FINITE GENERATION FOR GROMOV-HAUSDORFF LIMITS

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Abstract. We survey some aspects of the recent work [DS12] by Donaldson and Sun proving that Gromov-Hausdorff limits of projective manifolds with uniformly bounded Ricci curvature are normal projective varieties.

1. Introduction

As a matter of terminology, a *polarized manifold* will mean a pair (X, L) consisting of a complex manifold X together with a Hermitian holomorphic line bundle L with positive curvature form ω, which we use to view X as a Kähler manifold. By the Kodaira embedding theorem, a compact polarized manifold is automatically projective algebraic, with L ample.

In their recent work [DS12], Donaldson and Sun study the Gromov-Hausdorff limit of a sequence of compact polarized manifolds (X_j, L_j) of fixed complex dimension n and with uniformly bounded Ricci curvature. Since $n!\operatorname{vol}(X_j) = c_1(L_j)^n$ is a positive integer and $\operatorname{diam}(X_j)$ is bounded by convergence, the lower bound on the Ricci curvature implies that the limit is automatically "non-collapsed" thanks the Bishop-Gromov comparison theorem, and it follows from the Cheeger-Colding theory that the limit compact metric space X_∞ has Hausdorff dimension $2n$, and that its $2n$-dimensional Hausdorff measure satisfies

$$C^{-1} r^{2n} \leq \mathcal{H}_{2n}(B_r) \leq C r^{2n}$$

for all balls $B_r \subset X_\infty$. By results of Cheeger-Colding-Tian, the upper bound on the Ricci curvature of X_j further guarantees the existence of a polarized manifold (X, L) of complex dimension n such that X embeds isometrically as an open subset of X_∞, and that (X_j, L_j) converges in $C^{1,\alpha}$ topology to (X, L) (see Definition 5.1 below). Finally, the closed subset $X_\infty \setminus X$ has Hausdorff codimension at least 4, which is more than enough to ensure that X is *parabolic* as a Riemannian manifold, cf. §2.1 below.

The purpose of these notes is to study the graded algebra

$$R_b(X, L) = \bigoplus_{k \in \mathbb{N}} H^0_b(X, kL)$$

of bounded holomorphic sections of tensor powers of L. Using only the normality of X, we first prove that this algebra, which is obviously an integral domain, is automatically normal, i.e. integrally closed in its fraction field (see Proposition 4.1).

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Following [Mok86], we then show (Theorem 4.2), using only that X is parabolic and of finite volume, that $H^0_b(X, kL)$ is finite dimensional for each k, with

$$\dim H^0_b(X, kL) = O(k^n).$$

Finally, in the case of a Gromov-Hausdorff limit as above, we prove that $R_b(X, L)$ is finitely generated, using the main result of [DS12] (the so-called partial C^0-estimate) and Skoda's L^2 division theorem, expanding a remark from [DS12] based on an observation of Chi Li.

2. Preliminary facts

2.1. **Parabolic Riemannian manifolds.** We recall the following standard definition (cf. [Gla83] and references therein), which is precisely the property proved for the regular part of the Gromov-Hausdorff limit in [DS12, Proposition 3.5].

Definition 2.1. A Riemannian manifold (M^m, g) is said to be *parabolic* if the following equivalent conditions hold:

(i) there exists an exhaustion function $\psi \in C^\infty(M)$ with $\|d\psi\|_{L^2} < +\infty$;
(ii) for each compact $K \subset M$ and each $\varepsilon > 0$, there exists a smooth cut-off function $\theta \in C^\infty_c(M)$ with $0 \le \theta \le 1$, $\theta \equiv 1$ on a neighborhood of K and $\|d\theta\|_{L^2} \le \varepsilon$;
(iii) every subharmonic function on M that is bounded above is constant.

Note that (M, g) is not required to be complete. By the Hopf-Rinow theorem, (M, g) is complete iff it admits an exhaustion function $\psi \in C^\infty(M)$ such that $\|d\psi\|_{L^\infty} < +\infty$, which shows that a complete Riemannian manifold of finite volume is parabolic. More generally, it is shown in [CY75] that a complete manifold with at most quadratic volume growth (i.e. $\operatorname{vol} B(x_0, r) = O(r^2)$ when $r \to \infty$ and $x_0 \in M$ is a fixed point) is still parabolic

For $m = 2$, parabolicity only depends on the conformal structure, and coincides with the usual notion from the function theory on Riemann surfaces. As a final side remark, [Gla83] shows that (M, g) is parabolic iff its boundary is negligible in the L^2 Stokes' theorem, in the sense that every square integrable $(m-1)$-form α on M with $d\alpha$ integrable satisfies $\int_M d\alpha = 0$.

Suppose that M is an open subset of a compact Riemannian manifold $\bar{M}$. Then characterization (ii) in Definition 2.1 means that M is parabolic iff ∂M has zero capacity. In that case, one can show that ∂M has Hausdorff codimension at least 2, compare [EG92, Theorem 4, p.156]. Conversely, if ∂M has finite $(m-2)$-dimensional Hausdorff measure, then M is parabolic. More generally:

Lemma 2.2. *Assume that (M^m, g) embeds isometrically as an open set of a compact metric space $(\bar{M}, d)$ whose m-dimensional Hausdorff measure satisfies $\mathcal{H}_m(B_r) = O(r^m)$ for all balls $B_r \subset \bar{M}$. If ∂M has finite $(m-2)$-dimensional Hausdorff measure, then (M, g) is parabolic.*

Proof. We follow the proof of [EG92, Theorem 3, p.154], which is closely related to that of [DS12, Proposition 3.5].

Step 1. We first claim that there exists $C > 0$ such that for any compact set $K \subset M$, we can find $\theta \in C_c^\infty(M)$ with $0 \le \theta \le 1$, $\theta = 1$ on a neighborhood of K, and $\|\theta\|_{L^2} \le C$.

Since ∂M is compact and has finite $\mathcal{H}_{m-2}$-measure, there exists a constant $C > 0$ such that ∂M can be covered by finitely many open balls $B(x_i, r_i)$ of radius

$$r_i \le d(K, \partial M)/4$$

and such that $\sum_i r_i^{m-2} \le C$.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be the piecewise affine function defined by $\chi(t) = 0$ for $t \le 1$, $\chi(t) = t - 1$ for $1 \le t \le 2$, and $\chi(t) = 1$ for $t \ge 2$, and consider the Lipschitz continuous function

$$\theta_i(x) = \chi\left(r_i^{-1} d(x, x_i)\right).$$

Then $\theta := \max_i \theta_i$ is also Lipschitz continuous, it has compact support in M, and $\theta = 1$ on a neighborhood of K. Further, we have

$$|d\theta| \le \max_i |d\theta_i| \le \max_i r_i^{-1} \mathbf{1}_{\{x | r_i \le d(x,x_i) \le 2r_i\}}$$

a.e. on M, hence

$$\int_M |d\theta|^2 dV \le \sum_i r_i^{-2} \mathcal{H}_m(B(x_i, 2r_i)) \le C' \sum_i r_i^{m-2} \le C'C$$

with $C' > 0$ independent of K. This proves the claim, after regularizing θ on M.

Step 2. Thanks to the first step, we can construct an exhaustion of M by compact sets K_j and a sequence $\theta_j \in C_c^\infty(M)$ with $0 \le \theta_j \le 1$, $\theta_j = 1$ on K_j, $\operatorname{supp} \theta_j \subset K_{j+1}$, and $\|\theta_j\|_{L^2} \le C$. Then

$$\psi := \sum_{j \ge 1} j^{-1}(1 - \theta_j)$$

is a smooth exhaustion function, since we have $\psi \ge S_k$ outside K_{k+1} with $S_k = \sum_{j=1}^k j^{-1} \to +\infty$. On the other hand, since the supports of the gradients $d\theta_j$ are disjoint, we have

$$\int_M |d\psi|^2 dV \le \sum_{j \ge 1} j^{-2} \int_M |d\theta_j|^2 < +\infty,$$

which shows that (M, g) is parabolic. □

2.2. **Mean value inequalities for holomorphic sections.** The following result (and its proof!) corresponds to [DS12, Proposition 2.1].

Proposition 2.3. *Let (X, L) be a compact polarized manifold of (complex) dimension n, and assume given $C > 0$ such that*

- $\mathrm{Ric}(\omega) \ge -C\omega$;
- $\mathrm{diam}(X, \omega) \le C$.

Then there exists a constant $A > 0$ only depending on C, n such that for all $k \geq C$ and all holomorphic sections $s \in H^0(X, kL)$ we have

$$\|s\|_{L^\infty} \leq Ak^{n/2}\|s\|_{L^2}$$

and

$$\|\nabla s\|_{L^\infty} \leq Ak^{(n+1)/2}\|s\|_{L^2}.$$

The proof of the proposition will rely on the following result:

Lemma 2.4. *Let (M^m, g) be a compact Riemannian manifold, and assume given $C > 0$ such that*

- $\mathrm{Ric}(g) \geq -Cg$;
- $\mathrm{vol}(M, g) \geq C^{-1}$;
- $\mathrm{diam}(M, g) \leq C$.

Then there exists a constant $A = A(C, m)$ with the following property: each function $f \geq 0$ on M such that $\Delta f \leq \lambda f$ with $\lambda \geq 1$ (and $\Delta = d^*d$) satisfies a mean value inequality

$$\|f\|_{L^\infty} \leq A\lambda^{m/4}\|f\|_{L^2}.$$

Proof. We use the Moser iteration technique. By Croke and Gallot, the operator norm of the Sobolev injection $L^2_1 \hookrightarrow L^{\frac{2m}{m-2}}$ is under control, i.e. we have a uniform Sobolev inequality

$$\left(\int |g|^{\frac{2m}{m-2}} dV\right)^{\frac{m-2}{m}} \leq A\left(\int |g|^2 dV + \int |dg|^2 dV\right)$$

with $A = A(C, m)$. For each $p \geq 2$, $g := f^{p/2}$ satisfies $\Delta g \leq \frac{\lambda p}{2} g$ in the sense of distributions. Injecting

$$\int |dg|^2 dV = \int (g \cdot \Delta g) dV \leq \frac{\lambda p}{2} \int g^2$$

in the Sobolev inequality, we get

$$\|f\|_{L^{\frac{pm}{m-2}}} \leq A^{1/p}\left(1 + \frac{\lambda p}{2}\right)^{1/p} \|f\|_{L^p} \leq A^{1/p}(\lambda p)^{1/p}\|f\|_{L^p}.$$

since $\lambda \geq 1$ and $p \geq 2$. If we set $p_j := 2\left(\frac{m}{m-2}\right)^j$ then $\sum_{j=0}^\infty 1/p_j = m/4$, and we get

$$\|f\|_{L^\infty} \leq A^m B\lambda^{m/4}\|f\|_{L^2}$$

with $B := \prod_{j\geq 0} p_j^{1/p_j} < \infty$ only depending on m. □

Proof of Proposition 2.3. Note that $n!\,\mathrm{vol}(X) = c_1(L)^n$ is a positive integer, hence bounded below by 1, and we may thus apply Lemma 2.4. We have the Bochner-Weitzenböck type formulas

$$\Delta = 2\Delta_{\overline{\partial}} + k$$

on smooth sections of kL, and

$$\Delta = \Delta_\partial - \mathrm{Ric}(\omega) + k$$

on smooth sections of $\Omega^{1,0}(kL)$. For a holomorphic section $s \in H^0(X, kL)$ this implies that

$$\nabla^*\nabla s = ks$$

and

$$\langle \nabla^*\nabla \partial s, \partial s\rangle \leq (k+C)|\partial s|^2,$$

from which one easily infers

$$\Delta|s| \leq k|s|$$

and

$$\Delta|\partial s| \leq (k+C)|\partial s|$$

in the sense of distributions. By Lemma 2.4, it follows that

$$\|s\|_{L^\infty} \leq Ak^{n/2}\|s\|_{L^2}$$

and

$$\|\nabla s\|_{L^\infty} \leq Ak^{n/2}\|\nabla s\|_{L^2}.$$

Finally, we use once more

$$\|\nabla s\|_{L^2}^2 = \langle \nabla^*\nabla s, s\rangle = k\|s\|_{L^2}^2,$$

to conclude the proof. □

2.3. **The Hörmander inequality.** The following result is a direct consequence of Hörmander's L^2-estimates for the $\overline{\partial}$-equation, see for instance [BDIP].

Theorem 2.5. *Let (X, L) be a compact polarized manifold, and assume that $\mathrm{Ric}(\omega) \geq -C\omega$. For each $k > C$ and each L^2 section s of kL with L^2 orthogonal projection $P(s) \in H^0(X, kL)$ we have*

$$\|P(s) - s\|_{L^2} \leq (k-C)^{-1/2}\|\overline{\partial}s\|_{L^2}.$$

3. Quantitative finite generation

Skoda's division theorem, as stated in [PAG, Theorem 9.6.31], immediately implies the following algebro-geometric result:

Proposition 3.1. *Let X be a smooth projective variety of dimension n and L an ample line bundle on X. Assume given $a, k_0 \in \mathbb{N}$ such that*

(i) *$aL - K_X$ is ample;*
(ii) *k_0L is base-point free.*

Then $R(X, L) = \bigoplus_{k\in\mathbb{N}} H^0(X, kL)$ is generated in degree $< a + (n+1)k_0$.

Remark 3.2. More generally, the result holds if (X, Δ) is a projective klt pair and (i) is replaced with (i)' $aL - (K_X + \Delta)$ is ample.

As observed in [Li, Proposition 7, p.32], this statement admits the following quantitative version.

Theorem 3.3. *Let (X, L) be a compact polarized manifold of dimension n, and assume given $C > 0$, $0 < c_- < 1 < c_+$ and $k_0 \in \mathbb{N}$ such that*

(i) *$\mathrm{Ric}(\omega) \geq -C\omega$;*
(ii) *$c_- \leq \inf_X \rho_{k_0L} \leq \sup_X \rho_{k_0L} \leq c_+$.*

If (s_i) denotes an orthonormal basis of $\bigoplus_{k\le C+(n+1)k_0} H^0(X,kL)$, then any $s\in H^0(X,kL)$ with $k>C+(n+1)k_0$ can be expressed as a polynomial in the s_i having coefficients bounded above by

$$(n+1)^{k/2}c_-^{-k/2}c_+^{n/2}\|s\|_{L^2}.$$

In order to prove this, we recall the version of Skoda's division theorem given in [Dem82, Théorème 6.2].

Theorem 3.4. *Let (X,H) be a compact polarized manifold of dimension n, and $s_1,...,s_p\in H^0(X,M)$ global sections of another holomorphic line bundle M on X. Then any $\sigma\in H^0(X,K_X+kM+H)$ with $k>n$ satisfying the L^2 condition*

$$\int_X \frac{|\sigma|^2}{\left(\sum_j |s_j|^2\right)^k}<+\infty$$

writes $\sigma=\sum_i h_i s_i$ with $h_i\in H^0(K_X+(k-1)M+H)$ satisfying

$$\int_X \frac{|h_i|^2}{\left(\sum_j |s_j|^2\right)^{k-1}}\le (n+1)\int_X \frac{|\sigma|^2}{\left(\sum_j |s_j|^2\right)^k}.$$

Here the integrals are defined without having to specify a volume form, by viewing h_i and s as holomorphic n-forms with values in $(k-1)M+H$ and $kM+H$ respectively.

Proof of Theorem 3.3. Set $a:=\lfloor C\rfloor+1$, so that $\mathrm{Ric}(\omega)>-a\omega$, and $M:=k_0L$, which is basepoint free by assumption. Let $k\ge a+(n+1)k_0$, so that $k-a=qk_0+r$ with $q>n$ and $0\le r<k_0$. We then have $kL=K_X+qM+H$ with $M:=k_0L$ and $H:=(aL-K_X)+rL$. Endow $-K_X$ with the metric induced by the volume form ω^n, and H with the corresponding metric, whose curvature $a\omega+\mathrm{Ric}(\omega)+r\omega$ is positive by assumption. Let also $s_1,...,s_p$ be an orthonormal basis of $H^0(X,k_0L)$.

Applying iteratively Theorem 3.4 shows that any $s\in H^0(X,kL)$ writes

$$s=\sum_{\alpha\in\mathbb{N}^p,|\alpha|=q-n} h_\alpha s_1^{\alpha_1}...s_p^{\alpha_p}$$

where $h_\alpha\in H^0(K_X+nM+H)=H^0((a+nk_0+r)L)$ satisfies

$$\int \frac{|h_\alpha|^2}{\left(\sum_j |s_j|^2\right)^n}\le (n+1)^{q-n}\int \frac{|s|^2}{\left(\sum_j |s_j|^2\right)^q}.$$

As explained above, the integrals are defined by viewing h_α and s as holomorphic n-forms with values in nk_0L+H and qk_0L+H respectively. Viewing them instead as sections of $(a+nk_0+r)L$ and $(a+qk_0+r)L$, the L^2 estimate becomes

$$\int \frac{|h_\alpha|^2}{\rho_{k_0L}^n}\omega^n\le (n+1)^{q-n}\int \frac{|\sigma|^2}{\rho_{k_0L}^q}\omega^n.$$

This implies

$$\|h_\alpha\|_{L^2}\le (n+1)^{k/2}c_-^{-k/2}c_+^{n/2}\|s\|_{L^2},$$

and the result follows. □

4. The algebra of bounded sections: general properties

4.1. **Normality.** Normality of the graded algebra $R_b(X, kL)$ is a general fact:

Proposition 4.1. *Let X be an arbitrary normal complex space and L be a holomorphic line bundle on X. Then the graded algebra $R(X,L) = \bigoplus_{k\in\mathbb{N}} H^0(X, kL)$ is normal.*

If L is endowed with a Hermitian metric and $R_b(X,L) \subset R(X,L)$ denotes the subalgebra of bounded sections, then $R_b(X,L)$ is normal as well.

Proof. Every section $s \in H^0(X, kL)$ induces a function $\tilde{s}$ on the total space L^* of the dual bundle, and $s \mapsto \tilde{s}$ identifies $H^0(X, kL)$ with the weight k eigenspace $\mathcal{O}(L^*)_k$ with respect to its natural $\mathbb{C}^*$-action.

If P/Q is integral over $R(X,L)$ for some non-zero $P, Q \in R(X,L)$, then the meromorphic function $\tilde{P}/\tilde{Q}$ on the total space of L^* is integral over $\mathcal{O}(L^*)$, hence defines a function $f \in \mathcal{O}(L^*)$ by normality of L^*. Since $\tilde{Q}$ and $f\tilde{Q}$ both belong to $\bigoplus_{k\in\mathbb{N}} \mathcal{O}(L^*)_k$, it is easy to see that the Taylor expansion of f along the fibers of $L^* \to X$ involves finitely many terms, i.e. $f \in \bigoplus_{k\in\mathbb{N}} \mathcal{O}(L^*)_k$. We thus have $P/Q \in R(X,L)$, which proves that $R(X,L)$ is integrally closed.

Assume now that L is endowed with a Hermitian metric. If P/Q is integral over $R_b(X,L)$ for some $P, Q \in R_b(X,L)$, what has just been proved shows that $\tilde{P} = \tilde{Q}\tilde{R}$ with $R \in R(X,L)$. Since R satisfies a unit polynomial equation with coefficients in $R_b(X,L)$, the usual estimate of the solutions of such a unit equation in terms of the coefficients shows that $\tilde{R}$ is bounded on the unit circle bundle of L^*. Taking L^2 averages on the fibers of the unit circle bundle and using Parseval's identity, we see that each homogeneous component R_k of R is bounded as well, i.e. $R \in R_b(X,L)$, which shows $R_b(X,L)$ is normal. □

4.2. **Polynomial growth.** As a special case of [Mok86], if (X,L) is a polarized manifold such that the Kähler manifold X is complete and of finite volume, then the dimension of the space of L^2 holomorphic sections of kL growth like k^n, with $n = \dim X$. When X is merely parabolic and of finite volume, we adapt his arguments to prove:

Theorem 4.2. *Let (X,L) be a polarized manifold such that the Kähler manifold (X,ω) is parabolic and of finite volume. Then we have*

$$\dim H^0_b(X, kL) = O(k^n).$$

As a consequence, if we denote by $\Phi_k : X \dashrightarrow \mathbb{P}H^0_b(X, kL)$ the meromorphic map defined by sections of kL, then the Zariski closure Y_k of the image of Φ_k has dimension at most n. This follows from the Hilbert-serre theorem, since the homogeneous coordinate ring of Y_k is by construction the graded subalgebra of $R_b(X,L)$ generated by $H^0_b(X, kL)$.

Proof. As already mentioned, the proof is a direct extension of the arguments given in [Mok86].

Step 1. We first claim that ∇s is in L^2 for each $s \in H^0_b(X, kL)$. To see this, let θ_ν be an exhaustive sequence of cut-off functions such that $\|d\theta_\nu\|_{L^2} \to 0$. By the Bochner-Kodaira-Nakano identity (which is always valid for smooth sections with compact support), we have for each ν

$$\|\nabla(\theta_\nu s)\|_{L^2}^2 = 2\|\overline{\partial}(\theta_\nu s)\|_{L^2}^2 + k\|\theta_\nu s\|_{L^2}^2.$$

Now $\overline{\partial}(\theta_\nu s) = (\overline{\partial}\theta_\nu)s$ and $\nabla(\theta_\nu s) = (d\theta_\nu)s + \theta_\nu \nabla s$, where both $(\overline{\partial}\theta_\nu)s$ and $(d\theta_\nu)s$ tend to 0 in L^2 since s in bounded. It follows that ∇s is L^2, with $\|\nabla s\|_{L^2} = k^{1/2}\|s\|_{L^2}$.

Step 2. We next show that the zero divisor $\operatorname{div}(s)$ of every non-zero section $s \in H^0_b(X, kL)$ has finite volume, with a linear estimate

$$\int_X [\operatorname{div}(s)] \wedge \omega^{n-1} \leq k \int_X \omega^n,$$

where $\int_X \omega^n = n!\operatorname{vol}(X)$ is finite by assumption. For each $\varepsilon > 0$, the smooth function $\varphi_\varepsilon := \log\left(|s|^2 + \varepsilon^2\right)$ is $k\omega$-psh, i.e. it satisfies

$$dd^c\varphi_\varepsilon \geq -k\omega,$$

and $k\omega + dd^c\varphi_\varepsilon$ converges weakly as $\varepsilon \to 0$ to the current of integration $[\operatorname{div}(s)]$, by the Poincaré-Lelong formula. Since the total mass of a positive measure is lower semicontinuous with respect to weak convergence, it will be enough to show that

$$\int_X (k\omega + dd^c\varphi_\varepsilon) \wedge \omega^{n-1} = k \int_X \omega^n$$

for each fixed $\varepsilon > 0$, which boils down to

$$\lim_{\nu\to\infty} \int_X \theta_\nu dd^c\varphi_\varepsilon \wedge \omega^{n-1} = 0 \tag{4.1}$$

for $\varepsilon > 0$ fixed. We have

$$d\varphi_\varepsilon \wedge d^c\varphi_\varepsilon = c\frac{|\langle s, \partial s\rangle|^2}{(|s|^2 + \varepsilon^2)^2}$$

for some numerical constant $c > 0$. Since s is bounded and ∂s is in L^2 thanks to Step 1, it follows that

$$\int_X d\varphi_\varepsilon \wedge d^c\varphi_\varepsilon \wedge \omega^{n-1} < +\infty.$$

On the other hand, integration by parts and the Cauchy-Schwarz inequality yields

$$\left(\int_X \theta_\nu dd^c\varphi_\varepsilon \wedge \omega^{n-1}\right)^2 = \left(\int_X d\theta_\nu \wedge d^c\varphi_\varepsilon \wedge \omega^{n-1}\right)^2$$

$$\leq \left(\int_X d\theta_\nu \wedge d^c\theta_\nu \wedge \omega^{n-1}\right)\left(\int_X d\varphi_\varepsilon \wedge d^c\varphi_\varepsilon \wedge \omega^{n-1}\right),$$

and (4.1) follows.

Step 3. We prove that $\dim H_b^0(X, kL) = O(k^n)$ using the classical Poincaré-Siegel argument, which goes as follows. Fix a point $x \in X$. The usual properties of Lelong numbers show that

$$\mathrm{ord}_x(s) \le C \int_X [\mathrm{div}(s)] \wedge \omega^{n-1}$$

for some constant $C > 0$ only depending on (X, ω). By Step 2, we thus have a linear bound $\mathrm{ord}_x(s) \le Ck$ for the vanishing orders at x of sections $s \in H_b^0(X, kL)$. As a consequence, the evaluation map

$$H_b^0(X, kL) \to \mathcal{O}_x(kL)/\mathfrak{m}_x^{Ck+1}$$

is injective, and hence

$$\dim H_b^0(X, kL) \le \binom{n + Ck}{n} = O(k^n).$$

□

5. Finite generation on Gromov-Hausdorff limits

5.1. **Donaldson-Sun's result.** Following [DS12], we first make precise the notion of convergence we use for polarized manifolds.

Definition 5.1. A sequence (X_j, L_j) of polarized manifolds of fixed complex dimension n converges in $C^{1,\alpha}$ topology to a polarized manifold (X, L) if, for each open $U \Subset X$, there exists $C^{2,\alpha}$ open embeddings

$$\tau_j : U \hookrightarrow X_j$$

and bundle isomorphisms $\tau_j^* L_j \simeq L|_U$, with respect to which the complex structures and the Hermitian metrics converge on compact subsets of U in $C^{1,\alpha}$ and $C^{2,\alpha}$ topology respectively.

We emphasize that, with this definition, (X_j, L_j) converges as well to $(U, L|_U)$ for any open subset $U \subset X$.

As an example, if X_∞ is the Gromov-Hausdorff limit of a sequence (X_j, L_j) of compact polarized manifolds of uniformly bounded Ricci curvature, then the Cheeger-Colding-Tian theory guarantees the existence of a polarized manifold (X, L) such that X embeds isometrically as an open subset of X_∞, and such that (X_j, L_j) converges in $C^{1,\alpha}$ topology to (X, L).

As an extra piece of notation, if (X, L) is a polarized manifold with finite volume, the L^2 norm is well defined on $H_b^0(X, kL)$ for each $k \in \mathbb{N}$, and we define the distorsion function of $H_b^0(X, kL)$ (aka density of states function, aka Bergman function) as the squared operator norm of the evaluation map, i.e.

$$\rho_{kL}(x) = \sup_{s \in H_b^0(X, kL) \setminus \{0\}} \frac{|s(x)|^2}{\|s\|_{L^2}^2}.$$

By the mean value inequality for holomorphic functions, ρ_{kL} is locally bounded on X.

Theorem 5.2. *Let (X_j, L_j) be a sequence of compact polarized manifolds such that*

- $\mathrm{Ric}(\omega_j) \geq -C\omega_j$,
- $\mathrm{diam}(X_j) \leq C$,

for some $C > 0$ independent of j. Assume also that (X_j, L_j) converges in $C^{1,\alpha}$ topology to a polarized manifold (X, L) such that X is parabolic as a Riemannian manifold. Then we have

(i) $\mathrm{vol}(X_j) = \mathrm{vol}(X)$ *for all* $j \gg 1$;
(ii) *for each* $k \geq C + 1$, *we have*

$$\dim H^0_b(X, kL) = \dim H^0(X_j, kL_j)$$

for all $j \gg 1$.
(iii) *If we further assume that*

$$\limsup_{j\to\infty} \left(\inf_{X_j} \rho_{k_0 L_j} \right) > 0$$

for some $k_0 \in \mathbb{N}$, then $\inf_X \rho_{kL} > 0$ for all mutiples k of k_0, and the graded algebra $R_b(X, L) = \bigoplus_{k\in\mathbb{N}} H^0_b(X, kL)$ is finitely generated.

5.2. **Remarks and questions.** In the context of Gromov-Hausdorff limits, the convergence of the volume in (i) already follows from the Cheeger-Colding theory.

By the Kodaira embedding theorem, each X_j is a projective variety and L_j is ample. The uniform lower bound on the Ricci curvature of ω_j further implies that $aL_j - K_{X_j}$ is ample for a fixed positive integer a, say $a = \lfloor C \rfloor + 1$. By the effective very ampleness results pioneered in [Dem93], it follows that kL_j is very ample for all $k \geq k_0$ for some k_0 only depending on a and n. In particular, $k_0 L_j$ is basepoint free, i.e. $\inf_{X_j} \rho_{k_0 L_j} > 0$ for each j, but the lower bound depends a priori on j. The main result in [DS12] guarantees that a uniform lower bound

$$\limsup_{j\to\infty} \left(\inf_{X_j} \rho_{k_0 L_j} \right) > 0$$

holds (for a possibly larger k_0), if we further impose a uniform upper bound on $\mathrm{Ric}(\omega_j)$.

Choose k such that the sections of kL_j embed X_j as a subvariety of a fixed projective space. These subvarieties belongs to finitely many components of the Hilbert scheme, since $c_1(L_j)^n = n!\,\mathrm{vol}(X_j)$ is uniformly bounded by the Bishop-Gromov comparison theorem. In particular, the X_j can only have finitely many diffeomorphism types. After passing to a subsequence, the Hilbert polynomial P_j of (X_j, L_j) may be assumed to be independent of j. Since we also have $P_j(k) = \dim H^0(X_j, kL_j)$ for $k \geq a$ by Kodaira vanishing, (ii) implies that $P_j(k) = \dim H^0_b(X, kL)$ for all $k \gg 1$, and hence

$$\dim H^0_b(X, kL) = \mathrm{vol}(X)k^n + O(k^{n-1}).$$

In the setting of (iii), the Hilbert-Serre theorem therefore shows that $\mathrm{Proj}\, R_b(X, L)$ is a projective variety of dimension n, which is also normal by Proposition 4.1.

Question 1. Is it true that $R_b(X, L)$ is finitely generated for every polarized manifold (X, L) which is parabolic, of finite volume and such that $\inf_X \rho_{k_0 L} > 0$ for some $k_0 \in \mathbb{N}$?

At least, we can produce an example of a polarized manifold (X, L) of finite volume, with $H^0_b(X, kL)$ is finite dimensional for all k, base-point free for some k, but such that $R_b(X, L)$ is not finitely generated. Indeed, let $(\bar{X}, L)$ be a projective manifold and a nef and big line bundle such that $R(\bar{X}, L)$ is not finitely generated (examples abound starting in every dimension at least 2, see [PAG]). By [BEGZ10, Theorem 5.1], we can find a singular semipositive metric on L with minimal singularities whose curvature current ω is such that ω^n is a smooth positive volume form on $\bar{X}$. If we denote by $X \subset \bar{X}$ the ample locus of L, a Zariski open subset, then ω is automatically smooth on X, hence a Kähler form, so that (X, L) defines a polarized manifold (probably parabolic...). Since the metric of L over $\bar{X}$ has minimal singularities, it is easy to check that $R_b(X, L) = R(\bar{X}, L)$, so that $R_b(X, L)$ is not finitely generated. However, the ample locus X is contained in the complement of the asymptotic base locus of L on $\bar{X}$, so that $H^0_b(X, kL)$ is base point free on X for some k. However, we do have $\inf_X \rho_{kL} = 0$, since kL cannot be basepoint free on $\bar{X}$.

In the next sections, we prove Theorem 3.3.

5.3. **Convergence of the volume.** It is immediate to check from the definition of $C^{1,\alpha}$ convergence that

$$\mathrm{vol}(X) \le \liminf_{j\to\infty} \mathrm{vol}(X_j). \tag{5.1}$$

Since $n! \,\mathrm{vol}(X_j)$ is an integer for each j, it is thus enough to show that

$$\limsup_{j\to\infty} \mathrm{vol}(X_j) \le \mathrm{vol}(X).$$

We will prove this by adapting the usual argument that relies on the Poincaré inequality to show that sets of zero capacity have measure zero, as in [EG92]. Let $\varepsilon > 0$ and let $\theta \in C^\infty_c(X)$ be a non-negative function with $\theta = 1$ on a given compact set $K \subset X$ with non-empty interior and $\|d\theta\|^2_{L^2} < \varepsilon$. Let τ_j be a sequence of open embeddings of a neighborhood of $K' = \operatorname{supp}\theta$ in X_j as in Definition 5.1, and $\theta_j \in C^{1,\alpha}_c(X_j)$ the corresponding functions. The uniform bounds on X_j yields a uniform positive lower bound on the first positive eigenvalue of the Laplacian [Yau75]. We thus have a uniform Poincaré inequality

$$\|\theta_j - \bar{\theta}_j\|^2_{L^2} \le C\|d\theta_j\|^2_{L^2}$$

with $\bar{\theta}_j$ the mean value of θ_j. Since θ_j vanishes outside $K'_j = \tau_j(K')$, this implies

$$\mathrm{vol}(X_j \setminus K'_j)\bar{\theta}_j^2 \le C\varepsilon$$

for $j \gg 1$, and hence

$$\mathrm{vol}(X_j) \le \mathrm{vol}(K'_j) + C\varepsilon\bar{\theta}_j^{-2}.$$

But

$$\lim_{j\to\infty} \mathrm{vol}(K'_j) = \mathrm{vol}(K') \le \mathrm{vol}(X) + \varepsilon,$$

and it is now enough to bound $\bar{\theta}_j$ from below. Using that $\theta_j = 1$ on K'_j, we have $\bar{\theta}_j \geq \mathrm{vol}(K_j)/\mathrm{vol}(X_j)$, which is uniformly bounded below since $\mathrm{vol}(X_j)$ is bounded above by Bishop-Gromov while $\mathrm{vol}(K_j) \to \mathrm{vol}(K)$.

5.4. **Isomorphism of spaces of sections.** The goal of this section is to provide a detailed proof of [DS12, Lemma 4.5], using only the assumptions of Theorem 3.3. Let us set some notation. Since X is a parabolic Kähler manifold, we may and do fix an exhaustion of X by open sets U_ν and smooth cut-off functions $\theta_\nu \in C_c^\infty(U_\nu)$, $\theta_\nu \equiv 1$ on a neighborhood of $\overline{U}_{\nu-1}$, such that $\|d\theta_\nu\|_{L^2} \to 0$ as $\nu \to \infty$. For each ν, let

$$\tau_{\nu,j} : U_\nu \hookrightarrow X_j$$

be a sequence of open embeddings as in Definition 5.1, and set

$$U_{\nu,j} = \tau_{\nu,j}(U_\nu) \subset X_j.$$

Since X is parabolic and has finite volume, we known from Theorem 4.2 that $H^0_b(X, kL)$ is finite dimensional for each $k \in \mathbb{N}$. We define linear maps

$$Q_{\nu,j} : H^0_b(X, kL) \to H^0(X_j, kL_j) \tag{5.2}$$

as follows. Given $s \in H^0_b(X, kL)$, denote by $(\theta_\nu s)_j$ the section of kL_j with compact support in $U_{\nu,j}$ induced by transporting $\theta_\nu s$ via $\tau_{\nu,j}$ and the bundle isomorphism $\tau^*_{\nu,j} L_j \simeq L|_{U_\nu}$. We then define $Q_{\nu,j}(s)$ as the L^2 orthogonal projection of $(\theta_\nu s)_j$ to the space of holomorphic sections.

In the other direction, we define an operator $P_{\nu,j}$ from $H^0(X_j, kL_j)$ to C_c^∞ sections of kL on X by setting

$$P_{\nu,j}(s) := \theta_\nu \tau^*_{\nu,j} s.$$

Lemma 5.3. *For each $k \geq C + 1$ fixed, the composition $P_{\nu,j} Q_{\nu,j}$ is close to the identity on $H^0_b(X, kL)$ for $j \gg \nu \gg 1$. More precisely, for each $\varepsilon > 0$, there exists ν_0 and a sequence j_ν such that*

$$\|s - P_{\nu,j} Q_{\nu,j}(s)\|_{L^2} \leq \varepsilon \|s\|_{L^2}.$$

for all $s \in H^0_b(X, kL)$, all $\nu \geq \nu_0$ and all $j \geq j_\nu$.

Proof. Given $\varepsilon > 0$, we choose ν_0 such that $\mathrm{vol}(X \setminus U_{\nu-1}) \leq \varepsilon^2$ and $\|d\theta_\nu\|_{L^2} \leq \varepsilon$ for all $\nu \geq \nu_0$. Pick $s \in H^0_b(X, kL)$ and $\nu \geq \nu_0$. For all j, $P_{\nu,j} Q_{\nu,j}(s)$ is supported in U_ν and $\theta_\nu = 1$ on $U_{\nu-1}$, hence

$$\|s - P_{\nu,j} Q_{\nu,j}(s)\|_{L^2} \leq 2\|s\|_{L^2(X \setminus U_{\nu-1})} + \|\theta_\nu s - P_{\nu,j} Q_{\nu,j}(s)\|_{L^2(U_\nu)}. \tag{5.3}$$

Now

$$\|s\|_{L^2(X \setminus U_{\nu-1})} \leq \varepsilon \|s\|_{L^\infty} \leq C_k \varepsilon \|s\|_{L^2}$$

for some constant $C_k > 0$, which takes care of the first term in the right-hand side of (5.3). Here we have used that the equivalence of the L^2 and L^∞ norms on $H^0_b(X, kL)$, which holds since $H^0_b(X, kL)$ is finite dimensional by Theorem 4.2.

Let us now consider the second term in the right-hand side of (5.3). Since $\tau_{\nu,j}$ gets $C^{1,\alpha}$ close to an isometry as $j \to \infty$ for each ν fixed, there exists a sequence j_ν such that

$$\|\theta_\nu s - P_{\nu,j} Q_{\nu,j}(s)\|_{L^2(U_\nu)} \leq 2\|(\theta_\nu s)_j - Q_{\nu,j}(s)\|_{L^2}$$

for all ν, j with $j \geq j_\nu$. Since $k \geq C+1$ and $\mathrm{Ric}(\omega_j) \geq -C\omega_j$, the Hörmander inequality (Theorem 2.5) yields

$$\|(\theta_\nu s)_j - Q_{\nu,j}(s)\|_{L^2} \leq \|\overline{\partial}(\theta_\nu s)_j\|_{L^2}.$$

After perhaps taking j_ν even larger, we may ensure that

$$\|\overline{\partial}(\theta_\nu s)_j\|_{L^2} \leq 2\|(\overline{\partial}(\theta_\nu s)\|_{L^2} + \varepsilon\|\nabla(\theta_\nu s)\|_{L^2}$$

for all $j \geq j_\nu$. Now $\overline{\partial}(\theta_\nu s) = (\overline{\partial}\theta_\nu)s$ since s is holomorphic, while $\|d\theta_\nu\|_{L^2} \leq \varepsilon$, so this is in turn bounded above by $C_k\varepsilon\|s\|_{L^2}$ for some constant $C_k > 0$, using this time the equivalence between the L^2 norm, the L^∞ norm and the Sobolev L^2_1 norm on $H^0_b(X, kL)$. Summing up, we have proved that the second term in the right-hand side of (5.3) satisfies

$$\|P_{\nu,j}Q_{\nu,j}(s) - \theta_\nu s\|_{L^2(U_\nu)} \leq C_k\varepsilon\|s\|_{L^2},$$

which concludes the proof of Lemma 5.3. □

Lemma 5.4. *$P_{\nu,j}$ is an almost isometry for $j \gg \nu \gg 1$. More precisely, for each $\varepsilon > 0$, there exist ν_0 and a sequence j_ν such that*

$$(1-\varepsilon)\|t\|_{L^2} \leq \|P_{\nu,j}(t)\|_{L^2} \leq (1+\varepsilon)\|t\|_{L^2}$$

for all ν, j with $\nu \geq \nu_0$, $j \geq j_\nu$ and all $t \in H^0(X_j, kL_j)$. Furthermore, we can also require that

$$\|\overline{\partial}P_{\nu,j}(t)\|_{L^\infty} \leq \varepsilon\|t\|_{L^2}$$

and

$$\|\nabla P_{\nu,j}(t)\|_{L^\infty} \leq C_k\|t\|_{L^2}.$$

Proof. By Proposition 2.3, there exists a constant $C_k > 0$ independent of j such that

$$\|\nabla t\|_{L^\infty} + \|t\|_{L^\infty} \leq C_k\|t\|_{L^2} \tag{5.4}$$

for all $t \in H^0(X_j, kL_j)$. For each ν fixed, $\tau_{\nu,j}$ is almost an isometry for $j \gg 1$, hence

$$\|P_{\nu,j}(t)\|_{L^2} \leq (1+\varepsilon)\|t\|_{L^2},$$
$$\|\overline{\partial}P_{\nu,j}(t)\|_{L^\infty} \leq 2\|\overline{\partial}\theta_\nu\|_{L^2}|t\|_{L^\infty} + \varepsilon\|\nabla t\|_{L^\infty},$$

and

$$\|\nabla P_{\nu,j}(t)\|_{L^\infty} \leq 2\|\nabla((\theta_\nu)_j t)\|_{L^\infty}.$$

Thanks to (5.4), this already proves the right-hand part of the first estimate, as well as last two ones. To get a lower bound on $\|P_{\nu,j}(t)\|_{L^2}$, we write

$$\|t\|^2_{L^2} \leq \mathrm{vol}(X_j \setminus \tau_{\nu,j}(U_{\nu-1}))\|t\|^2_{L^\infty} + \|t\|^2_{L^2(\setminus\tau_{\nu,j}(U_{\nu-1}))}.$$

Using (5.4) and

$$\lim_{\nu\to\infty}\lim_{j\to\infty}\mathrm{vol}(X_j \setminus \tau_{\nu,j}(U_{\nu-1})) = \mathrm{vol}(X) - \lim_{\nu\to\infty}\mathrm{vol}(U_{\nu-1}) = 0,$$

we get

$$\|t\|_{L^2} \leq (1+\varepsilon)\|t\|_{L^2(\tau_{\nu,j}(U_{\nu-1}))},$$

for $j \gg \nu \gg 1$, and the result follows easily. □

We are now in a position to prove:

Proposition 5.5. *For each $k \geq C+1$, $Q_{\nu,j} : H^0_b(X,kL) \to H^0(X_j, kL_j)$ is an isomorphism for all $j \gg \nu \gg 1$.*

Proof. Injectivity follows directly from Lemma 5.3. Assume now that for all ν large enough, there exists j_ν arbitrarily large such that $Q_\nu := Q_{\nu,j_\nu}$ is not surjective, and set for simplicity $P_\nu := P_{\nu,j_\nu}$ and $X_\nu := X_{j_\nu}$. By Lemma 5.4, we may assume that

$$(1-\varepsilon_\nu)\|t\|_{L^2} \leq \|P_\nu(t)\|_{L^2} \leq (1+\varepsilon_\nu)\|t\|_{L^2},$$

$$\|\overline{\partial} P_\nu(t)\|_{L^\infty} \leq \varepsilon_\nu \|t\|_{L^2}$$

and

$$\|\nabla P_\nu(t)\|_{L^\infty} \leq C_k \|t\|_{L^2}$$

for all $t \in H^0(X_\nu, kL_\nu)$, with $\varepsilon_\nu \to 0$.

For each ν, choose $t_\nu \in H^0(X_\nu, kL_\nu)$ orthogonal to the image of Q_ν and such that $\|t_\nu\|_{L^2} = 1$. By the above estimates, we may assume after perhaps passing to a subsequence of ν that $P_\nu(t_\nu)$ converges uniformly on compact sets of X to a non-zero bounded holomorphic section $s \in H^0_b(X,kL)$. Since t_ν is holomorphic and orthogonal to $Q_\nu(s)$, the orthogonal projection of $(\theta_\nu s)_{j_\nu}$ on holomorphic sections, we have $\langle t_\nu, (\theta_\nu s)_{j_\nu}\rangle = 0$, and hence

$$\langle P_\nu(t_\nu), \theta_\nu s\rangle \to 0.$$

But since $P_\nu(t_\nu)$ and $\theta_\nu s$ are uniformly bounded and both converge to s on compact sets, this implies by dominated convergence that $\langle s, s\rangle = 0$, which contradicts the fact that s is non-zero. □

5.5. **Lower bound for the distorsion function.** Let us show that for each $k \geq C+1$ we have

$$\inf_X \rho_{kL} \geq \limsup_{j\to\infty} \left(\inf_{X_j} \rho_{kL_j}\right).$$

Pick $x \in X$ and choose ν such that $x \in U_\nu$. We can then find an infinite sequence $s_j \in H^0(X_j, kL_j)$ such that

$$|s_j(\tau_{\nu,j}(x))|^2 = \rho_{kL_j}(\tau_{\nu,j}(x)) \geq \inf_{X_j} \rho_{kL_j}$$

and $\|s_j\|_{L^2} = 1$. We thus have

$$|P_{\nu,j}(s_j)(x)|^2 \geq \inf_{X_j} \rho_{kL_j}$$

and $\|P_{\nu,j}(s_j)\|_{L^2} \to 1$ for $j \gg \nu \gg 1$ by Lemma 5.4. Arguing as in the proof of Proposition 5.5, we may assume that $P_{\nu,j}(s_j)$ converges for $j \gg \nu \gg 1$ to a bounded holomorphic section $s \in H^0_b(X,kL)$ such that $\|s\|_{L^2} = 1$, and which satisfies

$$\rho_{kL}(x) \geq |s(x)|^2 \geq \limsup_{j\to\infty} \left(\inf_{X_j} \rho_{kL_j}\right).$$

5.6. **Finite generation.** Suppose given $k_0 \in \mathbb{N}$ such that

$$\limsup_{j\to\infty}\left(\inf_j \rho_{k_0L_j}\right) > 0.$$

After passing to a subsequence, we may assume that $\inf_{X_j} \rho_{k_0L_j} \geq c > 0$ for all j. On the other hand, by Proposition 2.3 there exists a constant $C_k > 0$ independent of j such that $\sup_{X_j} \rho_{kL_j} \leq C_k$ for all j, and it follows easily that for each k multiple of k_0 there exists $c_k > 0$ independent of j such that $\inf_{X_j} \rho_{kL_j} \geq c_k > 0$ for all j, cf. [DS12, Lemma 3.1].

Since a graded algebra is finitely generated iff it is noetherian, and since $R_b(X, L)$ is a finite module over $R_b(X, kL)$ for each k, we may replace L with kL in showing that $R_b(X, L)$ is finitely generated. We may thus assume that $\mathrm{Ric}(\omega_j) \geq -\omega_j/2$. As a consequence, for each $k \geq 1$, we may choose ν_k and j_k such that $Q_{\nu_k,j} : H^0_b(X, kL) \to H^0(X_j, kL_j)$ is an isomorphism for all $j \geq j_k$. Let (s_α) be an orthonormal basis of

$$\bigoplus_{k\leq(n+1)m} H^0_b(X, kL).$$

Their images $(Q_{\nu_k,j}(s_\alpha))_\alpha$ under the isomorphism form an almost orthonormal basis of

$$\bigoplus_{k\leq(n+1)m} H^0(X_j, kL_j)$$

for $j \gg 1$. Pick $k \geq 1 + (n+1)m$ and a section $s \in H^0_b(X, kL)$. By Theorem 3.3, there exists a constant $B_k > 0$ independent of j such that $Q_{\nu_k,j}(s) \in H^0(X_j, kL_j)$ can be expressed as a polynomial in $(Q_{\nu_k,j}(s_\alpha))_\alpha$ with coefficients bounded above by $B_k\|Q_{\nu_k,j}(s)\|_{L^2}$, hence bounded independently of $j \gg 1$. Upon taking a subsequence, we can assume that the coefficients of these polynomials converge as $j \to \infty$. Since $\tau^*_{\nu_k,j}Q_{\nu_k,j}(s) \to s$ and $\tau^*_{\nu_k,j}Q_{\nu_k,j}(s_\alpha) \to s_\alpha$ for each α, we easily get in the limit that s can be expressed as a polynomial in $(s_\alpha)_\alpha$, which shows as desired that $R_b(X, L)$ is finitely generated.

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