Lecture 1: Spaces of norms and GIT

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Growth of convex functions

- If \( f : \mathbb{R}_+ \to \mathbb{R} \) is convex, then either \( f(t) \geq \delta t - C \) with \( \delta, C > 0 \), or \( f \) is nonincreasing.
- Let \((X, d)\) be a complete metric space. A geodesic \((x_t)_{t \in I}\) of constant speed \( c \geq 0 \) satisfies \( d(x_s, x_t) = c|s - t| \). It is a geodesic segment if \( I = [a, b] \), and a geodesic ray if \( I = \mathbb{R}_+ \).
- Suppose \((X, d)\) is a geodesic space (i.e. any two points are connected by a geodesic segment). By Hopf–Rinow, \( X \) is locally compact iff each closed ball \( B \) is compact.

Theorem

Suppose \( f : X \to \mathbb{R} \cup \{+\infty\} \) is convex on geodesics and strongly lsc, i.e. \( B \cap \{f \leq c\} \) is compact for each closed ball \( B \) and \( c \in \mathbb{R} \). For any \( x_0 \in X \), one of the following holds:

(i) either \( f \) is coercive, i.e. \( f \geq \delta d(\cdot, x_0) - C \) for some \( \delta, C > 0 \);
(ii) or there exists a geodesic ray emanating from \( x_0 \in X \) along which \( f \) is nonincreasing.

Further, (i) implies that \( f \) admits a minimizer, and the converse holds if the minimizer is unique.
Proof

- Assume (i) fails, and pick a sequence \((x_j)\) such that \(f(x_j) \leq \delta_j d(x_j, x_0) - C_j\) with \(\delta_j \to 0\) and \(C_j \to +\infty\).
- \((x_j)\) unbounded, otherwise \(f(x_j) \to -\infty\), while \(f\) strongly lsc implies bounded below on balls \(\implies\) may assume \(T_j := d(x_j, x_0) \to +\infty\).
- Pick a unit-speed geodesic \((x_{j,t})_{t \in [0, T_j]}\) joining \(x_0\) to \(x_j\).
- By convexity, \(f(x_{j,t}) - f(x_0) \leq \frac{t}{T_j} (f(x_j) - f(x_0)) \leq \delta_j t\)
  \(\implies (x_{j,t})_{t \in [0, T]}\) stays in compact set \(B(x_0, T) \cap \{ f \leq C_T\}\)
  \(\implies\) can assume \((x_{j,t})\) converges on compacts sets to a unit-speed geodesic ray \((x_t)_{t \in [0, +\infty)}\) (Arzelà-Ascoli);
- \(f\) lsc \(\implies f(x_t) \leq f(x_0)\), hence (ii).
- (i) \(\implies\) any minimizing sequence is bounded \(\implies\) stays in compact set \(B \cap \{ f \leq c\}\) \(\implies\) limit point is a minimizer.
- If \(x_0\) unique minimizer of \(f\), then (ii) cannot hold, hence (i) does.
Busemann convexity and the asymptotic cone

Assume \((X, d)\) complete geodesic metric space.

- The space \((X, d)\) is **Busemann convex** if \(d\) is convex on geodesics, i.e. \(t \mapsto d(x_t, y_t)\) is convex for any pair of geodesics \((x_t), (y_t)\).

- This is a **nonpositive curvature** condition: suffices to consider geodesic segments with same origin, and convexity is then a sub-Thales’ theorem.

- Busemann convexity \(\implies\) uniqueness and continuity of geodesics segments with respect to end-points \(\implies\) \(X\) contractible.

- The **asymptotic cone** \((\hat{X}, \hat{d})\) is defined as the set \(\hat{X}\) of (constant-speed) geodesic rays \(\hat{x} = (x_t)\) modulo parallelism (bounded distance), endowed with the metric

\[
\hat{d}(\hat{x}, \hat{y}) := \lim_{t \to +\infty} \frac{1}{t} d(x_t, y_t).
\]

- \((\hat{X}, \hat{d})\) is also geodesic and complete, and \(\hat{X}\) is in 1–1 correspondence with rays emanating from any given base point.
Busemann convexity and the asymptotic cone

- Assume \( f : X \to \mathbb{R} \cup \{ +\infty \} \) is convex and lsc, and pick \( x_0 \in X \) with \( f(x_0) < \infty \). Then \( \hat{f} : \hat{X} \to \mathbb{R} \cup \{ +\infty \} \) defined by
  \[
  \hat{f}(\hat{x}) := \lim_{t \to +\infty} \frac{1}{t} f(x_t)
  \]
  is independent of \( x_0 \), convex, lsc.
- If \( f \) is strongly lsc, then theorem yields \( f \) coercive \( \iff \hat{f} \) coercive.
- All of the above holds when geodesics are restricted to a distinguished class \( \mathcal{G} \), closed under affine reparametrization and pointwise limits.

**Example**

Assume \((X, d)\) is normed vector space. Then:
- \((X, d)\) is Busemann convex with respect to class \( \mathcal{G} \) of affine geodesics;
- \((\hat{X}, \hat{d}) = (X, d)\);
- \( \hat{f} \) is homogenization of \( f \), i.e. smallest convex homogeneous function above \( f \).
The space of norms

Fix a complex vector space $V \simeq \mathbb{C}^d$.

- Denote by $\mathcal{N} = \mathcal{N}(V)$ the set of all norms $\chi: V \to \mathbb{R}_+$. Natural metric

  $$d_{\infty}(\chi, \chi') := \log \inf\{C \geq 1 \mid C^{-1}\chi \leq \chi' \leq C\chi\} = \sup_{V\setminus\{0\}} |\log \chi - \log \chi'|.$$

- $(\mathcal{N}, d_{\infty})$ is complete and locally compact. Topology = pointwise convergence on $V$.

- Denote by $\mathcal{H} \subset \mathcal{N}$ the subspace of Hermitian norms. It is closed (parallelogram identity), and hence $(\mathcal{H}, d_{\infty})$ complete and locally compact.

- Choice of $\chi \in \mathcal{H}$ yields canonical isomorphisms

  $$\mathcal{H} \simeq \text{Herm}^+(V, \chi) \quad \mathcal{H} \simeq \text{GL}(V)/\text{U}(V, \chi).$$

- Any $\chi \in \mathcal{H}$ can be diagonalized in some basis $e = (e_i)$ of $V$, i.e.

  $$\chi(v)^2 = \sum_i |a_i|^2 \chi(e_i)^2, \quad v = \sum_i a_i e_i.$$
The space of norms

- For any basis $e$ of $V$, define the **apartment** $\mathbb{A}_e \subset \mathcal{H}$ as the set of Hermitian norms diagonalized in $e$.
- It is closed in $\mathcal{H}$, and $\chi \mapsto (\log \chi(e_i))_i$ yields isometry $(\mathbb{A}_e, d_\infty) \cong (\mathbb{R}^d, \ell^\infty)$.
- Affine paths in apartments form a distinguished class $\mathcal{G}$ of geodesics in $(\mathcal{H}, d_\infty)$.
- Any pair of points in $\mathcal{H}$ lies in some common apartment $\mathbb{A}_e$, affine segment between them is independent of $e$.
- $(\mathcal{H}, d_\infty)$ is Busemann convex with respect to $\mathcal{G}$. More generally:

**Theorem (Bhatia, Darvas–Lu–Rubinstein, B–Eriksson)**

For any $p \in [1, \infty]$, there exists a unique metric $d_p$ on $\mathcal{H}$ inducing $(\mathbb{A}_e, d_p) \cong (\mathbb{R}^d, \ell^p)$ for all $e$. Furthermore, $(\mathcal{H}, d_p)$ is Busemann convex with respect to $\mathcal{G}$.

For $p = 2$: $(\mathcal{H}, d_2) \cong \text{GL}(V)/\text{U}(V)$ as Riemannian symmetric space.
Non-Archimedean norms

- Denote by $| \cdot |_0$ the *trivial absolute value* on $\mathbb{C}$. Note $| \cdot |_0 = \lim_{\varepsilon \to 0^+} | \cdot |^\varepsilon$.

- A **non-Archimedean norm** is a function $\chi : V \to \mathbb{R}_{\geq 0}$ such that
  \begin{itemize}
  \item $\chi(v + w) \leq \max\{\chi(v), \chi(w)\}$;
  \item $\chi(av) = |a|_0 \chi(v)$;
  \item $\chi(v) = 0$ iff $v = 0$.
  \end{itemize}

  Balls are linear subspaces, and data of $\chi$ is equivalent to the filtration $\{\chi \leq R\}_{R \geq 0}$.

- If $\chi = \lim_i \chi_i^{\varepsilon_i}$ pointwise with $\chi_i \in \mathcal{N}(V)$ and $\varepsilon_i \to 0^+$, then $\chi$ is a non-Archimedean (semi)norm.

- Denote by $\mathcal{N}^{NA}$ the set of all non-Archimedean norms $\chi$ on $V$. It is a cone with respect to the scaling action $\mathbb{R}_{>0} \times \mathcal{N}^{NA} \to \mathcal{N}^{NA}$ $(\varepsilon, \chi) \mapsto \chi^\varepsilon$, with apex the trivial norm.

- Can define $d_{\infty}^{NA}$ as before. Then $(\mathcal{N}^{NA}, d_{\infty}^{NA})$ complete, but not locally compact when $d = \dim V > 1$. 
Non-Archimedean norms as an asymptotic cone

- any $\chi \in \mathcal{N}^{\text{NA}}$ can be **diagonalized** in some basis $e = (e_i)$ of $V$, i.e.
  \[ \chi(v) = \max_i \chi(a_i e_i) = \max_{a_i \neq 0} \chi(e_i) \text{ for } v = \sum_i a_i e_i \quad (\Leftrightarrow \text{compatible basis for the corresponding filtration}). \]

- **apartement** $(\mathbb{A}^{\text{NA}}_e, d^{\text{NA}}_\infty) \simeq (\mathbb{R}^d, \ell^\infty)$.

- for any $p \in [1, \infty]$, there exists a unique metric $d^{\text{NA}}_p$ on $\mathcal{N}^{\text{NA}}$ such that $(\mathbb{A}^{\text{NA}}_e, d^{\text{NA}}_p) \simeq (\mathbb{R}^d, \ell^p)$ for all $e$.

- $p = 2$, $(\mathcal{N}^{\text{NA}}, d_2)$ Euclidean building.

**Theorem**

For any affine geodesic ray $\hat{\chi} = (\chi_t)$ in $\mathcal{H}$, $\chi_\infty := \lim_{t \to \infty} \chi_t^{1/t}$ defines a non-Archimedean norm. Further:

- $\hat{\chi}$ is uniquely determined by $\chi_0$ and $\chi_\infty$;

- $\hat{\chi} \mapsto \chi_\infty$ induces $(\hat{\mathcal{H}}, \hat{d}_p) \simeq (\mathcal{N}^{\text{NA}}, d^{\text{NA}}_p)$. 
The Hilbert–Mumford criterion

Consider a finite dimensional representation $W$ of $G := \text{GL}(V)$, with maximal compact subgroup $K = \text{U}(V)$.

- A (nonzero) $w \in W$ is **stable** if its $G$-orbit is closed, with finite stabilizer.
- This holds iff $f : G \to \mathbb{R}$ defined by $f(g) := \log \|g \cdot w\|$ is proper, for any choice of norm $\| \cdot \|$ on $W$.
- Choose a $K$-invariant norm $\Rightarrow$ induced function $f : G/K = \mathcal{H} \to \mathbb{R}$ is **convex** $\Rightarrow$ $w$ stable iff $\hat{f} > 0$ on each nontrivial ray in $\hat{\mathcal{H}} \simeq \mathcal{H}^{\text{NA}}$.
- Basis $e$ of $V \Leftrightarrow$ maximal torus $T \simeq (\mathbb{C}^\times)^d \subset G$ $\Rightarrow$ weight decomposition $W = \bigoplus_{\alpha \in \mathbb{Z}^d} W_\alpha$.
- Write $w = \sum_{\alpha} w_\alpha$. Then $f(\lambda) = \log \| \sum_{\alpha} e^{\langle \alpha, \lambda \rangle} w_\alpha \|$ on $\mathbb{A}_e \simeq \mathbb{R}^d$, hence $\hat{f}(\lambda) = \max_{w_\alpha \neq 0} \langle \alpha, \lambda \rangle$ on $\hat{\mathbb{A}}_e \simeq \mathbb{R}^d$.
- Conclusion: $w$ is stable iff $\hat{f} > 0$ on $\hat{\mathbb{A}}_e(\mathbb{Z}) \simeq \mathbb{Z}^d$ for all $e$ (Hilbert–Mumford criterion).