Lecture 2: Pluripotential theory and the YTD conjecture

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Kähler metrics and curvature

Let X be a compact connected complex manifold, $n = \dim X$.

- In local coordinates, a **Kähler metric** on X is of the form $(\frac{\partial\varphi}{\partial z_j \partial \bar{z}_k})_{1 \leq jk \leq n}$ for a smooth **plurisubharmonic (psh)** function φ.
- Equivalently given by a **Kähler form**, i.e. a closed positive $(1,1)$-form $\omega =_{\mathrm{loc}} i\partial\bar{\partial}\varphi$.
- Induced volume form $\omega^n \leftrightarrow$ metric on **canonical bundle** K_X, locally generated by $dz_1 \wedge \cdots \wedge dz_n$.
- Curvature form of dual bundle = **Ricci curvature** $\mathrm{Ric}(\omega) =_{\text{loc}} -i\partial\bar{\partial} \log \omega^n$.
  De Rham class $[\mathrm{Ric}(\omega)] = -c_1(K_X) = c_1(X) \in H^2(X, \mathbb{R})$.
- **Scalar curvature** $S(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega) =_{\mathrm{loc}} \Delta_\omega \log \omega^n$. Mean value $\bar{S} = n\frac{c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$.
- ω is a
  - **constant scalar curvature Kähler (cscK) metric** if $S(\omega) = \text{cst} = \bar{S}$;
  - **Kähler–Einstein (KE) metric** if $\mathrm{Ric}(\omega) = \lambda\omega$, $\lambda \in \mathbb{R} \iff \omega$ cscK and $c_1(X) = \lambda[\omega]$.
- If $n = 1$: cscK = KE = Poincaré metric.

Kähler potentials and energy functionals

Fix a Kähler form ω on X, of volume $V = \int \omega^n = [\omega]^n$.

- Space of **Kähler potentials** $\mathcal{K} = \mathcal{K}(\omega) := \{\varphi \in C^\infty(X) \mid \omega_\varphi := \omega + i\partial\bar\partial\varphi > 0\}$. $\partial\bar\partial$-Lemma $\Rightarrow \mathcal{K}/\mathbb{R} \simeq \{$Kähler forms in $[\omega]\}$.
- **Monge–Ampère operator** $\varphi \mapsto \mathrm{MA}(\varphi) := V^{-1}\omega_\varphi^n$. KE equation $\Leftrightarrow \mathrm{MA}(\varphi) = e^{-\lambda\varphi}dV$.
- MA operator admits a primitive $\mathrm{E}\colon \mathcal{K} \to \mathbb{R}$, the **Monge–Ampère energy**, i.e. $\frac{d}{dt}\mathrm{E}(\varphi_t) = \int_X \dot\varphi_t \, \mathrm{MA}(\varphi_t)$. Explicitly $\mathrm{E}(\varphi) = \frac{1}{(n+1)V}\sum_{j=0}^n \int \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$.
- $\varphi \mapsto S(\omega_\varphi) =_{\mathrm{loc}} \Delta_{\omega_\varphi} \log \mathrm{MA}(\varphi)$ fourth order differential operator.
- $\varphi \mapsto (\bar S - S(\omega_\varphi))\,\mathrm{MA}(\varphi)$ also admits a primitive $\mathrm{M}\colon \mathcal{K} \to \mathbb{R}$, the **Mabuchi K-energy** $\Rightarrow$ cscK potentials $=$ critical points of M.
- Explicitly (Chen–Tian) $\mathrm{M}(\varphi) = \mathrm{H}(\varphi) + \nabla_{-\mathrm{Ric}(\omega)}\,\mathrm{E}(\varphi) =$ **entropy** $+$ **energy** where
  - $\mathrm{H}(\varphi) := \int \log\left(\frac{\mathrm{MA}(\varphi)}{\mathrm{MA}(0)}\right)\mathrm{MA}(\varphi) \geq 0$;
  - $\nabla_\theta\,\mathrm{E}(\varphi) := \frac{d}{ds}\Big|_{s=0} \mathrm{E}_{\omega+s\theta}(\varphi)$ for any closed $(1,1)$-form.

Completion: potentials of finite energy

- Space of **ω-psh functions** $\mathrm{PSH} = \mathrm{PSH}(\omega) := \left\{\varphi \in L^1 \mid \omega_\varphi = \omega + i\partial\overline{\partial}\varphi \geq 0\right\}$.
  $\varphi \in \mathrm{PSH} \leftrightarrow \varphi = \lim_i \downarrow \varphi_i \in \mathcal{K}$ (Richberg, Demailly, Blocki–Kolodziej).
- $\mathrm{E}\colon \mathcal{K} \to \mathbb{R}$ nondecreasing $\rightsquigarrow$ uniquely extends to a nondecreasing, usc functional $\mathrm{E}\colon\ \mathrm{PSH} \to \mathbb{R} \cup \{-\infty\}$, with $\mathrm{E}(\varphi) = \lim_i \downarrow \mathrm{E}(\varphi_i)$.
- Space of **potentials of finite energy** $\mathcal{E}^1 := \{\varphi \in \mathrm{PSH} \mid \mathrm{E}(\varphi) > -\infty\}$.
  Strong topology of $\mathcal{E}^1$:= coarsest refinement of weak ($=L^1$) topology such that $\mathrm{E}\colon \mathcal{E}^1 \to \mathbb{R}$ continuous. $\mathcal{K}$ **dense** in $\mathcal{E}^1$.
- If $n = 1$: $\mathcal{E}^1 = \mathrm{PSH} \cap L^2_1$, strong topology = Sobolev topology.
- Monge–Ampère operator admits unique continuous extension to $\mathcal{E}^1$ (BBEGZ).

Theorem (Darvas–DiNezza–Lu)

The strong topology of $\mathcal{E}^1$ is defined by a unique metric d_1 such that

- $\varphi \geq \psi \implies \mathrm{d}_1(\varphi, \psi) = \mathrm{E}(\varphi) - \mathrm{E}(\psi)$;
- $\mathrm{d}_1(\varphi, \psi) = \inf\left\{\mathrm{d}_1(\varphi, \tau) + \mathrm{d}_1(\tau, \psi) \mid \tau \in \mathcal{E}^1,\ \tau \leq \varphi, \psi\right\}$;

Furthermore, $(\mathcal{E}^1, \mathrm{d}_1)$ is complete.

Psh geodesics

- A path $(\varphi_t)_{t\in I}$ in $\mathcal{E}^1$ is **psh** if $(t,x)\mapsto\varphi_t(x)$ is ω-psh on $(I+i\mathbb{R})\times X$. Then $t\mapsto \mathrm{E}(\varphi_t)$ convex.
- Each pair $\varphi_0,\varphi_1\in\mathcal{E}^1$ is joined by a (unique) maximal psh path $(\varphi_t)_{t\in[0,1]}$, called a **psh geodesic**. Characterized by $t\mapsto\mathrm{E}(\varphi_t)$ affine linear.
- If $\varphi_0,\varphi_1\in\mathcal{K}$, then $\varphi_t(x)$ is $C^{1,1}$ (X. Chen, Chu–Tosatti–Weinkove), but not C^2 in general (Lempert–Vivas).

Proposition (Berman–Darvas–Lu)

Psh geodesics are metric geodesics in $(\mathcal{E}^1,\mathrm{d}_1)$ with respect to which this space is Busemann convex.

Also holds for the subspace $\mathcal{E}^1_0:=\{\varphi\in\mathcal{E}^1\mid \mathrm{E}(\varphi)=0\}$ of **normalized potentials**.

Example

Pick finite dimensional vector space V, set $(X,L):=(\mathbb{P}(V),\mathcal{O}(1))$. Hermitian norm $\chi\in\mathcal{H}(V)\rightsquigarrow$ Fubini–Study metric $\mathrm{FS}(\chi)$ on $L\Longrightarrow$ isometric embedding $(\mathcal{H},\mathrm{d}_1)\hookrightarrow(\mathcal{E}^1,\mathrm{d}_1)$, takes affine geodesics to psh geodesics.

Proof of Proposition

- Pick psh geodesics $(\varphi_t)_{t\in[0,1]}$ and $(\psi_t)_{t\in[0,1]}$ in $\mathcal{E}^1$. Enough to show

$$\mathrm{d}_1(\varphi_t,\psi_t) \le (1-t)\mathrm{d}_1(\varphi_0,\psi_0) + t\mathrm{d}_1(\varphi_t,\psi_t) \tag{1}$$

  (this implies metric geodesics).
- When $\varphi_t \le \psi_t$, $\mathrm{d}_1(\varphi_t,\psi_t) = \mathrm{E}(\psi_t) - \mathrm{E}(\varphi_t)$ affine linear.
- In general, pick $\tau_0 \le \varphi_0, \psi_0$ and $\tau_1 \le \varphi_1, \psi_1$. $(\tau_t)_{t\in[0,1]}$ psh geodesic joining them. Maximality of (φ_t) and $(\psi_t) \Rightarrow \tau_t \le \varphi_t, \psi_t$ for all t. Thus

$$\mathrm{d}_1(\varphi_t,\psi_t) \le \mathrm{d}_1(\varphi_t,\tau_t) + \mathrm{d}_1(\tau_t,\psi_t)$$

$$= (1-t)\big[\mathrm{d}_1(\varphi_0,\tau_0) + \mathrm{d}_1(\tau_0,\psi_0)\big] + t\big[\mathrm{d}_1(\varphi_1,\tau_1) + \mathrm{d}_1(\tau_1,\psi_1)\big].$$

- Infimum over $\tau_0, \tau_1 \Rightarrow$(1).

The extended K-energy

- For any closed $(1,1)$-form θ, $\nabla_\theta \mathrm{E}$ admits a (unique) continuous to extension to $\mathcal{E}^1$, bounded on each ball.
- For $\varphi \in \mathcal{E}^1$ set $\mathrm{H}(\varphi) := \int \log\left(\frac{\mathrm{MA}(\varphi)}{\mathrm{MA}(0)}\right) \mathrm{MA}(\varphi) \in [0,+\infty]$ if $\mathrm{MA}(\varphi)$ absolutely continuous, and $+\infty$ otherwise. **Entropy functional** $\mathrm{H}\colon \mathcal{E}^1 \to [0,+\infty]$ is
  - **strongly lsc** on $\mathcal{E}^1$, i.e. $\{\mathrm{H} \le C\} \cap B$ is compact for any closed ball B and $C > 0$ (BBEGZ);
  - the **maximal lsc extension** from $\mathcal{K}$, i.e. $\varphi \in \mathcal{E}^1 \Rightarrow \varphi = \lim_i \varphi_i \in \mathcal{K}$ such that $\mathrm{H}(\varphi_i) \to \mathrm{H}(\varphi)$ (Berman–Darvas–Lu).
- Thus $\mathrm{M}(\varphi) := \mathrm{H}(\varphi) + \nabla_{-\mathrm{Ric}(\omega)} \mathrm{E}(\varphi) \rightsquigarrow$ maximal (strongly) lsc extension $\mathrm{M}\colon \mathcal{E}^1 \to \mathbb{R} \cup \{+\infty\}$ of Mabuchi K-energy.

Theorem (Berman–Berndtsson)

$\mathrm{M}\colon \mathcal{E}^1 \to \mathbb{R} \cup \{+\infty\}$ *is convex along psh geodesics.*
Furthermore, $\{$*cscK potentials*$\} = \mathcal{K} \cap \{$*minimizers of* M *in* $\mathcal{E}^1\}$.

Regularity of minimizers

Pick a closed $(1,1)$-form θ.

- **Twisted Ricci and scalar curvature** $\mathrm{Ric}^\theta(\omega) := \mathrm{Ric}(\omega) - \theta$, $S^\theta(\omega) := \mathrm{tr}_\omega \mathrm{Ric}^\theta(\omega)$. ω θ-**twisted cscK metric** if $S^\theta(\omega) = \text{cst}$.
- $\mathcal{C}_\theta := \{\varphi \in \mathcal{K} \mid \omega_\varphi\ \theta\text{-twisted cscK}\}$ = critical points (in $\mathcal{K}$) of θ-**twisted K-energy**

$$\mathrm{M}^\theta := \mathrm{H} + \nabla_{-\mathrm{Ric}^\theta(\omega)} \mathrm{E} = \mathrm{M} + \nabla_\theta \mathrm{E}\,.$$

- $\theta \geq 0 \Rightarrow \nabla_\theta \mathrm{E}$ convex $\Rightarrow \mathrm{M}^\theta$ convex $\Rightarrow \mathcal{C}_\theta = \mathcal{K} \cap \{\text{minimizers of } \mathrm{M}^\theta \text{ in } \mathcal{E}^1\}$. Thus

$$\varphi \in \mathcal{C}_\theta \Longrightarrow \mathrm{M}^\theta(\varphi) \leq \mathrm{M}^\theta(0) = 0 \Longleftrightarrow \mathrm{H}(\varphi) \leq \nabla_{-\mathrm{Ric}^\theta(\omega)} \mathrm{E}(\varphi).$$

So energy bound on φ implies entropy bound.

Theorem (Chen–Cheng)

The following holds:

(i) *entropy bound on $\varphi \in \mathcal{C}_\theta$ and C^∞-bound on $\theta \Longrightarrow C^\infty$-bound on φ;*

(ii) *if $\theta \geq 0$, any minimizer of M^θ lies in $\mathcal{C}_\theta$.*

Sketch of proof of (i)⇒(ii).

- Assume $\psi \in \mathcal{E}^1$ minimizes M^θ. Pick $\alpha = \omega_\varphi$, $\varphi \in \mathcal{K}$.
- $S := \{t > 0 \mid \mathcal{C}_{\theta+t\alpha} \neq \emptyset\}$ open and non-empty (Hashimoto, Zeng).
- $\varphi_t \in \mathcal{C}_{\theta+t\alpha} \Rightarrow$ minimizes $\mathrm{M}^{\theta+t\alpha} = \mathrm{M}^\theta + t\nabla_\alpha \mathrm{E}$
  $\Rightarrow \mathrm{M}^\theta(\varphi_t) + t\nabla_\alpha \mathrm{E}(\varphi_t) \leq \mathrm{M}^\theta(\psi) + t\nabla_\alpha \mathrm{E}(\psi) \leq \mathrm{M}^\theta(\varphi_t) + t\nabla_\alpha \mathrm{E}(\psi)$
  $\Rightarrow \nabla_\alpha \mathrm{E}(\varphi_t) \leq \nabla_\alpha \mathrm{E}(\psi)$.
- $\nabla_\alpha \mathrm{E}$ coercive on $\mathcal{E}^1_0 \Rightarrow$ energy bound for $\varphi_t \Rightarrow C^\infty$-bound for φ_t, by (i).
- Thus $\inf S = 0$ and $\varphi_{t_i} \to \tilde{\varphi} \in \mathcal{C}_\theta$ with $t_i \to 0$. $\nabla_\alpha \mathrm{E}(\tilde{\varphi}) \leq \nabla_\alpha \mathrm{E}(\psi)$.
- Write $\psi = \lim_j \varphi_j \in \mathcal{K}$, set $\alpha_j := \omega_{\varphi_j}$. For each j, get $\tilde{\varphi}_j \in \mathcal{C}_\theta$ with
  $\nabla_{\alpha_j} \mathrm{E}(\tilde{\varphi}_j) \leq \nabla_{\alpha_j} \mathrm{E}(\psi)$.
- $\varphi_j \to \psi \Rightarrow \varphi_j$ bounded in $\mathcal{E}^1 \Rightarrow \nabla_{\alpha_j} \mathrm{E} = \nabla_\omega \mathrm{E} + O(1) \Rightarrow$ energy bound for
  $\tilde{\varphi}_j \Rightarrow C^\infty$-bound for $\tilde{\varphi}_j$, by (i).
- $0 \leq \nabla_{\alpha_j} \mathrm{E}(\tilde{\varphi}_j) - \nabla_{\alpha_j} \mathrm{E}(\varphi_j) \leq \nabla_{\alpha_j} \mathrm{E}(\psi) - \nabla_{\alpha_j} \mathrm{E}(\varphi_j) \to 0$ implies $\mathrm{d}_1(\tilde{\varphi}_j, \varphi_j) \to 0$.
  Hence $\tilde{\varphi}_j \to \psi$ and $\psi \in \mathcal{C}_\theta$.

Analytic Yau–Tian–Donaldson conjecture

We have seen that

- $(\mathcal{E}^1, \mathrm{d}_1)$ Busemann convex with respect to the distinguished class of psh geodesics;
- $\mathrm{M}\colon \mathcal{E}^1 \to \mathbb{R} \cup \{+\infty\}$ strongly lsc, convex on psh geodesics;
- minimizers of M in $\mathcal{E}^1$ correspond to cscK metrics.

Theorem

For any compact Kähler manifold (X, ω), the following are equivalent:

(i) *there exists a* **unique** *cscK metric in $[\omega]$;*

(ii) M *is coercive on $\mathcal{E}^1_0$;*

(iii) $\mathrm{M} \geq \delta \nabla_\omega \mathrm{E} - C$ *on $\mathcal{E}^1$ for some $\varepsilon, C > 0$.*

(iv) $\hat{\varphi} = (\varphi_t)$ *nontrivial psh geodesic ray in $\mathcal{E}^1_0$* $\Rightarrow \widehat{\mathrm{M}}(\hat{\varphi}) := \lim_{t\to+\infty} \frac{1}{t}\mathrm{M}(\varphi_t) > 0.$