14/12/20

Essential Skeleton

I. Context

S : complex curve $\qquad$ $\begin{array}{c} X \\ \downarrow \\ S\setminus\{0\} \end{array}$ $\qquad$ Projective family

$0 \in S$ point $\qquad\qquad$ X_t smooth $t \neq 0$

Model: $\begin{array}{c} \mathcal{X} \\ \downarrow \\ S \end{array}$ $\qquad$ $\mathcal{X}$ smooth $\qquad$ $\mathcal{X}_0 = \sum_{i \in I} b_i E_i$

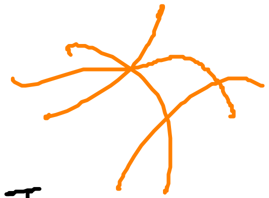
$\mathcal{X}_0$ SNC

Assumption $\quad$ X is CY. $\quad$ ω_X trivial $\qquad$ $\Omega \in H^0(X, \omega_X)$

$\Omega = dt \wedge \Omega_t \qquad \Omega_t \in H^0(X_t, \omega_{X_t})$

Question $\quad$ Describe the limit $\lim\limits_{t \to 0} d\mu_t$ $\qquad$ $d\mu_t = \Omega_t \wedge \overline{\Omega_t} \Big/ \int_{X_t} \Omega_t \wedge \overline{\Omega_t}$

$$\mathcal{X}_0 = \sum_I b_i E_i$$

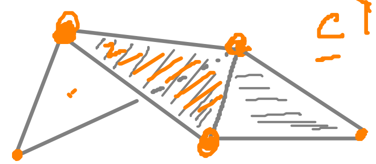


Dual complex

$\Sigma(\mathcal{X})$ with vertices I — Connected

edges $\{i,j\}$ $\quad E_i \cap E_j \neq \emptyset$

triangles $\{i,j,k\}$ $\quad E_i \cap E_j \cap E_k \neq \emptyset$



$\subseteq \mathbb{R}_+^I$

$J \subseteq I$ $\qquad \sigma_J = \left\{ (x_j)_{j\in J} \;\middle|\; \begin{array}{c} x_j \geq 0 \\ \sum x_j b_j = 1 \end{array} \right\} \subseteq \mathbb{R}_+^J$

$J = \{j\}$ $\qquad \sigma_j = \{ 1/b_j \, e_j \}$ $\qquad e_j = (0,0,\dots,0,1,0,\dots,0) \in \mathbb{R}_+^I$

$K_{\mathcal{X}} = \mathrm{div}(\Omega)$ Canonical divisor

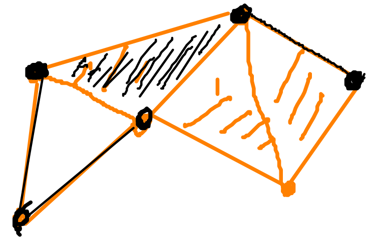
$$K_{\mathcal{X}} = \sum_{i\in I} (a_i + b_i - 1) E_i$$

$$K^{\log}_{\mathcal{X}/S} := K_{\mathcal{X}} - K_S + \mathcal{X}_{0,red} - \mathcal{X}_0 = \sum a_i E_i$$

Changing $\Omega \rightsquigarrow t^{-\ell}\Omega \Rightarrow K^{\log}_{\mathcal{X}/S} \rightsquigarrow \sum (a_i - \ell b_i) E_i$

Arrange to have $a_i - \ell b_i \geq 0$ with min achieved

$$\ell = \min\{ a_i / b_i \}$$



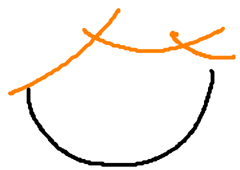
Kontsevich-Soibelman

$Sk_{KS}(\mathcal{X}) :=$ subcomplex of $\Sigma(\mathcal{X})$ formed by vertices j with $a_j/b_j = \min_{i \in I}\{a_i/b_i\}$

Thm (Boucksom-Jonsson)
2017

$$\lim_{t \to 0} (X_t, d\mu_t) = (Sk_{KS}(\mathcal{X}), d\mu_0).$$

Plan More conceptual definition and properties of $Sk(\mathcal{X})$



Ref

- Kontsevich-Soibelman
- Mustaţă-Nicaise
- Nicaise-Xu
- Temkin
- Baker-Nicaise

Mauri-Mazzon-Stevenson

II Reminder on Berkovich spaces

$K = \mathbb{C}((t))$

$R = \mathbb{C}[[t]]$ valuation ring

X/K smooth proj variety

$X^{an} := \{$ valuations on points of X which extend that of K $\}$

$x \in X^{an}$ corresponds thus to a pair (η_Y, ν_x)

$Y \subset X$

η_Y generic pt of Y

$= \operatorname{Spec} K(Y)$

$$\begin{tikzcd} \nu_x : K(Y)^\times \arrow[r] & \mathbb{R} \\ K^\times \arrow[u, hook] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

$\mathcal{H}(x)$: Completion of $(K(Y), |\cdot|_x = e^{-\nu_x(\cdot)})$

$\mathcal{H}(x)^0$: valuation ring

$X^{\mathrm{bir,an}} := \{$ valuations on $K(X)$ which extend that of K $\}$.

$\nu_E \in$

$$X^{\mathrm{bir,an}} \subset X^{\mathrm{an}}$$

$$\mathcal{X}_0 = \sum_i b_i E_i$$

Divisorial Points

$$\begin{array}{ccccc} E \hookrightarrow & \mathcal{X}_0 & \subset & \mathcal{X} \\ & \downarrow & & \downarrow \\ & 0 & \in & \operatorname{Spec} R \end{array}$$

$\mathcal{X}$ a model of X → 

$R = \mathbb{C}[[t]]$

E irreducible component of $\mathcal{X}_0$

η_E generic point of E

$\mathcal{O}_{\mathcal{X},\eta_E}$ is a discrete valuation ring

$\nu_{\eta_E} = \nu_E$ corresponding valuation

b mult. of E in $\mathcal{X}_0$

$\nu_E(t) = 1/b$

$X^{an}_{div} \subseteq X^{bir,an}$ the set of all divisorial pts. $\sim E$

$$E \hookrightarrow \mathcal{X} \to \operatorname{Spec} R \qquad X/K$$

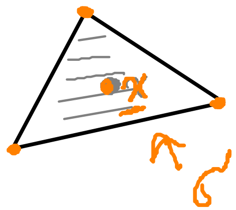
Monomial points

$\mathcal{X} \to \operatorname{Spec} R$ a model of X

$$\mathcal{X}_0 = \sum_{i \in I} b_i E_i$$

$J \subseteq I$ with $\bigcap_{j \in J} E_j \neq \emptyset$ and connected.

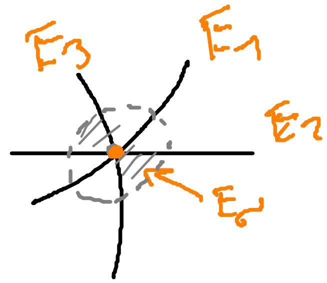
$\Sigma(\mathcal{X})$

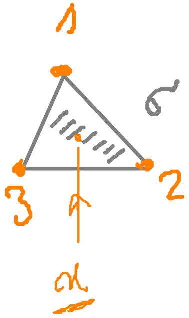


$$\sigma = \sigma_J = \left\{ (x_j) \in \mathbb{R}_+^J \;\middle|\; \sum_{j \in J} b_j x_j = 1 \right\}$$

$\underline{x} \in \sigma$ defines a valuation $v_{\underline{x}}$ on $K(X)$ as follows.

Let η_σ be the generic point of $E_\sigma = \bigcap_{j \in J} E_j$





Consider the local ring $\mathcal{O}_{X,\eta_\sigma}$

local parameters around η_σ $\quad z_j \,,\quad j \in J$

with $\prod_{j \in J} z_j^{b_j} = u\, t$

invertible

$f \in K(X)^\times$

$f \in \mathcal{O}_{X,\eta_\sigma}$ " $f = \sum_{\beta \in \mathbb{Z}_+^J} c_\beta\, z^\beta$ " $= \prod z_j^{\beta_j}$ with c_β invertible

" $(\beta_j)_{j \in J}$

admissible expansions

$f \in \mathcal{O}_{X,\eta_\sigma} \rightsquigarrow A_f \subseteq \mathbb{Z}_+^J$

Prop ① admissible expansion exists.

② $A_f = \min_{\leq_{cw}} \{\beta \mid c_\beta \neq 0\}$ is a well-defined finite set in $\mathbb{Z}_+^J$.

in(f) $\quad$ coordinatewise

$\sum_{\beta \in \mathbb{Z}_+^d} c_\beta x^\beta$

Def (Tropicalization) ① $f \in \mathcal{O}_{\mathfrak{X}, \eta_\sigma}$, then

$\mathrm{trop}(f) : \sigma \longrightarrow \mathbb{R}$

is defined by $\mathrm{trop}(f)(\underline{x}) = \min_{\beta \in A_f} \langle \underline{x}, \beta \rangle$

$A_f \subseteq \mathbb{Z}_+^d$, $\langle \underline{x}, \beta \rangle = \sum x_j \beta_j$

E_1, E_2, E_3

② more generally, $f = g/h$, and set $\mathrm{trop}(f) := \mathrm{trop}(g) - \mathrm{trop}(h)$.

$f \in K(X) \longrightarrow \mathrm{trop}(f)$

Def $\mathfrak{X} \to \operatorname{Spec} R$, $\mathfrak{X}_0 = \sum_I b_i E_i$, $J \subseteq I$, $\sigma = \sigma_J$, $\underline{x} \in \sigma$, $v_{\underline{x}} : K(X)^\times \to \mathbb{R}$

$v_{\underline{x}}(f) := \mathrm{trop}(f)(\underline{x})$.

is a valuation.

monomial

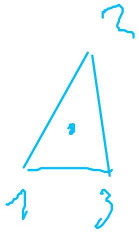
$\underline{x} = (x_j)$

$$X^{\mathrm{bir,an}}_{\mathrm{mon}} := \left\{ \nu_{\underline{x}} \;\middle|\; \underline{x} \in \sigma_J \quad \begin{array}{l} \text{for } J \subseteq I \\ \text{for some model } \begin{array}{c}\mathcal{X} \\ \downarrow \\ \operatorname{Spec} R\end{array} \\ \text{with } \mathcal{X}_0 = \sum_{i \in I} b_i E_i \end{array} \right\}.$$

$X^{\mathrm{bir,an}}_{\mathrm{div}}$

Prop $\nu_{\underline{x}}$ is divisorial iff $\underline{x}$ has rational coordinates in σ.

Proof $\underline{x} = (x_1, \ldots, x_r)$ with $x_j \geq 0$ $\quad \sum b_j x_j = 1$.



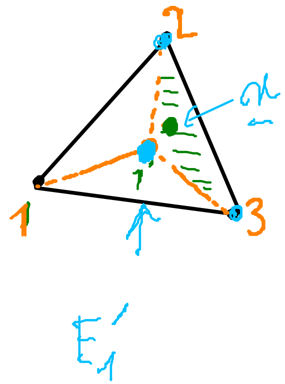
reduce to the case $x_j > 0$.

$Bl_{E_\sigma}(\mathcal{X})$

$x_1 \leq x_2, \ldots, x_r$

$(x_1, \ldots, x_r)$

$\underline{x} = (x_1, x_2 - x_1, \ldots, x_j - x_1)$



$\Sigma(\mathcal{X})$

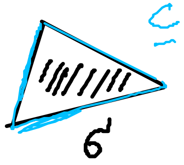
Def $\quad Sk(\mathcal{X}) := \{ \vartheta_x \mid x \in \sigma_J \text{ for } J \subseteq I \}$

skeleton

Prop • $Sk(\mathcal{X}) \hookrightarrow X^{\text{bir, an}}$ $\qquad f \quad \text{trop}(f)$

• $Sk(\mathcal{X}) \simeq_{\text{hom}} \Sigma(\mathcal{X})$

• $Sk(\mathcal{X})$ has a natural affine structure given by the sheaf $\mathcal{O}_{Sk(\mathcal{X})}$ of "tropicalization of rational functions".

Rk.  $\subseteq Sk(\mathcal{X})$

$\text{trop}(f) : \sigma \to \mathbb{R}$ is

• concave if $\eta_\sigma \notin \overline{\text{Poles of } f}$

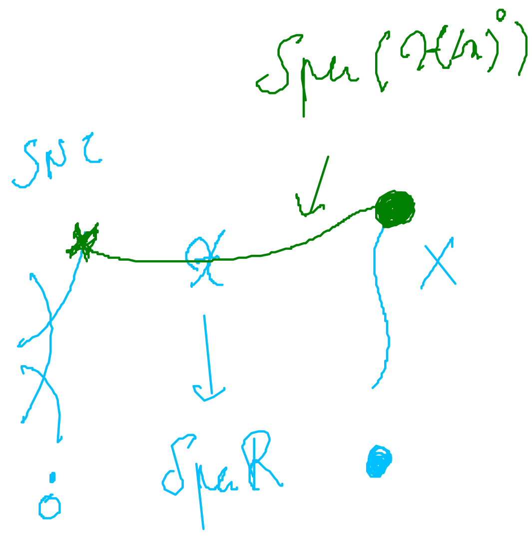
• convexe if $\eta_\sigma \notin \overline{\text{zeros of } f}$

• affine if $\eta_\sigma \notin \overline{\text{Poles + zeros of } f}$

η_σ generic pt of $\bigcap_{j \in \sigma} E_j$

Retraction

$$r_{\mathfrak{X}} : \underbrace{X^{an}}_{x\in} \longrightarrow Sk(\mathfrak{X}).$$



(1) Reduction map $\quad red_{\mathfrak{X}} : X^{an} \longrightarrow \mathfrak{X}_0$

$v_x : K(Y)^* \longrightarrow \mathbb{R}$

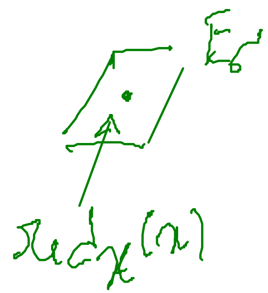
$K(Y) \subset \mathcal{H}(x)$

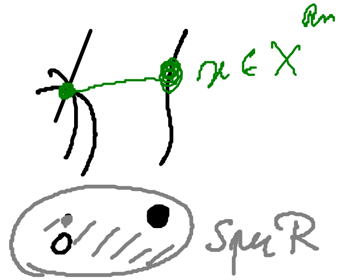
$x \in X^{an}$

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{H}(x)) \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\text{Proper}"] \\
\mathrm{Spec}(\mathcal{H}(x)^{\circ}) \arrow[r] \arrow[ur] & \mathrm{Spec}\, R
\end{tikzcd}$$

Valuative criterion

$\rightsquigarrow \quad red_{\mathfrak{X}}(x) = \mathrm{im}(\text{closed pt of } \mathrm{Spec}(\mathcal{H}(x)^{\circ}))$ in $\mathfrak{X}_0$.



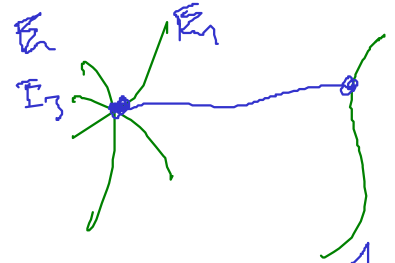


(2) Let $J \subseteq I$ all j with $red(x) \in E_j^{o}$

Let $w_{E_j} = \vartheta_x(E_j)$

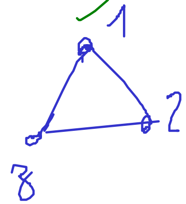
$x \in X^{an}$

ϑ_x corresponding valuation



$\Rightarrow \quad (w_{E_j})_{j\in J} \in Sk(\mathcal{X})$

and we define $\quad r_{\mathcal{X}}(x) = (w_{E_j})_{j \in J}.$

$r_{\mathcal{X}}(x) \in$ 

Prop (Retraction inequality) $\quad x \in X^{an} \quad \vartheta_x : K(X)^\times \to \mathbb{R}$

Let $f \in \mathcal{O}_{\mathcal{X}, \mathrm{red}(x)}$.

Then we have $\quad \boxed{\vartheta_x(f) \geq \vartheta_{r_{\mathcal{X}}(x)}(f).}$

Proof $\quad f = \sum_{\beta \in A_f} c_\beta z^\beta \Rightarrow \vartheta_x(f) \geq \min\{\langle \beta, \vartheta_x(z^\beta)\rangle\} = \vartheta_{r_{\mathcal{X}}(x)}(f).$

Thm $X^{an} = \varprojlim_{\substack{\mathfrak{X} \\ \downarrow \\ \operatorname{Spec} R}} Sk(\mathfrak{X})$

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[dr] & \mathfrak{X} \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$

$$\begin{tikzcd} \rho_{\mathfrak{X}'} : X^{an} \arrow[r] & Sk(\mathfrak{X}) \\ Sk(\mathfrak{X}') \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

III. Weight functions

Mustata – Nicaise

Temkin

$\omega_{X/K}^{\otimes m}$

Kähler metric

X/K smooth proj $/K$

$m \geq 1$

$\omega_{X/K}^{\otimes m}$

ω rational section of $\omega_{X/K}^{\otimes m}$.

regular

Aim

- Define wt_ω
- $Sk(\mathfrak{X}, \omega)$

$X^{an} \leftarrow$ (Divisorial pts, monomial pts

ω rational section
$\otimes m$
$\omega_{X/K}$

Def of weight at a divisorial pt

$$\begin{array}{c}\mathfrak{X}\\ \downarrow\\ \operatorname{Spec} R\end{array}$$ model of X. Let $E \hookrightarrow \mathfrak{X}_0$ irr. comp.

$x_E \in X^{\mathrm{bir,an}}_{\mathrm{div}} \leftarrow \subseteq X^{\mathrm{an}}$

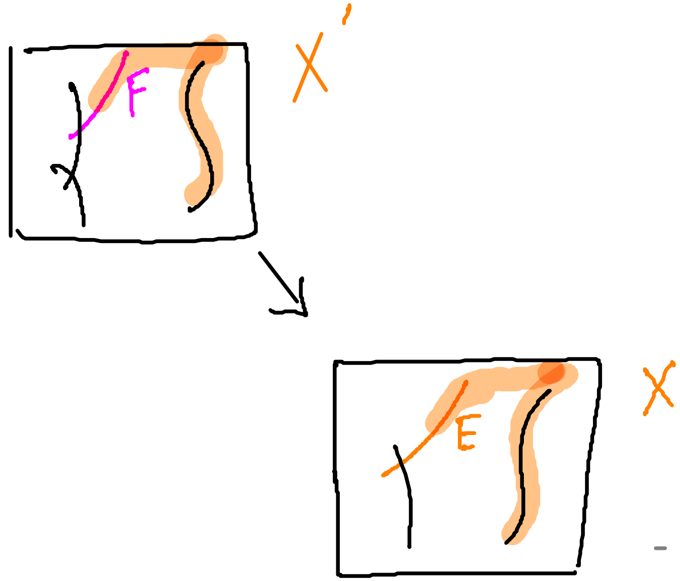
$\vartheta_E : K(X)^\times \longrightarrow \mathbb{R}$ corr. valuation.

Def $\quad wt_\omega(x_E) := \mu/b$ where $\quad \left[m/b + \dfrac{\operatorname{ord}_E \omega}{b} \right]$

$\mu = m +$ mult. of E in $\operatorname{div}_{\mathfrak{X}}(\omega)$

$b =$ multiplicity of E in $\mathfrak{X}_0$.

Prop $\quad wt_\omega(x_E)$ is well-defined.



Proof

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$

Properties

- $wt_{\omega^{\otimes d}}(\ ) = d\, wt_\omega(\ )$
- $wt_{f\omega} = wt_\omega + \operatorname{trop}(f)$

$f \ \omega$

$\uparrow$

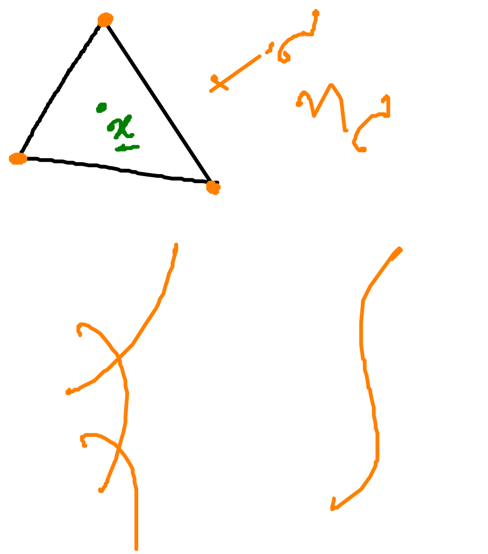
$K(X)^*$

$\rightsquigarrow$ wt_ω can be regarded as tropicalization of ω.

Monomial description of the weight at a divisorial point

$\mathcal{X} \to \operatorname{Spec} R$ model $\qquad \mathcal{X}_0 = \sum_{i \in I} b_i E_i$

$J \subseteq I \qquad \sigma = \sigma_J$



$\underline{x}$: rational pt of σ

$v_{\underline{x}}$ is divisorial.

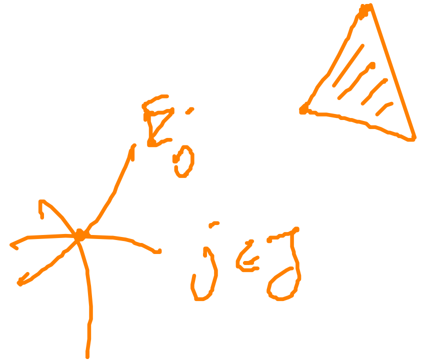
Prop ① $\mathrm{wt}_\omega(\underline{x}) = v_{\underline{x}}\left(\operatorname{div}_{\mathcal{X}}(\omega) + m\left(\underbrace{(\mathcal{X}_0)_{red}}_{\sum b_i}\right)\right)$

② if $\eta_\sigma \notin \overline{\text{Zeros and Poles of } \omega} \Rightarrow \mathrm{wt}_\omega(\underline{x})$ is affine

$= \sum_{j \in J} x_j \mu_j$

trop(f)?

for $j \in J$, let $\mu_j = m +$ mult. E_j in $\mathrm{div}_{\mathfrak{X}}(\omega)$



Proof of the proposition

* ① $\Rightarrow$ ②

In this case $\mathrm{div}_{\mathfrak{X}}(\omega)$ around η_σ

$$= \sum (\mu_j - m) E_j$$

$$\Rightarrow \mathrm{wt}_\omega(x) = \vartheta_x\Big(\mathrm{div}_{\mathfrak{X}}(\omega) + \underbrace{m\,(\mathfrak{X}_0)_{red}}_{m \sum E_j}\Big)$$

$$= \vartheta_x\Big(\sum \mu_j E_j\Big)$$

$$= \sum \mu_j \underbrace{\vartheta_x(E_j)}_{x_j} = \sum \mu_j x_j = \sum_j \frac{\mu_j}{b_j}(x_j b_j)$$

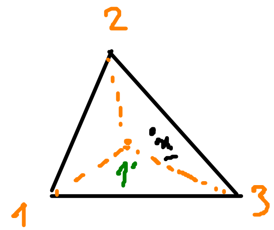
$$x = \sum x_j b_j$$

To show ①:

* if σ is a vertex $\Rightarrow$ definition

* $\underline{x} = (x_1, \dots, x_r)$

  $x_1 \leq x_j$



Consider the blow-up $Bl_{E_\sigma}(\mathcal{X}) = \mathcal{X}'$

$\underline{x} = (x_1, x_2 - x_1, \dots, x_r - x_1)$ in the new coordinates.

$= (x_1', x_2', \dots, x_r')$

$$\vartheta_{\underline{x}}(\operatorname{div}_{\mathcal{X}}(\omega)) + \cancel{m \sum x_i} = \vartheta_{\underline{x}}(\operatorname{div}_{\mathcal{X}'}(\omega)) + m \sum x_j'$$

$$= \vartheta_{\underline{x}}(\operatorname{div}_{\mathcal{X}'}(\omega)) + \cancel{m \sum x_j} - (m-1)\, x_1$$

The equality follows by $\underline{K_{\mathcal{X}'/\mathcal{X}} = (r-1) E_1'}$.

Def of weight function on $Sk(\mathcal{X})$

$\forall x \in Sk(\mathcal{X})$, let $\quad wt_{\mathcal{X},\omega}(x) := v_x\left(\operatorname{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_0)_{red}\right)$

$$wt_{\mathcal{X},\omega} : Sk(\mathcal{X}) \longrightarrow \mathbb{R}$$

Skeleton ineq.

Prop

① $wt_{\mathcal{X},\omega} = wt_\omega \big|_{Sk(\mathcal{X}) \cap X^{\mathrm{bir},\mathrm{an}}_{\mathrm{div}}}$

② $\begin{cases} x \text{ divisorial in } X^{\mathrm{bir},\mathrm{an}} \\ red_{\mathcal{X}}(x) \notin \text{Poles of } \omega \end{cases} \Longrightarrow wt_\omega(x) \geq wt_{\mathcal{X},\omega}(r_{\mathcal{X}}(x))$.

③ $wt_{\mathcal{X},\omega}(x) = wt_{\mathcal{Y},\omega}(x)$ if $x \in Sk(\mathcal{X}) \cap Sk(\mathcal{Y})$.

- $wt_{\mathcal{X},\omega^{\otimes d}} = d\, wt_{\mathcal{X},\omega}$
- $wt_{\mathcal{X},f\omega} = wt_{\mathcal{X},\omega} + \operatorname{trop}(f)$
- $wt_{\mathcal{X},\omega}$ is piecewise affine.

Corollary

There exists a weight function

$$\mathrm{wt}_\omega : X^{\mathrm{birj\,an}}_{\mathrm{mon}} \to \mathbb{Q}$$

such that $\mathrm{wt}_\omega(x) = v_x\big(\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_0)_{\mathrm{red}}\big)$.

Weight function on X^{an}

Def $\omega \in H^0(X, \omega_X^{\otimes m})$ regular

$$\mathrm{wt}_\omega(x) := \sup_{\mathcal{X}} \{ \mathrm{wt}_\omega(r_{\mathcal{X}}(x)) \} \in \mathbb{R} \cup \{+\infty\}.$$

$X^{an} = \varprojlim_{\mathcal{X}} Sk(\mathcal{X})$

$$X^{an} \to \varprojlim Sk(\mathcal{X})$$

$$X^{an} \xrightarrow{r_{\mathcal{X}}} Sk(\mathcal{X})$$

Properties

1. wt_ω is lower semi-continuous.

2. $wt_\omega \big|_{X^{\mathrm{birian}}_{\mathrm{mon}}}$ coincide with the previous definition

3. (Retraction inequality) $x \in X^{an}$

$$wt_\omega(x) \geq wt_\omega(r_{\mathcal{X}}(x)) \quad \text{with equality iff } x \in Sk(\mathcal{X}).$$

4. $wt_{\omega^{\otimes d}} = d\, wt_\omega$

$$\boxed{wt_{f\omega} = wt_\omega + trop(f)}$$

$$\omega \in H^0(X, \omega_X^{\otimes m})$$

IV Essential Skeleton

ess. sk. of X wrt ω

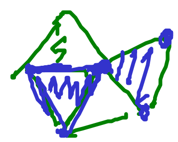
K-S skeleton

$$Sk(X,\omega) = \text{closure of } x \in X^{\text{birat}}_{\text{div}} \text{ with } wt_\omega(x) \text{ minimum}$$

$$= \text{closure of } x \in X^{\text{birat}}_{\text{mon}} \quad '' \quad '' \quad '' .$$

Thm $\omega \in H^0(X, \omega_X^{\otimes m})$

$$\begin{array}{c}\mathcal{X}\\ \downarrow \\ \operatorname{Spec} R\end{array}$$



$Sk(X,\omega)$ is the subcomplex of $Sk(\mathcal{X}) = \Sigma(\mathcal{X})$ with vertex set J consisting of $j \in I$ with

$$\mu_j / b_j = \min_{i \in I} \{ \mu_i / b_i \}.$$

$$\mathcal{X}_0 = \sum b_i E_i$$

$\mu_i = m +$ multiplicity of E_i in $\operatorname{div}_{\mathcal{X}}(\omega)$.

Proof $\quad wt_{\mathcal{X},\omega}(x) \geq \sum \mu_i \, v_x(E_i) \quad$ for $x \in Sk(\mathcal{X})$

with equality iff $red_{\mathcal{X}}(x)$ is not contained in the closure of zero locus of ω.

$H^0(X, \omega_X)$

$m = 1$

Rk. $X \subset Y$

$$K_{\mathcal{X}} = div(\Omega) = \sum (a_i + b_i - 1) E_j$$

$\mu_i = a_i + b_i \quad \rightsquigarrow \quad Sk(X, \Omega) = Sk_{KS}(\mathcal{X})$.

$\mu_i = m + mult_i$

$$\frac{a_i + b_i}{b_i} = 1 + a_i/b_i$$

Def • $\displaystyle Sk_{ess,m}(X) := \bigcup_{\omega \in H^0(X,\omega_X^{\otimes m})} Sk(X,\omega)$ $\subseteq Sk(\mathcal{X})$ $m \in \mathbb{N}$

$\subseteq Sk(\mathcal{X})$.

• $\displaystyle Sk_{ess}(X) := \bigcup_{m} Sk_{ess,m}(X)$.

$\cancel{Sk_{ess,m} \overset{?}{\subseteq} Sk_{ess,m+1}}$

RK. $Sk_{ess,m} \subseteq Sk_{ess,dm}$

min m?

s.t. $Sk_{ess,m} = Sk_{ess}$

$\boxed{m = f(\dim X)}$?

$d:\ H^0(X, \omega_X) = 1$

Properties

Temkin

* (Mustaţă–Nicaise) X geometric genus one $\Rightarrow$ $Sk_{ess}(X)$ is connected.

* (Baker–Nicaise) X curve of genus ≥ 1

? $X \downarrow$ Purely comb Sk_{ess}

$$Sk_{ess}(X) = Sk_{ess,2}(X) = \text{minimal dual graph of } X.$$

* (Nicaise–Xu) X CY maximal $\Rightarrow$ $Sk_{ess}(X)$ is a pseudo-manifold. with boundary

(1) every codim 1 simplex $\in$ one or two facets

(2) Connected through codim one.