Lecture 3: Non-Archimedean pluripotential theory and K-stability

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From psh rays to test configurations

Let (X,ω) be a compact Kähler manifold, $n=\dim X$, $V=[\omega]^n$.

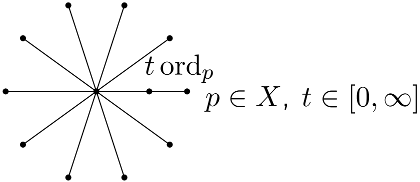
- Psh ray $(\varphi_t)_{t>0}$ in $\mathrm{PSH}(\omega) \leftrightarrow S^1$-invariant ω-psh function $\Phi(x,\tau)=\varphi_{-\log|\tau|}(x)$ on $X\times\mathbb{D}^\times$.
- Assume **linear growth** $\varphi_t \le at+b$ (automatic for psh geodesic rays in $\mathcal{E}^1$) $\Rightarrow \widetilde{\Phi} := \Phi + a\log|\tau| \le b$ (uniquely) extends to ω-psh function on $X\times\mathbb{D}$.
- Simplest case: **analytic singularities** along $X\times\{0\}$, i.e. $\widetilde{\Phi} =_{\mathrm{loc}} \frac{1}{m}\log\sum_i |f_i| + O(1)$ with $m\in\mathbb{Z}_{>0}$ and $(f_i)\subset \mathcal{O}_{X\times\mathbb{D}}$ trivial outside $X\times\{0\}$.
- Ideal sheaf $\mathfrak{a} := \{f\in\mathcal{O}_{X\times\mathbb{D}} \mid \log|f| \le m\widetilde{\Phi}+O(1)\}$ coherent and S^1-invariant $\Rightarrow$ **flag ideal** on $X\times\mathbb{P}^1$, i.e. $\mathfrak{a}=\sum_{i=0}^N \mathfrak{a}_i\tau^i$ for an increasing sequence of (coherent sheaves of) ideals $\mathfrak{a}_0\subset\mathfrak{a}_1\subset\cdots\subset\mathfrak{a}_N=\mathcal{O}_X$.
- Blowup $\mathcal{X}\to X\times\mathbb{P}^1$ along $\mathfrak{a}$ is a **test configuration** for X, i.e. $\mathbb{C}^\times$-equivariant compactification of $X\times(\mathbb{P}^1\setminus\{0\})$, with fiber $\mathcal{X}_0$ over $0\in\mathbb{P}^1$.
- Write $\mathfrak{a}\cdot\mathcal{O}_{\mathcal{X}}=\mathcal{O}_{\mathcal{X}}(-E) \rightsquigarrow$ singularities of Φ encoded in $D:=-\frac{1}{m}E+a\mathcal{X}_0$. Lies in $\mathrm{Div}_0(\mathcal{X})=\{D\in\mathrm{Div}(\mathcal{X}) \mid \operatorname{supp} D\subset\mathcal{X}_0\}$ (**vertical divisors**).

The Berkovich space

- **Berkovich space** $X^{\beth}$:= space of (normalized) **tropical characters** of semiring $\mathcal{I}$ of flag ideals $\mathfrak{a}$ on $X \times \mathbb{P}^1$, i.e. functions $v\colon \mathcal{I} \to \mathbb{R}_{\geq 0}$ such that

$$v(\mathfrak{a} \cdot \mathfrak{a}') = v(\mathfrak{a}) + v(\mathfrak{a}'), \quad v(\mathfrak{a} + \mathfrak{a}') = \min\{v(\mathfrak{a}), v(\mathfrak{a}')\}, \quad v(\tau) = 1.$$

  Space $X^{\beth}$ is compact Hausdorff for topology of pointwise convergence.
- X projective (algebraic) $\Rightarrow X^{\beth}$ = **Berkovich analytification** of X with respect to the trivial absolute value on $\mathbb{C}$ = natural compactification of the space of valuations $v\colon \mathbb{C}(X)^{\times} \to \mathbb{R}$.
- Space of **piecewise linear (PL)** functions $\mathrm{PL}(X^{\beth})$:= $\mathbb{Q}$-vector space generated by image of evaluation map $\mathcal{I} \to \mathrm{C}^0(X^{\beth})$. Stable under max and min, separates points $\Rightarrow \mathrm{PL}(X^{\beth})$ dense in $\mathrm{C}^0(X^{\beth})$ (Stone–Weierstrass).
- Space $X^{\beth}$ for $n = 1$:



The non-Archimedean Monge–Ampère operator

- $\mathcal{X}$ test configuration, $\mathcal{X}_0 = \sum_F b_F F \rightsquigarrow v_F \in X^\beth$ such that $v_F(\mathfrak{a}) := b_F^{-1} \operatorname{ord}_F(\mathfrak{a})$ for $\mathfrak{a} \in \mathcal{I} \rightsquigarrow$ dense subset $X^{\mathrm{div}} \subset X^\beth$ of **divisorial valuations** (coincides with usual definition when X projective).
- $\mathcal{X}$ test configuration $\rightsquigarrow$ unique linear map $\mathrm{Div}_0(\mathcal{X}) \to \mathrm{PL}(X^\beth)$ $D \mapsto \varphi_D$ such that $\varphi_D(v) = v(\mathfrak{a})$ for $\mathfrak{a} \in \mathcal{I}$ and $D \in \mathrm{Div}_0(\mathcal{X})$ such that $\mathfrak{a} \cdot \mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{X}(-D)$.
- Induces $\varinjlim_\mathcal{X} \mathrm{Div}_0(\mathcal{X}) \overset{\sim}{\to} \mathrm{PL}(X^\beth)$. Define space of **closed** (p,p)**-forms** $\mathcal{Z}^{p,p}(X^\beth) := \varinjlim_\mathcal{X} \mathrm{H}^{p,p}(\mathcal{X}, \mathbb{R})$.
- Intersection pairing $\mathrm{PL}(X^\beth) \times \mathcal{Z}^{n,n}(X^\beth) \to \mathbb{R} \rightsquigarrow$ linear map $\mathcal{Z}^{n,n}(X^\beth) \to \mathrm{C}^0(X^\beth)^\vee$.
- $\varphi \in \mathrm{PL}(X^\beth) \rightsquigarrow [\omega]_\varphi \in \mathcal{Z}^{1,1}(X^\beth)$, represented by $[\omega]_\mathcal{X} + [D] \in \mathrm{H}^{1,1}(\mathcal{X})$ if $\varphi = \varphi_D$ with $D \in \mathrm{Div}_0(\mathcal{X})$ and $[\omega]_\mathcal{X}$ pullback of $[\omega]$.
- $V^{-1}[\omega]_\varphi^n \in \mathcal{Z}^{n,n}(X^\beth) \rightsquigarrow$ **NA Monge–Ampère measure**

$$\mathrm{MA}^\beth(\varphi) = \mathrm{MA}^\beth_{[\omega]}(\varphi) = \sum_{F \subset \mathcal{X}_0} c_F\, \delta_{v_F}, \quad c_F := b_F F \cdot ([\omega]_\mathcal{X} + [D])^n.$$

NA functionals and K-stability

- NA MA operator admits a primitive $\mathrm{E}^{\beth} = \mathrm{E}^{\beth}_{[\omega]} \colon \mathrm{PL}(X^{\beth}) \to \mathbb{R}$, the **NA MA energy**. For $\varphi = \varphi_D$ with $D \in \mathrm{Div}_0(\mathcal{X})$, $\mathrm{E}^{\beth}(\varphi_D) = \frac{([\omega]_{\mathcal{X}} + [D])^{n+1}}{(n+1)V}$.
- For any $[\theta] \in \mathrm{H}^{1,1}(X, \mathbb{R})$ set $\nabla_{[\theta]} \mathrm{E}^{\beth} := \frac{d}{ds}\big|_{s=0} \mathrm{E}^{\beth}_{[\omega]+s[\theta]}$.
- **Log discrepancy** function $\mathrm{A}_X \colon X^{\mathrm{div}} \to \mathbb{Q}_+$ defined by $\mathrm{A}_X(v_F) := b_F^{-1} \mathrm{A}_{(X \times \mathbb{P}^1, X \times \{0\})}(F)$ (coincides with usual definition in projective case).
- **NA entropy** of $\varphi \in \mathrm{PL}(X^{\beth})$ defined as $\mathrm{H}^{\beth}(\varphi) := \int \mathrm{A}_X \, \mathrm{MA}^{\beth}(\varphi)$, and **NA Mabuchi K-energy** as

$$\mathrm{M}^{\beth}(\varphi) := \mathrm{H}^{\beth}(\varphi) + \nabla_{[-\mathrm{Ric}(\omega)]} \mathrm{E}^{\beth}(\varphi) = \mathrm{H}^{\beth}(\varphi) + \nabla_{K_X} \mathrm{E}^{\beth}(\varphi).$$

- Set of **NA Kähler potentials** $\mathcal{K}^{\beth} := \left\{ \varphi_D \in \mathrm{PL}(X^{\beth}) \mid [\omega]_{\mathcal{X}} + [D] > 0 \text{ on } \mathcal{X}_0 \right\}$.
- For $\varphi \in \mathcal{K}^{\beth}$, $\mathrm{M}^{\beth}(\varphi)$ essentially coincides with the **Donaldson–Futaki invariant**.

Definition

$(X, [\omega])$ is **uniformly K-stable** if $\mathrm{M}^{\beth} \geq \delta \nabla_{[\omega]} \mathrm{E}^{\beth}$ on $\mathcal{K}^{\beth}$ for some $\delta > 0$.

Slopes of functionals

Recall $(\varphi_t)_{t\geq 0}$ psh ray with analytic singularities $\rightsquigarrow D \in \mathrm{Div}_0(\mathcal{X}) \rightsquigarrow \varphi_\infty := \varphi_D \in \mathrm{PL}(X^\beth)$.

Theorem (Phong–Ross–Sturm, B–Hisamoto–Jonsson, Sjöström-Dyrefelt)

For each $\varphi \in \mathcal{K}^\beth$, the following holds:

(i) *there exists a unique psh geodesic ray (φ_t) in $\mathcal{E}^1$ with analytic singularities such that $\varphi_0 = 0$ and $\varphi_\infty = \varphi$;*

(ii) *each of the functionals $F = \mathrm{E}, \nabla_\theta \mathrm{E}, \mathrm{H}, \mathrm{M}$ satisfies $\lim_{t\to+\infty} \frac{1}{t} F(\varphi_t) = F^\beth(\varphi_\infty)$.*

Corollary

$(X, [\omega])$ *uniquely cscK $\Longrightarrow$ uniformly K-stable.*

Proof.

Uniquely cscK $\Longrightarrow \mathrm{M} \geq \delta \nabla_\omega \mathrm{E} - C$ on $\mathcal{E}^1$ for some $\delta, C > 0$. Pick (φ_t) as in (i). Then $\mathrm{M}(\varphi_t) \geq \delta \nabla_\omega \mathrm{E}(\varphi_t) - C \Longrightarrow \mathrm{M}^\beth(\varphi_\infty) \geq \delta \nabla_{[\omega]} \mathrm{E}^\beth(\varphi_\infty)$. □

NA potentials of finite energy

- Denote by $\mathrm{PSH}^{\daleth} = \mathrm{PSH}^{\daleth}([\omega])$ the space of $[\omega]$-**psh functions** $\varphi\colon X^{\daleth} \to \mathbb{R} \cup \{-\infty\}$, i.e. $\varphi = \lim_i \downarrow \varphi_i \in \mathcal{K}^{\daleth}$. Then φ is uniquely determined by restriction to X^{div}, which is finite valued $\rightsquigarrow$ **(weak) topology** of $\mathrm{PSH}^{\daleth}$ = pointwise convergence on X^{div}.
- $\mathrm{E}^{\daleth}\colon \mathcal{K}^{\daleth} \to \mathbb{R}$ nondecreasing $\rightsquigarrow$ uniquely extends to a nondecreasing, usc functional $\mathrm{E}^{\daleth}\colon\ \mathrm{PSH}^{\daleth} \to \mathbb{R} \cup \{-\infty\}$, with $\mathrm{E}^{\daleth}(\varphi) = \lim_i \downarrow \mathrm{E}^{\daleth}(\varphi_i)$.
- Space of **NA finite energy potentials** $\mathcal{E}^{1,\daleth} := \{\varphi \in \mathrm{PSH}^{\daleth} \mid \mathrm{E}^{\daleth}(\varphi) > -\infty\}$. **Strong topology** of $\mathcal{E}^{1,\daleth}$:= coarsest refinement of weak topology such that $\mathrm{E}^{\daleth}\colon \mathcal{E}^{1,\daleth} \to \mathbb{R}$ continuous.
- NA MA operator and $\nabla_{[\theta]} \mathrm{E}^{\daleth}$ continuously extend to $\mathcal{E}^{1,\daleth}$ (B–Favre–Jonsson).
- Denote by $\mathrm{A}_X : X^{\daleth} \to [0, +\infty]$ maximal lsc extension of log discrepancy function. Define $\mathrm{H}^{\daleth}\colon \mathcal{E}^{1,\daleth} \to [0, +\infty]$, $\mathrm{M}^{\daleth}\colon \mathcal{E}^{1,\daleth} \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathrm{H}^{\daleth}(\varphi) := \int_{X^{\daleth}} \mathrm{A}_X \, \mathrm{MA}(\varphi), \quad \mathrm{M}^{\daleth}(\varphi) := \mathrm{H}^{\daleth}(\varphi) + \nabla_{K_X} \mathrm{E}^{\daleth}(\varphi).$$

Maximal geodesic rays

- Each psh ray (φ_t) in PSH of linear growth determines an $[\omega]$-psh function $\varphi_\infty \in \mathrm{PSH}^{\daleth}$. Restriction to X^{div} encodes **Lelong numbers** of the corresponding ω-psh function Φ on $X \times \mathbb{D}^\times$ (Berman–B–Jonsson).
- If (φ_t) psh geodesic ray in $\mathcal{E}^1$, then $\varphi_\infty \in \mathcal{E}^{1,\daleth}$ and $\mathrm{E}^{\daleth}(\varphi_\infty) \geq \lim_{t\to+\infty} \frac{1}{t}\,\mathrm{E}(\varphi_t)$. Say (φ_t) **maximal ray** if equality holds. Then (φ_t) uniquely determined by φ_0 and φ_∞.
- Conversely, any $\varphi \in \mathcal{E}^{1,\daleth}$ determines a unique maximal ray emanating from any given $\varphi_0 \in \mathcal{E}^1 \implies \mathcal{E}^{1,\daleth} \simeq \{$maximal geodesic rays in $\widehat{\mathcal{E}}^1\}$.
- It is a strict closed subset: assume $n = 1$, pick $V \in \mathrm{PSH}(\omega)$ such that $\omega + i\partial\bar\partial V$ is polar measure with no atom. Then $\max\{V, -t\} \rightsquigarrow$ nontrival psh geodesic ray (φ_t) with $\varphi_0 = 0$, $\varphi_\infty = 0$ (Darvas).

Theorem (Berman–B–Jonsson, C. Li)

Assume (φ_t) maximal geodesic ray on $\mathcal{E}^1$. Then:

(i) $\lim_{t\to+\infty} \frac{1}{t}\nabla_\theta\,\mathrm{E}(\varphi_t) = \nabla_{[\theta]}\,\mathrm{E}(\varphi_\infty)$;

(ii) $\lim_{t\to+\infty} \frac{1}{t}\mathrm{H}(\varphi_t) \geq \mathrm{H}^{\daleth}(\varphi_\infty)$.

Slope of entropy and YTD conjecture

Theorem (C. Li)

Any psh geodesic ray (φ_t) *in* $\mathcal{E}^1$ *such that* $\mathrm{H}(\varphi_t) \leq Ct$ *is maximal.*

Corollary

Consider the following properties:

(i) $(X, [\omega])$ **strongly K-stable**, i.e. $\mathrm{M}^{\beth} \geq \delta \nabla_{[\omega]} \mathrm{E}^{\beth}$ *on* $\mathcal{E}^{1,\beth}$ *for some* $\delta > 0$;

(ii) $(X, [\omega])$ *uniquely cscK;*

(iii) $(X, [\omega])$ *uniformly K-stable*, i.e. $\mathrm{M}^{\beth} \geq \delta \nabla_{[\omega]} \mathrm{E}^{\beth}$ *on* $\mathcal{K}^{\beth}$ *for some* $\delta > 0$.

Then (i)⇒(ii)⇒(iii).

Conjecture (Entropy approximation conjecture)

Each $\varphi \in \mathcal{E}^{1,\beth}$ *can be written as* $\varphi = \lim_i \varphi_i \in \mathcal{K}^{\beth}$ *such that* $\mathrm{H}^{\beth}(\varphi_i) \to \mathrm{H}^{\beth}(\varphi)$
$\Leftrightarrow$ $\mathrm{M}^{\beth} \colon \mathcal{E}^{1,\beth} \to \mathbb{R} \cup \{+\infty\}$ **maximal** *lsc extension from* $\mathcal{K}^{\beth}$.

Conjecture implies $(X, [\omega])$ uniquely cscK $\Leftrightarrow$ uniformly K-stable (YTD conjecture).

Sketch of proof of Theorem

- Assume (φ_t) geodesic ray in $\mathcal{E}^1$ with $\mathrm{H}(\varphi_t) \leq Ct \rightsquigarrow \varphi_\infty \in \mathcal{E}^{1,\beth}$.
- Let (ψ_t) be maximal geodesic ray with $\psi_0 = \varphi_0$ and $\psi_\infty = \varphi_\infty$. Then $\psi_t \geq \varphi_t$ and $\mathrm{E}(\psi_t) - \mathrm{E}(\varphi_t) = at$ with $a \geq 0$. Need to show $a = 0$.
- Induced ω-psh functions $\Psi \geq \Phi$ on $X \times \mathbb{D}^\times$ same Lelong numbers. For each $p > 0$, Demailly's multiplier ideal techniques
  $\Rightarrow \int_{X \times \mathbb{D}^\times} e^{p(\Psi - \Phi)} dV |d\tau|^2 < \infty \Leftrightarrow \int_0^\infty e^{-2t} dt \int_X e^{p(\psi_t - \varphi_t)} dV < \infty$.
- Jensen's inequality $\Rightarrow$

$$\mathrm{H}(\varphi_t) = \int_X \log\left(\frac{\mathrm{MA}(\varphi_t)}{dV}\right) \mathrm{MA}(\varphi_t) \geq \int_X p(\psi_t - \varphi_t)\,\mathrm{MA}(\varphi_t) - \log \int_X e^{p(\psi_t - \varphi_t)} dV.$$

- $\int_X (\psi_t - \varphi_t)\,\mathrm{MA}(\varphi_t) \geq \mathrm{E}(\psi_t) - \mathrm{E}(\varphi_t) = at$.
- Since $\mathrm{H}(\varphi_t) \leq Ct$ get $\int_0^\infty e^{(pa - (C+2))t} dt < \infty$ for all $p > 0 \Rightarrow a = 0$.