This implies in particular that $\bar{I}$ is an ideal of A, and we say of course that I is integrally closed if $\bar{I} = I$.

Example 8.3. If $I \subset A$ is either principal or reduced, then I is integrally closed.

If now $\mathfrak{a} \subset \mathcal{O}_X$ is a coherent ideal sheaf on a normal complex space X, the integral closure of $\mathfrak{a}$ is the ideal sheaf $\bar{\mathfrak{a}} \subset \mathcal{O}_X$ defined by taking the integral closure stalkwise. Coherence of $\bar{\mathfrak{a}}$ is non-obvious, and a consequence of the following geometric characterization of $\bar{\mathfrak{a}}$.

Theorem 8.4. *Let $\pi : \widetilde{X} \to X$ be the normalized blowup of the normal complex X along a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, and denote by D the effective Cartier divisor it determines on $\widetilde{X}$. Then*

$$\pi_* \mathcal{O}_{\widetilde{X}}(-D) = \bar{\mathfrak{a}}$$

Remark 8.5. For each $m \geq 1$, $\widetilde{X}$ is also the normalized blowup of $\mathfrak{a}^m$, and hence $\pi_* \mathcal{O}_{\widetilde{X}}(-mD) = \overline{\mathfrak{a}^m}$. In more algebraic terms, the result thus means that $\bigoplus_{m \in \mathbb{N}} \overline{\mathfrak{a}^m}$ is the integral closure of the graded domain $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m$.

As a consequence of Theorem 8.4, we get the following neat analytic characterization of the integral closure.

Corollary 8.6. *Let $\mathfrak{a}$ be an ideal with local generators (f_i). Then $f \in \mathcal{O}_X$ belongs to $\bar{\mathfrak{a}}$ if and only if $|f| \leq O(\max_i |f_i|)$.*

Proof of Theorem 8.4. Recall that $\mathfrak{b} := \pi_* \mathcal{O}_{\widetilde{X}}(-D)$ is a coherent ideal sheaf on X, and

$$\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathfrak{b} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-D).$$

If $f \in \mathcal{O}_X$ is integral over $\mathfrak{a}$, then $f \circ \pi$ is integral over the principal ideal $\mathfrak{b} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-D)$. By normality of $\widetilde{X}$, it follows that $f \circ \pi \in \mathcal{O}_{\widetilde{X}}(-D)$, i.e. $f \in \pi_* \mathcal{O}_{\widetilde{X}}(-D) = \mathfrak{b}$, and we have proved that $\bar{\mathfrak{a}} \subset \mathfrak{b}$. The converse inclusion is more difficult, and we reproduce the elegant geometric argument of [Laz, II.11.1.7]. The statement being local over X, we may choose a system of generators $(f_1, \dots, f_r)$ for $\mathfrak{a}$. This defines a surjection $\mathcal{O}_X^{\oplus r} \to \mathfrak{a}$, which induces, after pulling-back to $\widetilde{X}$ and twisting by $-mD$, a surjection

$$\mathcal{O}_{\widetilde{X}}(-mD)^{\oplus r} \to \mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}(-mD) = \mathcal{O}_{\widetilde{X}}(-(m+1)D)$$

for any $m \geq 1$. Set $\mathfrak{b}_m := \pi_* \mathcal{O}_{\widetilde{X}}(-mD)$, so that $\mathfrak{b} = \mathfrak{b}_1$. Since $-D$ is π-ample, Serre vanishing implies that the induced map

$$\mathfrak{b}_m^{\oplus r} = \pi_* \mathcal{O}_{\widetilde{X}}(-mD)^{\oplus r} \to \mathfrak{b}_{m+1} = \pi_* \mathcal{O}_{\widetilde{X}}(-(m+1)D)$$

is surjective for $m \gg 1$, i.e. $\mathfrak{a} \cdot \mathfrak{b}_m = \mathfrak{b}_{m+1}$. Since $\mathfrak{b} \cdot \mathfrak{b}_m \subset \mathfrak{b}_{m+1}$, $\mathfrak{b}$ thus acts on the nonzero ideal $\mathfrak{b}_m$ by multiplication with $\mathfrak{a}$, and hence $\mathfrak{b} \subset \overline{\mathfrak{a}}$ by the determinantal trick recalled above. □

Remark 8.7. The above proof shows that $\mathfrak{a} \cdot \overline{\mathfrak{a}^m} = \overline{\mathfrak{a}} \cdot \overline{\mathfrak{a}^m} = \overline{\mathfrak{a}^{m+1}}$ for all $m \gg 1$.

Proof of Corollary 8.6. If $f \in \mathcal{O}_X$ satisfies an integral equation $f^d + \sum_{i=1}^d a_i f^{d-i} = 0$ with $a_i \in \mathfrak{a}^i$, we have at each point $|f|^d \leq d|a_i||f|^{d-i}$ for some i, and hence

$$|f| \leq \max_i (d|a_i|)^{1/i} = O(\max_i |f_i|).$$

If conversely $|f| = O(\max_i |f_i|)$, then $|f \circ \pi| = O(|f_D|)$ locally on $\widetilde{X}$, with f_D a local equation of D. By normality of $\widetilde{X}$, it follows that $f \circ \pi \in \mathcal{O}_{\widetilde{X}}(-D)$, i.e. $f \in \pi_* \mathcal{O}_{\widetilde{X}}(-D) = \overline{\mathfrak{a}}$. □

Example 8.8. If φ is a quasi-psh function on a complex manifold X, then the multiplier ideal sheaf $\mathcal{J}(\varphi)$ is integrally closed. Indeed, if $f \in \mathcal{O}_X$ satisfies $|f| = O(\max_i |f_i|)$ for a set of generators (f_i) of $\mathcal{J}(\varphi)$, then $|f_i|e^{-\varphi} \in L^2_{\mathrm{loc}}$ implies $|f|e^{-\varphi} \in L^2_{\mathrm{loc}}$, i.e. $f \in \mathcal{J}(\varphi)$.

8.3. **Valuative ideals.** In what follows, X denotes a normal complex space, a condition which implies in particular that the singular locus X_{sing} has codimension at least 2.

Let $Z \subset X$ be an irreducible closed analytic subset, and assume that Z is not entirely contained in X_{sing}. For each $f \in \mathcal{O}_X$,

$$\mathrm{ord}_Z(f) := \inf_{z \in Z \cap X_{\mathrm{reg}}} \mathrm{ord}_z(f)$$

satisfies $\mathrm{ord}_Z(f) = \mathrm{ord}_z(f)$ for a general point $z \in Z$ (i.e. outside a strict closed analytic subset), and ord_Z thus defines a valuation on $\mathcal{O}_X$.

The condition $Z \not\subset X_{\mathrm{sing}}$ is in particular satisfied if Z has codimension 1, i.e. is a prime divisor. More generally, consider a modification (i.e. a proper, bimeromorphic morphism) $\mu : Y \to X$ with Y a complex manifold, and a prime divisor E on Y. We then say that E is a *divisor over* X, and attach to it a valuation ord_E on germs $f \in \mathcal{O}_X$ by setting

$$\mathrm{ord}_E(f) := \mathrm{ord}_E(f \circ \mu) \in \mathbb{N}.$$

The *center* of E on X is the irreducible closed analytic subset

$$c_X(E) := \mu(E) \subset X.$$

If $\rho : Y' \to Y$ is another modification and E' is the strict transform, then $c_X(E) = c_{X'}(E')$ and $\mathrm{ord}_E = \mathrm{ord}_{E'}$ as functions on $\mathcal{O}_X$.

Given a coherent ideal $\mathfrak{a} \subset \mathcal{O}_X$, we set

$$\mathrm{ord}_E(\mathfrak{a}) := \inf_{f \in \mathfrak{a}_x} \mathrm{ord}_E(f)$$

for a general point x in $\mu(E)$. It is easy to see that the infimum is achieved among any given set of local generators of $\mathfrak{a}_x$. For any two coherent ideals $\mathfrak{a}, \mathfrak{b}$, the valuation property of ord_E yields

$$\mathrm{ord}_E(\mathfrak{a} \cdot \mathfrak{b}) = \mathrm{ord}_E(\mathfrak{a}) + \mathrm{ord}_E(\mathfrak{b})$$

and
$$\operatorname{ord}_E(\mathfrak{a}+\mathfrak{b}) = \min\{\operatorname{ord}_E(\mathfrak{a}), \operatorname{ord}_E(\mathfrak{b})\}.$$

Definition 8.9. A coherent ideal $\mathfrak{a} \subset \mathcal{O}_X$ is *valuative* if
$$\mathfrak{a} = \{f \in \mathcal{O}_X \mid \operatorname{ord}_{E_\alpha}(f) \geq c_\alpha \text{ for all } \alpha\}$$
for a (possibly infinite) family of prime divisors E_α over X and constants $c_\alpha \in \mathbb{R}$.

Clearly, the constants can always be taken to be $c_\alpha = \operatorname{ord}_{E_\alpha}(\mathfrak{a})$. The next result is simply a reformulation of Theorem 8.4. The next result is basically a reformulation of Theorem 8.4.

Theorem 8.10. *Let $\mathfrak{a} \subset \mathcal{O}_X$ be a coherent ideal, and denote by E_α the irreducible components of the divisor D lying over $\mathfrak{a}$ in the normalized blowup $\pi : \widetilde{X} \to X$ along $\mathfrak{a}$. Then*
$$\overline{\mathfrak{a}} = \{f \in \mathcal{O}_X \mid \operatorname{ord}_{E_\alpha}(f) \geq \operatorname{ord}_{E_\alpha}(\mathfrak{a})\}.$$

The divisors E_α are known as the *Rees divisors* of $\mathfrak{a}$, and the corresponding valuations $\operatorname{ord}_{E_\alpha}$ are the *Rees valuations.*

Proof. By definition, we have
$$\operatorname{ord}_{E_\alpha}(\mathfrak{a}) = \operatorname{ord}_{E_\alpha}(\mathfrak{a}\cdot\mathcal{O}_{\widetilde{X}}) = \operatorname{ord}_{E_\alpha}(\mathcal{O}_{\widetilde{X}}(-D)) = \operatorname{ord}_{E_\alpha}(D),$$
and hence $D = \sum_\alpha \operatorname{ord}_{E_\alpha}(\mathfrak{a})E_\alpha$. But then $f \in \mathcal{O}_X$ belongs to the ideal $\overline{\mathfrak{a}} = \pi_*\mathcal{O}_{\widetilde{X}}(-D)$ if and only if $\operatorname{ord}_{E_\alpha}(f) = \operatorname{ord}_{E_\alpha}(f\circ\pi) \geq \operatorname{ord}_{E_\alpha}(\mathfrak{a})$ for all α. □

Corollary 8.11. *A coherent ideal is valuative if and only if it is integrally closed.*

Proof. Theorem 10.13 implies that any integrally closed ideal is valuative. The converse is elementary, and similar to the easy direction in Corollary 8.6. Assume that $\mathfrak{a} = \{f \in \mathcal{O}_X \mid \operatorname{ord}_{E_\alpha}(f) \geq \operatorname{ord}_{E_\alpha}(f) \text{ for all } \alpha\}$ for some family (E_α) of prime divisors over X, and pick $f \in \mathcal{O}_X$ such that $f^d + \sum_{i=1}^d a_i f^{d-i} = 0$ with $a_i \in \mathfrak{a}^i$. Since $\operatorname{ord}_{E_\alpha}$ is a valuation, we get
$$d\operatorname{ord}_{E_\alpha}(f) = \operatorname{ord}_{E_\alpha}(f^d) \geq \min_{1\leq i\leq d}(\operatorname{ord}_{E\alpha}(a_i) + (d-i)\operatorname{ord}_{E_\alpha}(f)).$$
It we pick i achieving the minimum, we get
$$i\operatorname{ord}_{E_\alpha}(f) \geq \operatorname{ord}_{E_\alpha}(a_i) \geq \operatorname{ord}_{E_\alpha}(\mathfrak{a}^i) = i\operatorname{ord}_{E_\alpha}(\mathfrak{a}),$$
and hence $f \in \mathfrak{a}$. □

Example 8.12. Let $Y \subset X$ be an irreducible closed analytic subset. For each $m \in \mathbb{N}$, the ideal sheaf
$$\mathcal{I}_Y^{\langle m\rangle} := \{f \in \mathcal{O}_X \mid \operatorname{ord}_Y(f) \geq m\}$$
is called the *m-th symbolic power of* $\mathcal{I}_Y$. If we denote by $\mu : \widetilde{X} \to X$ the normalized blowup of X along Y, the restriction of μ over the manifold $X_{\mathrm{reg}} \setminus Y_{\mathrm{sing}}$ is the blowup along the smooth connected submanifold $X_{\mathrm{reg}} \cap Y_{\mathrm{reg}}$, and $\mu^{-1}(X_{\mathrm{reg}} \cap Y_{\mathrm{reg}})$ is thus irreducible. As a consequence, there exists a unique Rees divisor E_α of Y with $c_X(E_\alpha)$ meeting $X_{\mathrm{reg}} \cap Y_{\mathrm{reg}}$, and then $\operatorname{ord}_Y = \operatorname{ord}_{E_\alpha}$. This shows that $\mathcal{I}_Y^{\langle m\rangle}$ is (coherent and) valuative, and hence integrally closed. In particular, $\overline{\mathcal{I}_Y^m} \subset \mathcal{I}_Y^{\langle m\rangle}$, the inclusion being strict in general, even when X is smooth.

For later use, we now discuss an alternative valuative characterization of integral closure. The data of a germ of holomorphic curve $\gamma : (\mathbb{C}, 0) \to (X, x)$ defines a (semi)valuation $\operatorname{ord}_\gamma$ on $\mathcal{O}_{X,x}$ by setting for $f \in \mathcal{O}_{X,x}$

$$\operatorname{ord}_\gamma(f) := \operatorname{ord}_0(f \circ \gamma) \in \mathbb{N} \cup \{\infty\}$$

For a coherent ideal sheaf $\mathfrak{a}$, we set as above

$$\operatorname{ord}_\gamma(\mathfrak{a}) := \inf_{f \in \mathfrak{a}_x} \operatorname{ord}_\gamma(f),$$

which is again achieved among any given set of generators of $\mathfrak{a}_x$.

Proposition 8.13. *Let $\mathfrak{a} \subset \mathcal{O}_X$ be a coherent ideal sheaf. A germ $f \in \mathcal{O}_X$ belongs to $\overline{\mathfrak{a}}$ if and only if $\operatorname{ord}_\gamma(f) \geq \operatorname{ord}_\gamma(\mathfrak{a})$ for all germs $\gamma : (\mathbb{C}, 0) \to X$.*

In view of Corollary 8.6, the result says that for any function $f \in \mathcal{O}_X$ that does not satisfy $|f| = O(\sum_i |f_i|)$ near $x \in X$, there exists $\gamma : (\mathbb{C}, 0) \to (X, x)$ such that $f \circ \gamma$ is not in the ideal of $\mathcal{O}_{\mathbb{C},0}$ generated by the $f_i \circ \gamma$, a result known as the *curve selection lemma*.

Proof. Since $\operatorname{ord}_\gamma$ is a (semi)valuation, $f \in \overline{\mathfrak{a}}$ implies $\operatorname{ord}_\gamma(f) \geq \operatorname{ord}_\gamma(\mathfrak{a})$ just as in the proof of Corollary 10.15. Assume conversely that this condition holds for all $\gamma : (\mathbb{C}, 0) \to X$, and denote by (E_α) the Rees divisors of $\mathfrak{a}$, i.e. the irreducible components of D on the normalized blowup $\pi : \widetilde{X} \to X$. At a general point of E_α, X is smooth, and E_α is a smooth hypersurface. It is thus easy to find a local gern $\tilde{\gamma} : (\mathbb{C}, 0) \to \widetilde{X}$ transverse to E_α at a general point of X with $\operatorname{ord}_{\tilde{\gamma}}(f \circ \pi) = \operatorname{ord}_{E_\alpha}(f)$ and $\operatorname{ord}_{\tilde{\gamma}}(\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}) = \operatorname{ord}_{E_\alpha}(\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}})$. Setting $\gamma := \pi \circ \gamma : (\mathbb{C}, 0) \to X$, we get

$$\operatorname{ord}_{E_\alpha}(f) = \operatorname{ord}_\gamma(f) \geq \operatorname{ord}_\gamma(\mathfrak{a}) = \operatorname{ord}_{E_\alpha}(\mathfrak{a}).$$

This is true for all α, and hence $f \in \overline{\mathfrak{a}}$. $\square$

9. Fundamental properties of multiplier ideals

9.1. **Singularities of psh functions.** Given two quasi-psh functions φ, ψ, we say that φ is *more singular than* ψ if $\varphi \leq \psi + O(1)$ locally on X.

Definition 9.1. A quasi-psh function φ on a complex manifold is said to have *analytic singularities* if there exists $c > 0$ and a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ such that we locally have

$$\varphi = c \log \left(\sum_j |f_j| \right) + O(1)$$

for some (and hence, any) choice of local generators (f_j) of $\mathfrak{a}$.

We sometimes say more precisely that φ has analytic singularities of type $\mathfrak{a}^c$. If φ is less singular than a quasi-psh function ψ, we simply say that $\mathfrak{a}^c$ is less singular than ψ.

9.2. **Multiplier ideals.** In what follows, X denotes a complex manifold.

Definition 9.2. The *multiplier ideal (sheaf)* of a quasi-psh function φ on a complex manifold X is the ideal sheaf $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ of germs of holomorphic function f with $|f|e^{-\varphi} \in L^2_{\mathrm{loc}}$.

As a first key property, we have:

Theorem 9.3. *Multiplier ideal sheaves are coherent.*

A we shall see later, a similar result actually holds for all L^p_{loc} with $1 \le p < \infty$, cf. Theorem 10.20.

Lemma 9.4. *If $\nu_x(\varphi) \ge n + k$ with $k \in \mathbb{N}$, then $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_x^{k+1}$.*

A partial converse to this elementary result is provided by Corollary 9.18 below.

Proof. Arguing in local coordinates, we may assume that $(X, x) = (\mathbb{C}^n, 0)$. By (10.1), we have $\varphi(z) \le (n+k)\log|z| + O(1)$, and hence $\mathcal{J}(\varphi)_0 \subset \mathcal{J}((n+k)\log|z|)_0$, i.e.

$$\int \frac{|f(z)|^2}{|z|^{2(n+k)}} < +\infty$$

on a small enough polydisc. By Lemma 9.5 below, we have

$$\int \frac{\prod_i |z_i|^{2\alpha_i}}{(\sum_i |z_i|^2)^{n+k)}} < +\infty$$

for each monomial z^α occuring in the Taylor expansion of f, and it is easy to see that this implies $|\alpha| = \sum_i \alpha_i \ge k + 1$ (a special case of Theorem 9.9 below). $\square$

Lemma 9.5. *Let $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha$ be a holomorphic function on $\mathbb{D}^n$, and assume that $h \ge 0$ is measurable on $\mathbb{D}^n$ and radial, i.e. $h(z_1, \dots, z_n)$ only depends on $|z_i|$. Then*

$$\int_{\mathbb{D}^n} |f|^2 h = \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|^2 \int_{\mathbb{D}^n} |z_\alpha|^2 h(z).$$

Proof. Pass to polar coordinates and use the Parseval formula for the L^2-norm of a Fourier series. $\square$

Lemma 9.6. *Let A be an arbitrary set of holomorphic functions, and $\mathfrak{a} \subset \mathcal{O}_X$ the ideal sheaf locally generated by A. Then $\mathfrak{a}$ is coherent.*

Proof. Let I be the directed set of finite subsets of A. For each $i \in I$, the corresponding (globally) finitely generated sheaf $\mathfrak{a}_i$ is coherent. We then have $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$, and for each compact set $K \subset X$, the strong noetherian property of coherent sheaves shows that $\mathfrak{a} = \mathfrak{a}_i$ near K for some i, hence the result. $\square$

Proof of Theorem 9.3. Arguing locally, we may argue in the ball $B \subset \mathbb{C}^n$, and assume that φ is psh. Let $\mathcal{H} = \mathcal{H}(B, \varphi)$ be the Bergman space of holomorphic functions $f \in \mathcal{O}(B)$ such that $\int_B |f|^2 e^{-\varphi} < +\infty$. By Lemma 9.6, the ideal sheaf $\mathfrak{a} \subset \mathcal{O}_B$ generated by $\mathcal{H}$ is coherent, and it is of course contained in $\mathcal{J}(\varphi)$. It will thus be enough to show that $\mathcal{J}(\varphi) = \mathfrak{a}$. By the Rees lemma, it will be enough to show that $\mathcal{J}(\varphi)_x \subset \mathfrak{a}_x + \mathfrak{m}_x^k$ for any given $x \in B$ and $k \in \mathbb{N}$. Pick $f \in \mathcal{J}(\varphi)_x$, and let

$U \subset B$ be an open neighborhood of x such that $f \in \mathcal{O}(U)$ and $\int_U |f|^2 e^{-2\varphi} < +\infty$. Let $\chi \in C_c^\infty(U)$ be a cut-off function with $\chi \equiv 1$ near x. For $C \gg 1$, the function

$$\psi(z) := \varphi(z) + (n+k)\chi(z)\log|z-x| + C|z|^2$$

is satisfies $i\partial\overline{\partial}\psi + \mathrm{Ric}(\omega) = i\partial\overline{\partial}\psi \geq \varepsilon\omega$, where $\omega = \frac{i}{2}\partial\overline{\partial}|z|^2$ is the Kähler form of $\mathbb{C}^n$. Note that

$$v := \overline{\partial}(\chi f) = f\overline{\partial}\chi$$

is L^2 with respect to ψ. By Theorem 6.5, we may thus find a function u such that $\int_B |u|^2 e^{-2\psi} dV < +\infty$ and $\overline{\partial} u = v$. Note that $g := u - \chi f$ belongs to $\mathcal{H}$. Since u is holomorphic near x and $\nu_x(\psi) \geq n+k$, Lemma 9.4 shows that $u \in \mathfrak{m}_x^k$, and we are done. □

9.3. **An example: the toric case.** The results in this section are due to Guenancia [Gue12]. A function defined on the polydisc $\mathbb{D}^n$ is *toric* if it invariant under the compact $(S^1)^n$, i.e. its value at $(z_1, \dots, z_n)$ only depends on $|z_i|$. By Lemma 2.3, a toric function φ on $\mathbb{D}^n$ is psh if and only if

$$\varphi(z_1, \dots, z_n) = \chi(\log|z_1|, \dots, \log|z_n|)$$

where $\chi : \mathbb{R}_-^n \to \mathbb{R}$ is convex and non-decreasing in each variable.

Definition 9.7. The *Newton set* $P(\chi)$ of a convex function $\chi : \mathbb{R}_-^n \to \mathbb{R}$ is the set of $\alpha \in \mathbb{R}^n$ such that $\langle \alpha, \cdot \rangle \leq \chi + O(1)$ on $\mathbb{R}_-^n$.

It is easy to see that $P(\chi)$ is convex and invariant under positive translation $P(\chi) + \mathbb{R}_+^n \subset P(\chi)$. It is however neither open nor closed in general, and we denote by $\bar{P}(\chi)$ and $\mathring{P}(\chi)$ its closure and interior. By definition, $P(\chi)$ is the domain of the Legendre transform

$$\chi^\star(\alpha) = \sup_{x \in \mathbb{R}_-^n} \left(\langle \alpha, x \rangle - \chi(x) \right),$$

and hence

$$\chi(x) = \sup_{\alpha \in P(\chi)} (\langle \alpha, x \rangle - \chi^\star(\alpha)), \tag{9.1}$$

by Legendre duality. In particular, χ is non-decreasing in each variable if and only if $P(\chi) \subset \mathbb{R}_+^n$.

Example 9.8. If χ is convex and piecewise affine, i.e. $\chi = \max_i (\langle \alpha_i, \cdot \rangle + c_i)$ with $\alpha_i \in \mathbb{R}_+^n$ and $c_i \in \mathbb{R}$, then $P(\chi)$ is the Newton polyhedron of the α_i, i.e. the convex envelope of $\bigcup_i (\alpha_i + \mathbb{R}_+^n)$. For $\varphi(z) = \log|z| = \max_i \log|z_i| + O(1)$, we get for instance

$$P(\log|z|) = \left\{ \alpha \in \mathbb{R}_+^n \mid |\alpha| := \sum_i \alpha_i \geq 1 \right\}.$$

Set $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}_+^n$.

Theorem 9.9. *If $\varphi = \chi(\log|z_1|, \dots, \log|z_n|)$ is a toric psh function on $\mathbb{C}^n$, then $\mathcal{J}(\varphi)_0$ is generated by monomials, and $z^\beta \in \mathcal{J}(\varphi)_0$ if and only iff $\beta + \mathbf{1} \in \mathring{P}(\varphi)$.*

As a consequence, we recover the following description of algebraic multiplier ideals in the toric case, originally due to Howald.

Corollary 9.10. *If $\mathfrak{a}$ is a monomial ideal generated by a (possibly infinite) family of monomials $(z^\alpha)_{\alpha\in A}$ with $A \subset \mathbb{N}^n$ and $c > 0$, then $z^\beta \in \mathcal{J}(\mathfrak{a}^c)_0$ if and only $\beta + \mathbf{1} \in c\mathring{P}$, with P the Newton polyhedron of A, i.e. the convex envelope of $\bigcup_{\alpha\in A}(\alpha + \mathbb{R}^n_+)$.*

Example 9.11. We have $\mathcal{J}(c\log|z|)_0 = \mathfrak{m}_0^{\lceil c\rceil - n+1}$.

As we shall see, passing to polar coordinates easily reduces Theorem 9.9 to the following convex-analytic discussion. Define the *homogeneization* of χ as the convex, positively homogeneous function

$$\chi^{\mathrm{hom}}(x) := \sup_{\alpha\in P(\chi)} \langle \alpha, x\rangle.$$

Using (9.1), one easily checks that for each $a, x \in \mathbb{R}^n_-$,

$$\chi^{\mathrm{hom}}(x) = \lim_{t\to+\infty} \frac{\chi(a+tx)}{t}$$

computes the slope at infinity of χ along the ray $a + \mathbb{R}_+ x$.

Proposition 9.12. *If $\chi : \mathbb{R}^n_- \to \mathbb{R}$ is convex, the following conditions are equivalent.*

(i) *$e^{-\chi}$ is integrable;*
(ii) *$0 \in \mathring{P}(\varphi)$;*
(iii) *χ decays at least linearly at infinity, i.e. $\chi(x) \le -\delta|x| + C$ for some $\delta, C > 0$;*
(iv) *$\chi^{\mathrm{hom}}(x) < 0$ for each nonzero $x \in \mathbb{R}^n_+$.*

Proof. It is clear that (ii)$\Longleftrightarrow$(iii)$\Longrightarrow$(i). Assume that $e^{-\chi} \in L^1$, and pick $x \in \mathbb{R}^n_-$. By Fubini-Study, there exists $a \in \mathbb{R}^n_-$ such that the restriction of $e^{-\chi}$ to the ray $R = a + \mathbb{R}_+ x$ is integrable. By convexity of the one-variable function $\chi|_R$, this clearly implies that the slope at infinity $\chi^{\mathrm{hom}}(x)$ is negative, and we infer (i)$\Longrightarrow$(iv). By homogeneity and compactness of $\{x \in \mathbb{R}^n_- \mid |x| = 1\}$, (iv) is equivalent to $\chi^{\mathrm{hom}}(x) \le -\delta|x|$ for some $\delta > 0$, which is in turn equivalent to (iii). □

9.4. **The Nadel vanishing theorem.**

Theorem 9.13 (Nadel vanishing)**.** *Let (X, ω) be Kähler manifold, and assume that X is weakly pseudonvex and contains a Stein Zariski open subset (e.g. X projective). Let L be a Hermitian holomorphic line bundle on X with curvature form θ, and let also φ be a quasi-psh function such that $\theta + i\partial\overline{\partial}\varphi \ge \varepsilon\omega$ for some $\varepsilon > 0$. Then*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(\varphi)) = 0$$

for all $q \ge 1$.

Proof. For each $q \ge 0$, denote by $\mathcal{A}^q$ the sheaf of germs u of measurable sections of $\Lambda^{n,q}T^\star_X \otimes L$ such that both $ue^{-\varphi}$ and $\overline{\partial}ue^{-\varphi}$ are L^2_{loc}. The $\overline{\partial}$ operator induces a complex of sheaves

$$0 \to \mathcal{O}(K_X + L) \otimes \mathcal{J}(\varphi) \to \mathcal{A}^0 \to \cdots \to \mathcal{A}^n.$$

An application of Theorem 6.5 to a small ball in X shows that this complex is exact. Since each $\mathcal{A}^q$ is a C^∞-module, the de Rham-Weil theorem implies that $H^q(X, \mathcal{O}(K_X+L)\otimes\mathcal{J}(\varphi))$ is isomorphic to the q-th cohomology group of the complex of global sections of $\mathcal{A}^q$. Let v be a $\overline{\partial}$-closed global section on X of $\mathcal{A}^q$, $q \geq 1$, denote by U the Stein Zariski open subset of X, and pick a (strictly) psh exhaustion function $\psi : X \to \mathbb{R}_+$. For each $t \geq 0$, $\int_{\{\psi<t\}} |v|^2 e^{-2\varphi} dV$ is finite, and we may thus choose $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ convex, increasing such that $\int_X |v|^2 e^{-2(\varphi+\chi\circ\psi)} dV < +\infty$. By Theorem 6.5, there exists $u \in L^2_{\mathrm{loc}}(X, \Lambda^{n,q-1}T^\star_X \otimes L)$ such that $\overline{\partial}u = v$ and $\int_X |u|^2 e^{-2(\varphi+\chi\circ\psi)} dV < +\infty$. In particular, u defines a global section of $\mathcal{A}^{q-1}$ on X, and the result follows. □

9.5. **Restriction property and subadditivity.** A fairly direct application of the 'local' Ohsawa–Takegoshi theorem yields the following qualitative counterpart of it.

Theorem 9.14. *If φ is quasi-psh on a complex manifold X, then $\mathcal{J}(\varphi|_Y) \subset \mathcal{J}(\varphi)|_Y$ for each complex submanifold $Y \subset X$.*

Applying this to the diagonal in $X \times X$, one can show (see [DEL, Theorem 2.6]):

Theorem 9.15. *For any two quasi-psh functions φ, ψ, we have $\mathcal{J}(\varphi + \psi) \subset \mathcal{J}(\varphi) \cdot \mathcal{J}(\psi)$.*

Corollary 9.16. *If k divides m, then $\mathcal{J}(m\varphi)^{1/m}$ is more singular than $\mathcal{J}(k\varphi)^{1/k}$.*

9.6. **Demailly's and Skoda's theorem.** The following result is a 'qualitative' version of a famous regularization theorem due to Demailly [Dem92].

Theorem 9.17. *Let φ be a quasi-psh function on a complex manifold. Then the multiplier ideal $\mathcal{J}(\varphi)$ is less singular than φ.*

Proof. Ohsawa–Takegoshi extension from a point. □

Corollary 9.18. *If $\nu_x(\varphi) < 1$, then $\mathcal{J}(\varphi)_x = \mathcal{O}_{X,x}$.*

Proof. By Theorem 9.17, we have $\nu_x(\varphi) \geq \mathrm{ord}_x \mathcal{J}(\varphi)$, and hence $\mathrm{ord}_x \mathcal{J}(\varphi) = 0$ since the latter is an integer. □

9.7. **Openness property.** Let φ be a quasi-psh function on a complex manifold X, and set

$$\mathcal{J}_+(\varphi) := \bigcup_{\varepsilon>0} \mathcal{J}((1+\varepsilon)\varphi) \subset \mathcal{O}_X.$$

Since the singularities of $(1+\varepsilon)\varphi$ decrease as $\varepsilon \searrow 0$, the strong noetherian property shows that $\mathcal{J}_+(\varphi) = \mathcal{J}(((1+\varepsilon)\varphi)$ for all $0 < \varepsilon \ll 1$ near any given compact subset of X.

The following result is known as the 'strong openness conjecture' of Demailly–Kollár [DK01], and was established by Guan–Zhou [GZ15].

Theorem 9.19. *For each quasi-psh function φ on a complex manifold, we have $\mathcal{J}_+(\varphi) = \mathcal{J}(\varphi)$. Equivalently, for any holomorphic function f with $|f|e^{-\varphi}$ L^2 in a neighborhood of a compact $K \subset X$, there exists $\varepsilon > 0$ such that $|f|e^{-(1+\varepsilon)\varphi}$ is L^2 in a neighborhood of K.*

The proof presented below, which is basically that of Guan and Zhou, relies on the following two elementary facts.

Lemma 9.20. *If g is an integrable function near $0 \in \mathbb{R}^d$, then there exists a sequence $x_j \to 0$ in $\mathbb{R}^d$ such that $|g(x_j)| = o\left(|x_j|^{-d}\right)$.*

Proof. Applying the Fubini-Study in polar coordinates yields $v \in S^{d-1}$ such that $h(r) := r^{d-1} g(rv)$ is integrable near $0 \in \mathbb{R}$, and we are thus reduced to the case $d = 1$. If the result fails, then $|h(r)| \geq c/r$ near 0 for some $c > 0$, contradicting the integrability at 0. □

Lemma 9.21. *Let a, b be holomorphic functions near $\overline{\mathbb{D}} \subset \mathbb{C}$ such that $\operatorname{ord}_0(a) > \operatorname{ord}_0(b)$, and assume that b is nowhere zero on $\overline{\mathbb{D}} \setminus \{0\}$. Assume also given $t_0 \in \mathbb{D}^\star$ with $a = b$ on all k-th roots of t_0. Then*

$$|t_0| \sup_{\partial\mathbb{D}} |a| \geq \inf_{\partial\mathbb{D}} |b| > 0.$$

Proof. After replacing a with $a(t)/b(t)$, we can assume that $b = 1$. As in [Lem14, Proposition 2.2], after averaging a with respect to the action of the k-th roots of unity we may further asume $k = 1$. By assumption, a vanishes as 0 and satisfies $a(t_0) = 1$. Applying the maximum principle to the holomorphic function $a(t)/t$, we get as desired

$$|t_0|^{-1} = |t_0^{-1} a(t_0)| \leq \sup_{\partial\mathbb{D}} |t^{-1} a(t)| = \sup_{\partial\mathbb{D}} |a|.$$

□

Proof of Theorem 9.19. The result being local, we can assume that φ is psh and $\varphi \leq 0$. Pick a germ f such that $|f|e^{-\varphi}$ is L^2_{loc} at a point $x \in X$. Since $\mathcal{J}_+(\varphi)$ locally coincides with $\mathcal{J}((1+\varepsilon)\varphi)$ for $0 < \varepsilon \ll 1$, it is integrally closed (cf. Example 8.8). By Proposition 8.13, it is thus enough to show that $\operatorname{ord}_\gamma(f) \geq \operatorname{ord}_\gamma(\mathcal{J}_+(\varphi))$ for each germ of curve $\gamma : (\mathbb{C}, 0) \to (X, x)$. Arguing by contradiction, we assume $\operatorname{ord}_\gamma(f) < \operatorname{ord}_\gamma(\mathcal{J}_+(\varphi))$, which implies in particular that $f \circ \gamma$ has isolated zeroes.

Pick local coordinates $z = (z_1, \dots, z_n)$ at x. After permuting the indices, we can assume that the image of γ is not entirely contained in $\{z_1 = 0\}$. Reparametrizing γ and scaling the coordinates, we then arrange the following:

(i) $f \in \mathcal{O}(\mathbb{D}^n)$ satisfies $\int_{\mathbb{D}^n} |f|^2 e^{-2\varphi} < +\infty$;
(ii) $\gamma : U \to \mathbb{D}^n$ is defined on a neighborhood U of $\bar{\mathbb{D}}$;
(iii) $\gamma_1(s) := z_1 \circ \gamma(s) = s^k$ for some $k \in \mathbb{N}^\star$;
(iv) $f \circ \gamma$ is nowhere zero on $\bar{\mathbb{D}} \setminus \{0\}$.

By (i) and Lemma 9.20, there exists a sequence $t_j \in \mathbb{D}$ converging to 0 such that

$$\int_{\{z_1 = t_j\} \cap \mathbb{D}^n} |f|^2 e^{-2\varphi} = o\left(|t_j|^{-2}\right).$$

Since $f \circ \gamma$ is nonzero on any k-th root of t_j, $f|_{\{z_1=t_j\}\cap\mathbb{D}^n}$ is not identically zero, and hence $\int_{\{z_1=t_j\}\cap\mathbb{D}^n} |f|^2 e^{-2\varphi} > 0$. Pick $0 < r < 1$ such that $\gamma(\bar{\mathbb{D}}) \subset \mathbb{D}^n_r$. Arguing by induction on the dimension, we get $\varepsilon_j > 0$ such that

$$\int_{\{z_1=t_j\}\cap\mathbb{D}^n_r} |f|^2 e^{-2(1+\varepsilon_j)\varphi} \le 2\int_{\{z_1=t_j\}\cap\mathbb{D}^n} |f|^2 e^{-2\varphi} = o\left(|t_j|^{-2}\right).$$

By the 'local version' of the Ohsawa–Takegoshi extension theorem, there exists $g_j \in \mathcal{O}(\mathbb{D}^n_r)$ such that $g_j|_{\{z_1=t_j\}\cap\mathbb{D}^n_r} = f|_{\{z_1=t_j\}\cap\mathbb{D}^n_r}$ and

$$\int_{\mathbb{D}^n_r} |g_j|^2 e^{-2(1+\varepsilon_j)\varphi} \le C\int_{\{z_1=t_j\}\cap\mathbb{D}^n_r} |f|^2 e^{-2(1+\varepsilon_j)\varphi} = o\left(|t_j|^{-2}\right)$$

for some uniform constant $C > 0$. By (iii), we have $(g_j \circ \gamma)(s) = (f \circ \gamma)(s)$ for all $s \in \mathbb{D}$ such that $\gamma_1(s) = s^k = t_j$. Also $g_j \in \mathcal{J}_+(\varphi)$, and hence

$$\mathrm{ord}_0(g_j \circ \gamma) \ge \mathrm{ord}_\gamma(\mathcal{J}_+(\varphi)) > \mathrm{ord}_0(f \circ \gamma).$$

By Lemma 9.21, it follows that

$$|t_j| \sup_{\partial\mathbb{D}} |g_j \circ \gamma| \ge \inf_{\partial\mathbb{D}} |f \circ \gamma| > 0,$$

and hence $\sup_{\partial\mathbb{D}} |g_j \circ \gamma|^2 \ge c|t_j|^{-2}$ for some constant $c > 0$. On the other hand, since $\gamma(\partial\mathbb{D})$ is compact in $\mathbb{D}^n_r$, the mean value inequality gives

$$\sup_{\partial\mathbb{D}} |g_j \circ \gamma|^2 \le C\int_{\mathbb{D}^n_r} |g_j|^2 \le C\int_{\mathbb{D}^n_r} |g_j|^2 e^{-2(1+\varepsilon_j)\varphi} = o\left(|t_j|^{-2}\right),$$

contradiction. □

Corollary 9.22. *Let φ, ψ be quasi-psh functions, and assume that ψ has analytic singularities. If $e^{\psi-\varphi}$ is L^2 on a neighborhood a compact $K \subset X$, then there exists $\varepsilon > 0$ such that $e^{\psi-(1+\varepsilon)\varphi}$ is L^2 on a neighborhood of K.*

Proof. After passing to a log resolution, we may assume that $X = \mathbb{D}^n$ and $\psi = \sum_{i=1}^n c_i \log|z_i|$ with $c_i \in \mathbb{R}_+$. Writing $c_i = m_i + r_i$ with $m_i \in \mathbb{N}$, we see that $e^{\psi-\varphi} \in L^2_{\mathrm{loc}}$ iff $f := \prod_i z_i^{m_i}$ belongs to $\mathcal{J}(\varphi + \sum_i r_i \log|z_i|)$. By openness, we get $\varepsilon > 0$ such that

$$f \in \mathcal{J}((1+\varepsilon)(\varphi + \sum_i r_i \log|z_i|)) \subset \mathcal{J}((1+\varepsilon)\varphi + \sum_i r_i \log|z_i|),$$

i.e. $e^{\psi-(1+\varepsilon)\varphi} \in L^2_{\mathrm{loc}}$. □

10. Valuative description of multiplier ideals

10.1. **Analytic semicontinuity of Lelong numbers.** Recall from Section 2.3 that the Lelong number of a psh function u near $0 \in \mathbb{C}^n$ satisfies

$$\nu_0(u) = \max\{\gamma \in \mathbb{R}_+ \mid u(z) \le \gamma \log|z| + O(1) \text{ near } 0\}. \tag{10.1}$$

Let now φ be a quasi-psh function on a complex manifold X. Near each point $x \in X$, we have by definition $\varphi = u + f$ with u psh and f smooth. Choosing local coordinates z centered at $x \in X$, we can view u as a psh function near $0 \in \mathbb{C}^n$ and we set $\nu_x(\varphi) := \nu_0(u)$. Clearly,

$$\nu_x(\varphi) = \max\{\gamma \in \mathbb{R}_+ \mid \varphi \le \gamma \log|z| + O(1) \text{ near } x\},$$

and the definition is thus independent of the choice of local coordinates at x, any two choices local coordinates being Lipschitz equivalent.

Example 10.1. If φ has analytic singularities of type $\mathfrak{a}^c$, then $\nu_x(\varphi) = c \operatorname{ord}_x(\mathfrak{a})$.

By Lemma 2.6, $\nu_x(\varphi)$ is a usc function of $x \in X$, and also a usc function of φ with respect to the L^1_{loc} topology. The following deep result was originally proved by Siu [Siu74]. The much simpler argument we present here is due to Demailly.

Theorem 10.2. *If φ is a quasi-psh function on a complex manifold X, then $x \mapsto \nu_x(\varphi)$ is usc in the analytic Zariski topology. In other words, $\{x \in X \mid \nu_x(\varphi) \ge c\}$ is a closed analytic subset of X for each $c \in \mathbb{R}$.*

The key point is the following consequence of the Ohsawa–Takegoshi theorem.

Lemma 10.3. *For each $x \in X$ we have $\nu_x(\varphi) \ge \operatorname{ord}_x \mathcal{J}(\varphi) \ge \nu_x(\varphi) - n$.*

Proof. By Theorem 9.17, φ is more singular than $\mathcal{J}(\varphi)$, and hence $\nu_x(\varphi) \ge \operatorname{ord}_x \mathcal{J}(\varphi)$. To prove the converse, pick coordinates z at x, and set $\gamma := \nu_x(\varphi)$. Since $\varphi \le \gamma \log|z| + C$ near x, we have

$$\mathcal{J}(\varphi)_x \subset \mathcal{J}(\gamma \log|z|)_x = \mathfrak{m}_x^{\lceil \gamma \rceil - n + 1},$$

and hence $\operatorname{ord}_x \mathcal{J}(\varphi) \ge \nu_x(\varphi) - n$. □

Proof of Theorem 10.2. When φ has analytic singularities, the result amounts to the fact that $\{x \in X \mid \operatorname{ord}_x(\mathfrak{a}) \ge c\}$ is closed analytic for a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, which is easy to see. Applying Lemma 10.3 to $k\varphi$ with $k \in \mathbb{N}^\star$, we get

$$|\nu_x(\varphi) - k^{-1} \operatorname{ord}_x \mathcal{J}(k\varphi)| \le n/k,$$

which proves that $x \mapsto \nu_x(\varphi)$ is the uniform limit of $x \mapsto k^{-1} \operatorname{ord}_x \mathcal{J}(k\varphi)$. By coherence of $\mathcal{J}(k\varphi)$, $x \mapsto k^{-1} \operatorname{ord}_x \mathcal{J}(k\varphi)$ is usc in the analytic Zariski topology for each k, and hence so is their uniform limit $x \mapsto \nu_x(\varphi)$. □

10.2. **Generic Lelong numbers.** As above, φ is a quasi-psh function on a complex manifold X.

Definition 10.4. The *generic Lelong number of* φ along an irreducible closed analytic subset $Y \subset X$ is defined as

$$\nu_Y(\varphi) := \inf_{y \in Y} \nu_y(\varphi).$$

The terminology is justified by the following consequence of the analytic semi-continuity theorem.

Lemma 10.5. *We have $\nu_y(\varphi) = \nu_Y(\varphi)$ for a very general point $y \in Y$, i.e. any point outside a countable union of strict closed analytic subsets Z_j of Y.*

Proof. For each $j \ge 1$, $Z_j := \{y \in Y \mid \nu_y(\varphi) \ge \nu_Y(\varphi) + 1/j\}$ is a strict closed anaytic subset of Y, and $\nu_y(\varphi) = \nu_Y(\varphi)$ for each $x \in Y \setminus \bigcup_j Z_j$. □

As a consequence, for any two quasi-psh functions φ, ψ we get the valuative formulae

$$\nu_Y(\varphi + \psi) = \nu_Y(\varphi) + \nu_Y(\psi)$$

and

$$\nu_Y(\max\{\varphi, \psi\}) = \min\{\nu_Y(\varphi), \nu_Y(\psi)\}.$$

Proposition 10.6. *For each regular point $y \in Y_{\mathrm{reg}}$, $\nu_Y(\varphi)$ is the largest $c \in \mathbb{R}_+$ such that φ is more singular then $\mathcal{I}_Y^c$ near y. When Y has codimension 1, this is also true when y is a singular point of Y.*

Proof. If φ is more singular than $\mathcal{I}_Y^c$ near y, then $\nu_{y'}(\varphi) \geq c \operatorname{ord}_{y'} \mathcal{I}_Y \geq c$ for all $y' \in Y$ near y. Since $\nu_{y'}(\varphi) = \nu_Y(\varphi)$ for a very general $y' \in Y$, we infer $\nu_Y(\varphi) \geq c$. Conversely, pick $y \in Y_{\mathrm{reg}}$ a regular point, and choose ocal coordinates $z = (z_1, \dots, z_n)$ at x such that $z_1, \dots, z_p$ generate $\mathcal{I}_Y$ near y. By (2.4), there exists $C > 0$ such that

$$\varphi(z) \leq \nu_y(\varphi) \log |z - y| + C \leq \nu_Y(\varphi) \log |z - y| + C$$

for all z, y near $x = 0$. Choosing $y = (0, \dots, 0, z_{p+1}, \dots, z_n) \in E$, we get $\varphi \leq \nu_Y(\varphi) \log(|z_1|^2 + \cdots + |z_p|^2)^{1/2} + C$ near x, i.e. φ is more singular then $\mathcal{I}_Y^{\nu_Y(\varphi)}$ near y.

Assume now that Y is a prime divisor. Since X is smooth, Y admits a local equation f_Y near any given $y \in Y$. Arguing locally, we can assume that φ is psh. Since $\log |f_Y|$ is pluriharmonic outside Y, $\varphi - \nu_Y(\varphi) \log |f_Y|$ is psh outside Y. Being locally bounded above near each point of Y_{reg}, it extends to a psh function ψ outside Y_{sing}. But the latter has codimension at least 2 in X, and ψ is thus automatically bounded above near near Y_{sing}. In particular, $\varphi \leq \nu_Y(\varphi) \log |f_Y| + O(1)$ near y. □

Remark 10.7. The second part of Proposition 10.6 fails in general when Y has codimension at least 2. Indeed, a holomorphic function $f \in \mathcal{O}_X$ satisfies $\operatorname{ord}_Y(f) = \nu_Y(\log |f|) \geq m$ if and only f belongs to the *symbolic power* $\mathcal{I}_Y^{\langle m \rangle}$. On the other hand, $\log |f|$ is more singular than $\mathcal{I}_Y^m$ if and only if f belongs to the integral closure $\overline{\mathcal{I}_Y^m}$, which is strictly contained in $\mathcal{I}_Y^{\langle m \rangle}$ is general.

10.3. **The case of analytic singularities.** We start with a few preliminary facts. Let $\mu : Y \to X$ be a modificatin with Y smooth. The Jacobian of μ, locally defined in coordinate charts, yields a global section of the *relative canonical bundle* $K_{Y/X} := K_Y - \mu^\star K_X$. The corresponding divisor is identified with $K_{Y/X}$, and its support coincides with the exceptional locus $\operatorname{Exc}(\mu)$. We may then show that $A := \mu(\operatorname{Exc}(\mu))$ has codimension at least 2 in X, and μ induces an isomorphism over $X \setminus A$.

The *log discrepancy* of a prime divisor $E \subset Y$ is defined as

$$A_X(E) := 1 + \operatorname{ord}_E(K_{Y/X}).$$

If Y' is a modification of Y, the strict transform E' of E on Y', then $A_X(E) = A_X(E')$.

Example 10.8. Let $\mu : Y \to X$ be the blowup of a submanifold $Z \subset X$. Then $\operatorname{Exc}(\mu) = E = \mathbb{P}(N_{Z/X})$, and $K_{X'/X} = (\operatorname{codim} Z - 1)E$, i.e. $A_X(E) = \operatorname{codim} Z$.

Given a psh function φ on X, we define $\nu_E(\varphi)$ as the generic Lelong number of $\varphi \circ \mu$ along E.

Example 10.9. If φ has analytic singularities of type $\mathfrak{a}^c$, then we have for each prime divisor E over X

$$\nu_E(\varphi) = \mathrm{ord}_E(\mathfrak{a}^c) := c \min_{f \in \mathfrak{a}} \mathrm{ord}_E(f \circ \mu),$$

Theorem 10.10. *Let φ, ψ be two quasi-psh functions on a complex manifold X, and assume that ψ has analytic singularities. For each compact subset $K \subset X$, we then have*

$$\inf \left\{ \frac{\nu_E(\varphi)}{\nu_E(\psi)} \mid \nu_E(\psi) > 0,\, c_X(E) \cap K \neq \emptyset \right\} = \max\{\gamma \geq \mathbb{R}_+ \mid \varphi \leq \gamma\psi + O(1) \text{ near } K\},$$

where the left-hand infimum ranges over all prime divisors E over X. Further, the infimum on the left-hand is achieved among the Rees valuations of ψ.

Definition 10.11. A *log resolution* of a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ is a proper bimeromorphic morphism $\mu : X' \to X$ with X' a complex manifold, such that

(i) $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-D)$ with D an effective divisor on X';
(ii) $D + K_{X'/X}$ has snc[1] support.

The existence of a log resolution is guaranteed by Hironaka, later simplfiied by Bierstone-Milman, Wlodarczyk, etc...

Proposition 10.12. *Assume that φ has analytic singularities of type $\mathfrak{a}^c$, and let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$, with snc divisor $\sum_i E_i$. Then*

$$\mathcal{J}(\varphi) = \mathcal{J}(\mathfrak{a}^c) = \{f \in \mathcal{O}_X \mid \mathrm{ord}_{E_i}(f) > \mathrm{ord}_{E_i}(\mathfrak{a}^c) - A_X(\mathrm{ord}_{E_i}) \text{ for all } i\}.$$

Proof. Change of variable. $\mathcal{J}(\varphi) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor D \rfloor)$.

□

10.4. **Valuative characterization of integrability.** Let X be a complex manifold.

Theorem 10.13. *Let φ, ψ be psh functions near $0 \in X$.*

(i) *If $e^{\psi - \varphi} \in L^2_{\mathrm{loc}}$ near $0 \in X$, then $\nu_E(\psi) \geq \nu_E(\varphi) - A_X(E)$ for all prime divisors E over X such that $0 \in c_X(E)$.*
(ii) *Assume conversely that there exists $\varepsilon > 0$ such that*

$$\nu_E(\psi) \geq (1+\varepsilon)\nu_E(\varphi) - A_X(E)$$

for all prime divisors E over X such that $0 \in c_X(E)$. Then $e^{\psi-\varphi} \in L^2_{\mathrm{loc}}$ at 0.

Remark 10.14. Arguing by approximation as in [BFJ08, Proposition 3.7], one shows that it is enough to check (ii) for prime divisors E such that $c_X(E) = \{0\}$.

Corollary 10.15. *If ψ further has analytic singularities, then $e^{\psi-\varphi} \in L^2_{\mathrm{loc}}$ iff there exists $\varepsilon > 0$ such that $\nu_E(\psi) \geq (1+\varepsilon)\nu_E(\varphi) - A_X(E)$ for all prime divisors E over X such that $0 \in c_X(E)$.*

[1]Short for simple normal crossing.

Proof. When ψ has analytic singularities, the condition $e^{\psi-\varphi} \in L^2_{\rm loc}$ implies $e^{\psi-(1+\varepsilon)\varphi} \in L^2_{\rm loc}$ for some $\varepsilon > 0$, by Corollary 9.22. The result is now a direct consequence of Theorem 10.13. $\square$

As the next example shows, the assumption on analytic singularities in Corollary 10.15 cannot be dispensed with.

Example 10.16. Take $\psi(z) = -\log(-\log|z|)$ and $\varphi = \log|z|$ on the unit disc $X = \mathbb{D}$. Then $\int e^{2(\psi-\varphi)} = \int \frac{|dz|^2}{|z|^2(\log|z|)^2}$ converges near 0, but $\nu_0(\psi) = 0$ and $\nu_0(\varphi) = A_X(0) = 1$.

Lemma 10.17. *Let φ, ψ be psh functions near $0 \in \mathbb{C}^n$, and assume that $e^{\psi-\varphi} \in L^2_{\rm loc}$ at 0. Then $\nu_H(\psi) \geq \nu_H(\varphi) - 1$ with $H := (z_1 = 0)$.*

Proof. Set $c := \nu_H(\varphi)$. Since φ is more singular than $c\log|z_1|$, we may assume wlog that $\varphi = c\log|z_1|$. After scaling, we may also assume that $e^{\psi-\varphi} = e^{\psi}|z_1|^{-c}$ is L^2 on the unit polydisc $\mathbb{D}^n$.

Assume first that $n = 1$, and hence $H = \{0\}$. For each $t \in \mathbb{R}$ large enough set $\chi(t) := \fint_{S(0,e^{-t})} \psi$. By Lemma 2.8,

$$\lim_{t\to+\infty} \chi(t)/t = -\nu_0(\psi),$$

and we thus need to show that $\lim_{t\to+\infty} \chi(t)/t \leq 1 - c$. By Jensen's inequality, $\fint_{S(0,e^{-t})} e^{2\psi} \geq e^{2\chi(t)}$, and hence

$$+\infty > \frac{1}{2\pi}\int_{\mathbb{D}} e^{2\psi}|z|^{-2c}|dz|^2 = \int_0^{+\infty} e^{2(c-1)t}dt \fint_{S(0,e^{-t})} e^{2\psi} \geq \int_0^{+\infty} e^{2(\chi(t)+(c-1)t)}dt.$$

We may thus find a sequence $t_i \to +\infty$ such that $\chi(t_i) + (c-1)t_i \leq 0$, and hence $\lim_{t\to+\infty} \chi(t)/t = \lim \chi(t_i)/t_i \leq 1 - c$.

In the general case, the Fubini–Torelli theorem yields

$$\int_{\mathbb{D}} e^{2\psi(z_1,z')}|z_1|^{-2c}|dz_1|^2 < +\infty$$

for a.e. $z' \in \mathbb{C}^{n-1}$ near 0. By the one-variable case, we thus have $\nu_0(\psi(\cdot, z')) \geq c - 1$ for a.e. z', and (2.4) thus yields

$$\psi(z_1, z') \leq (c-1)(\log|z_1| - \log 2) + \sup_{|z_1|\leq 1/2} \psi(z_1, z') \leq (c-1)\log|z_1| + C$$

for some uniform constant C and a.e. z'. As a result, this holds for all z' near 0, and we conclude as desired that $\nu_H(\psi) \geq c - 1$. $\square$

Proof of Theorem 10.13. Assume that $e^{\psi-\varphi} \in L^2_{\rm loc}$ near $0 \in X$. Let $\mu : Y \to X$ be a modification and E a prime divisor over X with $0 \in c_X(E) = \mu(E)$. Choose local coordinates $(z_1, \ldots, z_n)$ on Y near a regular point of E close to a preimage of 0. The change of variable formula yields

$$e^{\psi\circ\mu-\varphi\circ\mu}|z_1|^a \in L^2_{\rm loc}$$

with $a := \mathrm{ord}_E(K_{Y/X}) = A_X(E) - 1$. By Lemma 10.17, we infer

$$\nu_E(\psi\circ\mu) + a \geq \nu_E(\varphi\circ\mu) - 1,$$

i.e. $\nu_E(\psi) \geq \mu_E(\varphi) - A_X(E)$, which proves (i).

Conversely, assume given $\varepsilon > 0$ such that $\nu_E(\psi) \geq (1+\varepsilon)\nu_E(\varphi) - A_X(E)$ for all prime divisors E over X with $c_X(E) = \{0\}$. Assume first that φ, ψ have analytic singularities. We can then find a modification $\psi : X' \to X$ and an snc divisor $D = \sum_i E_i$ supporting $K_{X'/X}$ and such that φ and ψ have divisorial singularities along D. The change of variable formula then shows that $e^{\psi-\varphi} \in L^2_{\mathrm{loc}}$ iff $\nu_{E_i}(\psi) > \nu_{E_i}(\varphi) - A_X(E_i)$ for all i, and we are done.

Assume now that φ, ψ are arbitrary psh functions. For each $k \in \mathbb{Z}_{>0}$, let φ_k (resp. ψ_k) be a psh function with analytic singularities of type $\mathcal{J}(k\varphi)^{1/k}$ (resp. $\mathcal{J}(k\psi)^{1/k}$). By Theorem 9.17, φ_k (resp. ψ_k) is less singular then φ (resp. ψ). For each prime E over X we thus have

$$\nu_E(\varphi) \geq \nu_E(\varphi_k) \geq \nu_E(\varphi) - k^{-1}A_X(E),$$

$$\nu_E(\psi) \geq \nu_E(\psi_k) \geq \nu_E(\psi) - k^{-1}A_X(E).$$

Thus

$$\nu_E(\psi_k) \geq (1+\varepsilon)\nu_E(\varphi_k) - (1+k^{-1})A_X(E),$$

and hence

$$\nu_E(\psi_k) \geq \frac{1}{1+k^{-1}}\nu_E(\psi_k) \geq \frac{1+\varepsilon}{1+k^{-1}}\nu_E(\varphi_k) - A_X(E) \geq (1+\varepsilon/2)\nu_E(\varphi_k) - A_X(E)$$

for all $k \gg 1$. By the case of functions with analytic singularities, we infer $e^{\psi_k-(1+\varepsilon_k)\varphi_k} \in L^2_{\mathrm{loc}}$ for any $\varepsilon_k \in [0, \varepsilon/2)$, hence also $e^{\psi-(1+\varepsilon_k)\varphi_k} \in L^2_{\mathrm{loc}}$, since ψ is more singular than ψ_k. To conclude the proof of (i), it remains to note, following Demailly, that

$$e^{\psi-(1+\varepsilon_k)\varphi_k} \in L^2_{\mathrm{loc}} \Longrightarrow e^{\psi-\varphi} \in L^2_{\mathrm{loc}}$$

for any $\varepsilon_k \geq \frac{1}{k-1}$. To see this, note that $e^{k(\varphi_k-\varphi)} \in L^2_{\mathrm{loc}}$, by definition of the multiplier ideal $\mathcal{J}(k\varphi)$. Since ψ is locally bounded above, we have

$$\begin{aligned}\int e^{2(\psi-\varphi)} &\leq \int_{\{\varphi \geq (1+\varepsilon_k)\varphi_k\}} e^{2(\psi-(1+\varepsilon_k)\varphi_k)} + C\int_{\{\varphi \leq (1+\varepsilon_k)\varphi_k\}} e^{-2\varphi} \\ &\leq \int e^{2(\psi-(1+\varepsilon_k)\varphi_k)} + C\int_{\{\varphi \leq (1+\varepsilon_k)\varphi_k\}} e^{2(-\varphi+k(\varphi-\varphi_k))}e^{2k(\varphi_k-\varphi)} \\ &\leq \int e^{2(\psi-(1+\varepsilon_k)\varphi_k)} + C'\int e^{2k(\varphi_k-\varphi)} < \infty,\end{aligned}$$

since $\varphi + k(\varphi - \varphi_k)$ is bounded above when $\varphi \leq (1+\varepsilon_k)\varphi_k \leq \frac{k}{k-1}\varphi_k + O(1)$. $\square$

Corollary 10.18. *A function $f \in \mathcal{O}_X$ belongs to $\mathcal{J}(\varphi)$ iff there exists $\varepsilon > 0$ such that $\mathrm{ord}_E(f) \geq (1+\varepsilon)\nu_E(\varphi) - A_X(E)$ for all prime divisors E over X.*

Remark 10.19. Interestingly, the ideal sheaf of germs $f \in \mathcal{O}_X$ such that $\mathrm{ord}_E(f) \geq \nu_E(\varphi) - A_X(E)$ for all prime divisors E over X is not coherent in general. Indeed, Corollary 10.18 implies that this ideal coincides with $\cap_{\delta>0}\mathcal{J}((1-\delta)\varphi)$, which is not always coherent, by [Wu19, Proposition 2].

The following result is based on [Cao, Lemma 3.4] and [BFJ08].

Theorem 10.20. *For each $p \in (0, \infty)$, the sheaf*

$$\mathcal{L}^p(\varphi) := \{f \in \mathcal{O}_X \mid |f|e^{-\varphi} \in L^p_{\mathrm{loc}}\}$$

is coherent. Further, a germ $f \in \mathcal{O}_X$ is in $\mathcal{L}^p(\varphi)$ iff there exists $\varepsilon > 0$ such that

$$\mathrm{ord}_E(f) \geq (1+\varepsilon)\nu_E(\varphi) - p^{-1}A_X(E)$$

for all prime divisors E over X.

Lemma 10.21. *If φ_k has analytic singularities of type $\mathcal{J}(k\varphi)^{1/k}$, then there exists a sequence $\varepsilon_k \to 0$ such that*

$$\mathcal{L}^p(\varphi) = \bigcup_k \mathcal{L}^p((1+\varepsilon_k)\varphi_k).$$

Proof. Note that $e^{2k(\varphi_k - \varphi)} \in L^1_{\mathrm{loc}}$. As in [Dem12, (16.12)], this implies

$$\int |f|^p e^{-p\varphi} = \int_{\{\varphi \geq (1+\varepsilon_k)\varphi_k\}} |f|^p e^{-p\varphi} + \int_{\{\varphi < (1+\varepsilon_k)\varphi_k\}} |f|^p e^{2k(\varphi_k - \varphi)} e^{(2k-p)\varphi - 2k\varphi_k}$$

$$\leq \int |f|^p e^{-(1+\varepsilon_k)\varphi_k} + C \int e^{2k(\varphi - \varphi_k)}$$

whenever $\varepsilon_k \geq \frac{p}{2k-p}$, i.e. $(1+\varepsilon_k) \geq \frac{2k}{2k-p}$, because $(2k-p)\varphi - 2k\varphi_k$ is then bounded above on $\{\varphi < (1+\varepsilon_k)\varphi_k\}$. This proves

$$\mathcal{L}^p((1+\varepsilon_k)\varphi_k) \subset \mathcal{L}^p(\varphi).$$

In the other direction, pick $f \in \mathcal{L}^p(\varphi)$. Pick a positive integer m such that $2m > p$, and note that

$$\int |f|^p e^{-p\varphi} = \int |f|^{2m} e^{-(p\varphi + (2m-p)\log|f|)} < +\infty,$$

i.e. $f^d \in \mathcal{J}(p\varphi + (2m-p)\log|f|)$. The openness property of multiplier ideals yields $0 < \varepsilon \ll 1$ such that

$$f^d \in \mathcal{J}((1+\varepsilon)p\varphi + (2m-p)\log|f|),$$

i.e. $\int |f|^p e^{-(1+\varepsilon)p\varphi} < +\infty$. It follows that $f \in \mathcal{L}^p((1+\varepsilon)\varphi) \subset \mathcal{L}^p((1+\varepsilon_k)\varphi_k)$ for $k \gg 1$, which concludes the proof. □

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