Lecture 1: Spaces of norms and GIT

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Growth of convex functions

- If $f\colon \mathbb{R}_+ \to \mathbb{R}$ is convex, then either $f(t) \geq \delta t - C$ with $\delta, C > 0$, or f is nonincreasing.
- Let (X, d) be a complete metric space. A **geodesic** $(x_t)_{t \in I}$ of constant speed $c \geq 0$ satisfies $d(x_s, x_t) = c|s-t|$. It is a **geodesic segment** if $I = [a, b]$, and a **geodesic ray** if $I = \mathbb{R}_+$.
- Suppose (X, d) is a **geodesic space** (i.e. any two points are connected by a geodesic segment). By Hopf–Rinow, X is locally compact iff each closed ball B is compact.

Theorem

Suppose $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is **convex** *on geodesics and* **strongly lsc**, i.e. $B \cap \{f \leq c\}$ *is compact for each closed ball B and $c \in \mathbb{R}$. For any $x_0 \in X$, one of the following holds:*

(i) *either f is* **coercive**, i.e. $f \geq \delta\, d(\cdot, x_0) - C$ *for some $\delta, C > 0$;*

(ii) *or there exists a geodesic ray emanating from $x_0 \in X$ along which f is* **nonincreasing**.

Further, (i) implies that f admits a minimizer, and the converse holds if the minimizer is unique.

Proof

- Assume (i) fails, and pick a sequence (x_j) such that $f(x_j) \leq \delta_j d(x_j, x_0) - C_j$ with $\delta_j \to 0$ and $C_j \to +\infty$.
- (x_j) unbounded, otherwise $f(x_j) \to -\infty$, while f strongly lsc implies bounded below on balls $\implies$ may assume $T_j := d(x_j, x_0) \to +\infty$.
- Pick a unit-speed geodesic $(x_{j,t})_{t\in[0,T_j]}$ joining x_0 to x_j.
- By convexity, $f(x_{j,t}) - f(x_0) \leq \frac{t}{T_j}(f(x_j) - f(x_0)) \leq \delta_j t$
  $\implies (x_{j,t})_{t\in[0,T]}$ stays in compact set $B(x_0, T) \cap \{f \leq C_T\}$
  $\implies$ can assume $(x_{j,t})$ converges on compacts sets to a unit-speed geodesic ray $(x_t)_{t\in[0,+\infty)}$ (Arzelà-Ascoli);
- f lsc $\implies f(x_t) \leq f(x_0)$, hence (ii).
- (i) $\implies$ any minimizing sequence is bounded $\implies$ stays in compact set $B \cap \{f \leq c\} \implies$ limit point is a minimizer.
- If x_0 unique minimizer of f, then (ii) cannot hold, hence (i) does.

Busemann convexity and the asymptotic cone

Assume (X, d) complete geodesic metric space.

- The space (X, d) is **Busemann convex** if d is convex on geodesics, i.e. $t \mapsto d(x_t, y_t)$ is convex for any pair of geodesics (x_t), (y_t).
- This is a **nonpositive curvature** condition: suffices to consider geodesic segments with same origin, and convexity is then a sub-Thales' theorem.
- Busemann convexity $\implies$ uniqueness and continuity of geodesics segments with respect to end-points $\implies$ X contractible.
- The **asymptotic cone** $(\hat{X}, \hat{d})$ is defined as the set $\hat{X}$ of (constant-speed) geodesic rays $\hat{x} = (x_t)$ modulo parallelism (bounded distance), endowed with the metric

$$\hat{d}(\hat{x}, \hat{y}) := \lim_{t \to +\infty} \frac{1}{t} d(x_t, y_t).$$

- $(\hat{X}, \hat{d})$ is also geodesic and complete, and $\hat{X}$ is in 1–1 correspondence with rays emanating from any given base point.

Busemann convexity and the asymptotic cone

- Assume $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is convex and lsc, and pick $x_0 \in X$ with $f(x_0) < \infty$. Then $\hat{f}\colon \hat{X} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\hat{f}(\hat{x}) := \lim_{t \to +\infty} \frac{1}{t} f(x_t)$$

  is independent of x_0, convex, lsc.
- If f is strongly lsc, then theorem yields f coercive $\Leftrightarrow$ $\hat{f}$ coercive.
- All of the above holds when geodesics are restricted to a **distinguished class** $\mathcal{G}$, closed under affine reparametrization and pointwise limits.

Example

Assume (X, d) is normed vector space. Then:

- (X, d) is Busemann convex with respect to class $\mathcal{G}$ of affine geodesics;
- $(\hat{X}, \hat{d}) = (X, d)$;
- $\hat{f}$ is homogenization of f, i.e. smallest convex homogeneous function above f.

The space of norms

Fix a a complex vector space $V \simeq \mathbb{C}^d$.

- Denote by $\mathcal{N} = \mathcal{N}(V)$ the set of all norms $\chi\colon V \to \mathbb{R}_+$. Natural metric

$$\mathrm{d}_\infty(\chi, \chi') := \log \inf\{C \geq 1 \mid C^{-1}\chi \leq \chi' \leq C\chi\} = \sup_{V\setminus\{0\}} |\log \chi - \log \chi'|.$$

- $(\mathcal{N}, \mathrm{d}_\infty)$ is complete and locally compact. Topology = pointwise convergence on V.
- Denote by $\mathcal{H} \subset \mathcal{N}$ the subspace of Hermitian norms. It is closed (parallelogram identity), and hence $(\mathcal{H}, \mathrm{d}_\infty)$ complete and locally compact.
- Choice of $\chi \in \mathcal{H}$ yields canonical isomorphisms

$$\mathcal{H} \simeq \mathrm{Herm}^+(V, \chi) \quad \mathcal{H} \simeq \mathrm{GL}(V)/\mathrm{U}(V, \chi).$$

- Any $\chi \in \mathcal{H}$ can be diagonalized in some basis $\mathbf{e} = (e_i)$ of V, i.e.

$$\chi(v)^2 = \sum_i |a_i|^2 \chi(e_i)^2, \quad v = \sum_i a_i e_i.$$

The space of norms

- For any basis $\mathbf{e}$ of V, define the **apartment** $\mathbb{A}_{\mathbf{e}} \subset \mathcal{H}$ as the set of Hermitian norms diagonalized in $\mathbf{e}$.
- It is closed in $\mathcal{H}$, and $\chi \mapsto (\log \chi(e_i))_i$ yields isometry $(\mathbb{A}_{\mathbf{e}}, \mathrm{d}_\infty) \xrightarrow{\sim} (\mathbb{R}^d, \ell^\infty)$.
- Affine paths in apartments form a distinguished class $\mathcal{G}$ of geodesics in $(\mathcal{H}, \mathrm{d}_\infty)$.
- Any pair of points in $\mathcal{H}$ lies in some common apartment $\mathbb{A}_{\mathbf{e}}$, affine segment between them is independent of $\mathbf{e}$.
- $(\mathcal{H}, \mathrm{d}_\infty)$ is Busemann convex with respect to $\mathcal{G}$. More generally:

Theorem (Bhatia, Darvas–Lu–Rubinstein, B–Eriksson)

For any $p \in [1, \infty]$, there exists a unique metric d_p on $\mathcal{H}$ inducing $(\mathbb{A}_{\mathbf{e}}, \mathrm{d}_p) \simeq (\mathbb{R}^d, \ell^p)$ for all $\mathbf{e}$. Furthermore, $(\mathcal{H}, \mathrm{d}_p)$ is Busemann convex with respect to $\mathcal{G}$.

For $p = 2$: $(\mathcal{H}, \mathrm{d}_2) \simeq \mathrm{GL}(V)/\mathrm{U}(V)$ as Riemannian symmetric space.

Non-Archimedean norms

- Denote by $|\cdot|_0$ the *trivial absolute value* on $\mathbb{C}$. Note $|\cdot|_0 = \lim_{\varepsilon\to 0_+} |\cdot|^\varepsilon$.
- A **non-Archimedean norm** is a function $\chi\colon V \to \mathbb{R}_{\geq 0}$ such that
  - $\chi(v+w) \leq \max\{\chi(v), \chi(w)\}$;
  - $\chi(av) = |a|_0\chi(v)$;
  - $\chi(v) = 0$ iff $v = 0$.

  Balls are linear subspaces, and data of χ is equivalent to the filtration $\{\chi \leq R\}_{R\geq 0}$.
- If $\chi = \lim_i \chi_i^{\varepsilon_i}$ pointwise with $\chi_i \in \mathcal{N}(V)$ and $\varepsilon_i \to 0_+$, then χ is a non-Archimedean (semi)norm.
- Denote by $\mathcal{N}^{\mathrm{NA}}$ the set of all non-Archimedean norms χ on V. It is a cone with respect to the scaling action $\mathbb{R}_{>0} \times \mathcal{N}^{\mathrm{NA}} \to \mathcal{N}^{\mathrm{NA}} \quad (\varepsilon, \chi) \mapsto \chi^\varepsilon$, with apex the trivial norm.
- Can define $\mathrm{d}_\infty^{\mathrm{NA}}$ as before. Then $(\mathcal{N}^{\mathrm{NA}}, \mathrm{d}_\infty^{\mathrm{NA}})$ complete, but not locally compact when $d = \dim V > 1$.

Non-Archimedean norms as an asymptotic cone

- any $\chi \in \mathcal{N}^{\mathrm{NA}}$ can be **diagonalized** in some basis $\mathbf{e} = (e_i)$ of V, i.e. $\chi(v) = \max_i \chi(a_i e_i) = \max_{a_i \neq 0} \chi(e_i)$ for $v = \sum_i a_i e_i$ ($\Leftrightarrow$ compatible basis for the corresponding filtration).
- **apartement** $(\mathbb{A}_{\mathbf{e}}^{\mathrm{NA}}, \mathrm{d}_\infty^{\mathrm{NA}}) \simeq (\mathbb{R}^d, \ell^\infty)$.
- for any $p \in [1, \infty]$, there exists a unique metric $\mathrm{d}_p^{\mathrm{NA}}$ on $\mathcal{N}^{\mathrm{NA}}$ such that $(\mathbb{A}_{\mathbf{e}}^{\mathrm{NA}}, \mathrm{d}_p^{\mathrm{NA}}) \simeq (\mathbb{R}^d, \ell^p)$ for all $\mathbf{e}$.
- $p = 2$, $(\mathcal{N}^{\mathrm{NA}}, \mathrm{d}_2)$ Euclidean building.

Theorem

For any affine geodesic ray $\hat{\chi} = (\chi_t)$ in $\mathcal{H}$, $\chi_\infty := \lim_{t\to\infty} \chi_t^{1/t}$ defines a non-Archimedean norm. Further:

- *$\hat{\chi}$ is uniquely determined by χ_0 and χ_∞;*
- *$\hat{\chi} \mapsto \chi_\infty$ induces $(\hat{\mathcal{H}}, \hat{\mathrm{d}}_p) \xrightarrow{\sim} (\mathcal{N}^{\mathrm{NA}}, \mathrm{d}_p^{\mathrm{NA}})$.*

The Hilbert–Mumford criterion

Consider a finite dimensional representation W of $G := \mathrm{GL}(V)$, with maximal compact subgroup $K = \mathrm{U}(V)$.

- A (nonzero) $w \in W$ is **stable** if its G-orbit is closed, with finite stabilizer.
- This holds iff $f\colon G \to \mathbb{R}$ defined by $f(g) := \log \|g \cdot w\|$ is proper, for any choice of norm $\|\cdot\|$ on W.
- Choose a K-invariant norm $\Longrightarrow$ induced function $f\colon G/K = \mathcal{H} \to \mathbb{R}$ is **convex** $\Longrightarrow$ w stable iff $\hat{f} > 0$ on each nontrivial ray in $\hat{\mathcal{H}} \simeq \mathcal{H}^{\mathrm{NA}}$.
- Basis $\mathbf{e}$ of V $\Leftrightarrow$ maximal torus $T \simeq (\mathbb{C}^\times)^d \subset G \Rightarrow$ weight decomposition $W = \bigoplus_{\alpha \in \mathbb{Z}^d} W_\alpha$.
- Write $w = \sum_\alpha w_\alpha$. Then $f(\lambda) = \log \|\sum_\alpha e^{\langle \alpha, \lambda \rangle} w_\alpha\|$ on $\mathbb{A}_{\mathbf{e}} \simeq \mathbb{R}^d$, hence $\hat{f}(\lambda) = \max_{w_\alpha \neq 0} \langle \alpha, \lambda \rangle$ on $\hat{\mathbb{A}}_{\mathbf{e}} \simeq \mathbb{R}^d$.
- Conclusion: w is stable iff $\hat{f} > 0$ on $\hat{\mathbb{A}}_{\mathbf{e}}(\mathbb{Z}) \simeq \mathbb{Z}^d$ for all $\mathbf{e}$ (Hilbert–Mumford criterion).