Some ingredients (& techniques) in pluripotential theory

(X^n, ω) compact Kähler manif

ω (1,1)-form real, d-closed and >0.

$\int_X \omega^n = V$

1. the first notion we are going into is the notion of QPSH - functions (or ω-PSH)

We say that $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is ω-psh if $\varphi \in L^1(\omega^n)$

$\varphi \underset{loc}{\sim}$ smooth + psh & $\omega + \underbrace{dd^c}_{i\partial\bar\partial} \varphi \geq 0$ weak sense

we define

$PSH(X, \omega)$ = ~~set~~ the space of ω-psh functions endowed with the L^1-topology

Examples : * φ = constant * $\varphi \in C^2$ the φ/A is ω-psh

* $\varphi \simeq \log|z|$, $-(-\log|z|)^\alpha$ $\alpha \in (0,1)$

Properties

* bdd by above (we can always normalise with $\sup_X \varphi = 0$)

* $\varphi \in L^q(\omega^n)$ $\forall q \geq 1$

* Skoda uniform (?) MAYBE YES! $\exists \alpha > 0, C > 0 : \forall \psi$ ω-psh $\sup_X \psi = 0$ $\int_X e^{-\alpha(\psi)} dV < C$

* $\forall \varphi \in PSH(X, \omega)$ we can define the non-pluripolar MA measure $(\omega + dd^c \varphi)^n := MA(\varphi)$

  - if $\varphi \in C^\infty$ then ok
  - if $\varphi \in L^\infty$ BT82.
  - $\varphi \notin L^\infty \Rightarrow MA(\varphi)$ defined by GZ

$$\overset{\shortparallel}{\lim_{j \to +\infty}} \nearrow \mathbb{1}_{\{\varphi > -j\}} (\omega + dd^c \max(\varphi, -j))^n \quad j \in \mathbb{N}$$

whose mass $\leq V$

Define class $\mathcal{E}$ (+ comPARISON PRINCIPLE!)

(2.) We are now ready to introduce the notion of the MA capacity Cap_ω

Let $E \subset X$ a Borel set, then

(INNER REGULAR)

$$Cap_\omega(E) = \sup \left\{ \int_E MA(u) \quad \begin{matrix} u \ \omega\text{-psh} \\ -1 \leq u \leq 0 \end{matrix} \right\}$$

[This is NOT a measure : it is ~~easy to check that~~ just subadditive]

$\inf \{ Cap(G) \ \substack{E \subseteq G \\ G \text{ open}} \}$ OUTER CAP

the reason is $Cap_\omega^{(*)}(P) = 0 \Leftrightarrow P$ is pluripolar

$P \subset \{ \psi = -\infty \}$

ψ qpsh

Another capacity (defined by means of global extremal function) is the Alexander-Taylor capacity

Given $E \subset X$ Borel

Def: We set

$$V_{E,\omega} = \sup \{ u(\cancel{x}) : u \ \omega\text{-psh} \quad u \leq 0 \text{ on } E \}$$

FACT • $V^*_{E,\omega} \equiv +\infty \Longleftrightarrow E$ is pluripolar

$\Longleftrightarrow \sup V^*_{E,\omega} = +\infty$

• if E is NOT pluripolar then

$V^*_{E,\omega} = 0$ on E & $V^*_{E,\omega} \geq 0$

The Alexander-Taylor capacity is then defined as

$$T_\omega(E) = \exp\left(-\sup_X V^*_{E,\omega} \right)$$

again $T_\omega(P) = 0 \Leftrightarrow P$ is pluripolar there is a link between these 2-capacities

<u>Lemma</u>

$$T_\omega(E) \leq \exp\left[1 - \frac{1}{\mathrm{Cap}_\omega(E)^{1/n}}\right]$$

We are now ready to state the result we want to look at whose proof is based on Kolodziej's technique.

<u>theorem</u> Let $\mu = f\,dV$ $\mu(X)=1$ $f \in L^p(dV)$ $p>1$
~~Assume~~ & let φ ($\sup_X \varphi = 0$) be the unique ω-psh function solution of $\boxed{(\omega+dd^c\varphi)^n = f\,dV}$.
Assume that

(H1) $\exists \alpha>0$ & $A_\alpha>0$: $\forall \psi$ ~~sup~~ $\psi \neq 0$ ω-psh

$$\int_X e^{-\alpha(\psi - \sup_X \psi)}\,dV \leq A_\alpha$$

(H2) $\exists C>0$: $\left(\int_X |f|^p\,dV\right)^{1/p} \leq C$

Then $-M \leq \varphi \leq 0$

where $M := 1 + C^{1/n} A_\alpha^{1/nq} e^{\alpha/nq} b_n \left[5 + e\alpha^{-1} C (q!)^{1/q} A_\alpha^{1/q}\right]$

where q s.t $\frac{1}{p}+\frac{1}{q}=1$ &

$$b_n : \quad \exp\left(\frac{-\alpha/q}{x}\right) \leq b_n^n x^{2n} \quad \forall x>0$$

these two condition in this setting (nothing is moving) that are clearly satisfied BUT we want to have a thm where all the constants are explicit in order to be able to apply it when we work in fam "proof" We are going to prove that

$$(\circledast) \quad \boxed{\mathrm{Cap}_\omega(\{\varphi < -M\}) = 0} \Rightarrow$$

$\varphi \geq -M$ almost everywhere, hence everywhere

To do that, we let $g : \mathbb{R}^+ \to \mathbb{R}^+$

$$g(t) := \mathrm{Cap}_\omega(\{\varphi < -t\})^{1/n}$$

FACTS • g is decreasing, right-continuous & $g(+\infty) = 0$

$\circledast$ is then equivalent to show that g reaches 0 in finite time

claim 1 $\forall t > 0$ & $s \in (0,1)$

$$s\, g(t+s) \leq D^{1/n} g(t)^2 \quad \text{where}$$

$$D = b_n^n\, C A_\alpha^{1/q} e^{\alpha/q}$$

(EXPLICIT!)

proof of claim 1

of the LHS we have

$$s^n \mathrm{Cap}_\omega(\{\varphi < -t-s\})$$

Take $-1 \leq u \leq 0$ a candidate for the capacity, then the quantity to look at is

$$s^n \int_{\{\varphi < -t-s\}} (\omega + dd^c u)^n \leq \int_{\{\varphi < -t-su\}} (\omega + dd^c su)^n \underset{CP}{\leq} \int_{\{\varphi < -t+su\}} (\omega + dd^c \varphi)^n \leq \int_{\{\varphi < -t\}} MA(\varphi)$$

Obs : $\{\varphi < -t-s\} \subseteq \{\varphi < -t-su\} \subseteq \{\varphi < -t\}$ $\oplus$ $(\omega + dd^c su)^n \geq s^n MA(u)$

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[3]

$$= \int_{\{\varphi<-t\}} f\, dV \overset{\text{hölder}}{\leq} \left(\int_X |f|^p dV\right)^{1/p} \left(\int_{\{\varphi<-t\}} dV\right)^{1/q}$$

$$\leq C \cdot \left(\int_{\{\varphi<-t\}} dV\right)^{1/q}$$

Now, ~~let~~ denote $E := \{\varphi<-t\}$ & remember that. $V^*_{E,\omega} = 0$ on E & ≥ 0

hence

where α is the α of assumption (H1)

$$\int_E dV \leq \int_X e^{-\alpha V^*_{E,\omega}} dV = \int_X e^{-\alpha(V^*_{E,\omega} - \sup V^*_{E,\omega})} \cdot e^{-\alpha \sup V^*_{E,\omega}} dV$$

$$\leq A_\alpha e^{-\alpha \sup_X V^*_{E,\omega}} = A_\alpha T_\omega(E)^\alpha \overset{\text{lemma}}{\leq} A_\alpha e^\alpha \cdot e^{-\frac{\alpha}{Cap_\omega(E)^{1/n}}}$$

therefore, combining all the previous inequality we get

$$s^n \int_{\{\varphi<-t-s\}} MA(u) \leq C \cdot A_\alpha^{1/q} e^{\alpha/q} e^{-\frac{\alpha/q}{Cap_\omega(E)^{1/n}}}$$

$$\leq C \cdot A_\alpha^{1/q} e^{\alpha/q} b_n^n Cap_\omega(E)^2$$

the last ineq. comes from the elementary ineq.
$\exp\left(-\frac{\alpha/q}{x}\right) \leq b_n^n x^{2n} \quad \forall x>0$

Taking the sup over u we get that

$$s^n Cap_\omega(\{\varphi<-t-s\}) \leq \underbrace{C A_\alpha^{1/q} e^{\alpha/q} b_n^n}_{=: D} Cap_\omega(\{\varphi<-t\})^2$$

& this is what we wanted

claim 2 (no proof) $g(t) = 0 \quad \forall t \geq 5D^{1/4} + t_0$

where $t_0 = \inf\{t > 0 : eD^{1/4} \mathrm{Cap}_\omega(\{\varphi < -t\}) < 1\}$

~~$\mathrm{Cap}_\omega(\{\varphi < -t\}) \leq \frac{1}{e}$~~

It then remains to get a uniform control on t_0 !

From the previous arguments we got

$$s^4 \mathrm{Cap}_\omega(\{\varphi < -t-s\}) \leq MA(\varphi)(\{\varphi < -s\})$$

taking $s = 1$ we then have

$$\mathrm{Cap}_\omega(\{\varphi < -t-1\}) \leq MA(\varphi)(\{\varphi < -t\})$$

$$\frac{(-\varphi)}{t} > 1 \quad \int_{\{\varphi < -t\}} MA(\varphi) \leq \int_X \frac{(-\varphi)}{t} MA(\varphi)$$

$$= \frac{1}{t}\int_X (-\varphi) f \, dV \underset{\text{Hölder}}{\overset{\varphi \in L^q}{\leq}} \frac{1}{t}\left(\int_X |f|^p dV\right)^{1/p}\left(\int_X |\varphi|^q\right)^{1/q}$$

$$\boxed{\forall x \geq 0 \quad x^q \leq \frac{q!}{\alpha^q} e^{\alpha x}}$$

$$\underset{\&(H2)}{\leq} \frac{C}{t}\left(\frac{q!}{\alpha^q}\int_X e^{-\alpha\varphi} dV\right)^{1/q} \underset{(H1)}{\leq} \frac{C}{t}\left(\frac{q!}{\alpha^q}\right)^{1/q} \cdot A_\alpha^{1/q}$$

If

$$\frac{eD^{1/4} C}{t} \frac{(q!)^{1/q}}{\alpha^q} A_\alpha^{1/q} \leq 1 \quad \text{then} \quad t + 1 = t_0$$

$$t = \frac{eD^{1/4} C (q!)^{1/q} A_\alpha^{1/q}}{\alpha^q} \quad \underline{\text{hence}} \quad t_0 = 1 + e\_\_\_\_$$

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<u>Conclusion</u> : claim 2 tells us to take

$$M = 5D^{1/4} + 1 + \frac{eD^{1/4}}{\alpha} C(9)^{1/9} A_\alpha^{1/9}$$

I would like to finish with a sort of $\boxed{\text{stability result}}$: Fix $\varepsilon > 0$

Let $\varphi, \psi \in PSH(X,\omega)$ ≤ 0 $\qquad -M \leq \varphi \leq 0$

& ψ: $\omega_\psi^n \leq A \, Cap_\omega^2$, then $\exists C > 0$:

$C(A, \omega, M)$

$$\sup_X (\varphi - \psi) \leq \varepsilon + C \, Cap_\omega(\{\psi - \varphi < -\varepsilon\})^{2/n}$$

If they are close in capacity, they are close in the C^0 norm

<u>proof</u>: the same type of arguments of before.

the only difference/observation to make is that

$\forall s \in (0,1)$ $\qquad \boxed{s = \frac{\tau}{1+\tau}}$ $\qquad \begin{array}{l} -M \leq \varphi \leq 0 \\ -1 \leq u \leq 0 \end{array}$

$$\{\psi - \varphi < -t - s\} \subseteq \left\{\psi < \frac{\varphi + \tau(u+1)}{1+\tau} - t - s\right\}$$

$$\subseteq \left\{\psi < \frac{\varphi}{1+\tau} + \cancel{s} - t - \cancel{s}\right\}$$

$$= \{\psi - \varphi < -s\varphi - t\}$$

$$\subseteq \{\psi - \varphi < sM - t\}$$

$\frac{\varphi + \tau(u+1)}{1+\tau} \geq \varphi$

$\frac{\tau}{1+\tau}(u+1) \geq \frac{\tau}{1+\tau}\varphi$

& $u+1 \geq 0$ while $\varphi \leq 0$

$-\varphi \leq M$

therefore $\forall u$ candidate in the capacity [15.]

$$s^n \int_{\{\psi-\varphi<-t-s\}} \omega_u^n \leq s^n \int_{\{\psi < \underbrace{\frac{\varphi}{1+\tau}+\frac{\tau}{1+\tau}(u+1)-t-s}_{:=\tilde{\varphi}}\}} \omega_u^n$$

$$\leq \int_{\{\psi<\tilde{\varphi}\}} [s\omega_{\frac{u+1}{\tau}}+(1-s)\omega_\varphi]^n = \int_{\{\psi<\tilde{\varphi}\}} \omega_{\tilde{\varphi}}^n \overset{CP}{\leq} \int_{\{\psi<\tilde{\varphi}\}} \omega_\psi^n$$

$$\leq \int_{\{\psi-\varphi<sM-t\}} \omega_\psi^n$$

$$\boxed{\begin{array}{l}\text{Substituting} \\ t \longrightarrow t-sM = t' \text{ we get}\end{array}} \qquad \boxed{t = t'+sM}$$

$$s^n \int_{\{\psi-\varphi<-t'-sM-s\}} \omega_u^n \leq \int_{\{\psi-\varphi<-t'\}} \omega_\psi^n$$

$$\boxed{s \longrightarrow (1+M)s = s'}$$

$$\frac{s'}{(1+M)^n} \int_{\{\psi-\varphi<-t'-s'\}} \omega_u^n \leq \int_{\{\psi-\varphi<-t'\}} \omega_\psi^n$$

hence

$$s' \int_{\{\psi-\varphi<-t-s\}} \omega_u^n \leq (1+M)^n \int_{\{\psi-\varphi<-t\}} \omega_\psi^n$$