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Non-Archimedean Monge-Ampère measures (NAMA)

Goal: define Monge-Ampère measures in a non-Archimedean context $\rightsquigarrow$ Follow Yang Li §3

references

Kontsevich - Tschinkel

Boucksom - F. - Jonsson

Gubler - Martin

Boucksom Gubler - Martin

Boucksom - Jonsson

Chambert-Loir Ducros

The $\mathbb{C}$ setting

(X, ω) compact Kähler $\dim = n$

$\mu \geq 0$ measure $\mu(X) = \int_X \omega^n$

(MA_μ) Find $\varphi : X \to [-\infty, \infty)$

- φ is ω-psh

  locally $\mathcal{C}^\infty$ + psh

  $\omega + dd^c \varphi \geq 0$

- $MA_\omega(\varphi) := (\omega + dd^c \varphi)^d$ is well defined and $MA_\omega(\varphi) = \mu$

1. $\mu = h\omega^d$ $\quad h \in \mathcal{C}^\infty$ $\quad h > 0$

Yau : may choose $\varphi \in \mathcal{C}^\infty$

$$MA_\omega(\varphi) := (\omega + dd^c\varphi)^{\wedge d}$$

method : continuity method

2. $\mu = h\omega^d$ $\quad h \in L^{1+\varepsilon}$ $\quad \varepsilon > 0$

Kolodziej : $\varphi \in \mathcal{C}^0 \cap PSH_\omega$

$$MA_\omega(\varphi) := \lim_n (\omega + dd^c\varphi_n)^{\wedge d}$$

$$\varphi_n \in \mathcal{C}^\infty \cap PSH_\omega \xrightarrow{\sup} \varphi$$

method : capacity estimates

3. μ : measure of finite energy

BBEGZ : φ finite energy

$MA_\omega(\varphi)$ = non pluripolar product

(Bedford-Taylor, Guedj-Zeriahi)

<u>method</u> = variational in nature

$$E(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$$

$$\omega_\varphi = \omega + dd^c \varphi$$

Look for maximizer $\varphi \longmapsto E(\varphi) - \int \varphi\, d\mu$

□ Maximizer exists in the space of ω-psh functions with $E > -\infty$

compactness in PSH_ω

□ Maximizer satisfies (MA_μ)

differentiability of volume function

The Non-Archimedean case

$(K, |\cdot|) = (\mathbb{C}((t)), |\cdot|_t)$

$K^\circ = \{ |\cdot| \leq 1 \} = \mathbb{C}[[t]]$

- Not clear what a Kähler manifold is!

  $\rightarrow$ X/K projective mfd smooth

- Not clear what a smooth measure is!

  $\rightarrow$ μ supported on $\Delta_{\mathcal{X}}$

- The measure $MA_\omega(\varphi)$ will represent intersection theoretic quantities of cycles in models $\mathcal{X} \rightarrow \mathbb{D}$

(alternative approach CLD using ana. logs of $\partial, \bar\partial$ ...)

Gen setting

$\pi : \mathcal{X} \to \mathbb{D}$ hol.

$\mathcal{X}$ dim $= n+1$, smooth

$\mathcal{X}_* \to \mathbb{D}^\times$ submersion

$X_t = \pi^{-1}(t)$ connected

$\mathcal{L} \to \mathcal{X}$ relatively ample line bundle

i.e.

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{P}^N \times \mathbb{D} \arrow[d, "pr_1"] \\ & \mathbb{D} \end{tikzcd} \qquad \mathcal{L} = \mathcal{O}(1)|_{\mathcal{X}}$$

this gives $\left\{ \begin{array}{l} X/\mathbb{C}((t)) \text{ smooth projective} \\ L \to X \text{ ample line bundle} \\ (L, |.|) \text{ metrization} \end{array} \right.$

Thm $\mu \geq 0$ positive measure on $\Delta_{\mathcal{X}}$

$\exists!\ \varphi \in \mathcal{C}^0(X^{an}) \cap \mathcal{L}\text{-PSH}$

$$MA_{\mathcal{L}}(\varphi) = \mu$$

proof follows BBEGZ (variationnal)
and relies on crucial ingredients

- ☐ Compacity of $\mathcal{L}$-PSH functions
- ☐ Continuity of envelopes
  (multiplier ideals, Nadel vanishing)
- ☐ Differentiability of volumes
  (Siu's inequalities)
  Boucksom - Favre - Jonsson

Explain :

- $\mathcal{Z}$-PSH
- $MA_{\mathcal{Z}}(\varphi)$ ⇝ connection with real Monge-Ampère measures

Plan

1. Model functions and metrics
2. The case $n = 1$ (linear)
3. Semi-positive metrics and NA
4. Connection with real MA

① Model functions and metrics

def. A model is a hol. family

$$\begin{tikzcd} \mathcal{X}' \arrow[rr, "\pi'"] \arrow[dr, "\mu"'] & \arrow[d, phantom, "\circ"] & \mathbb{D} \\ & \mathcal{X} \arrow[ur, "\pi"'] & \end{tikzcd}$$

X'_0 snc

μ bimeromorphic

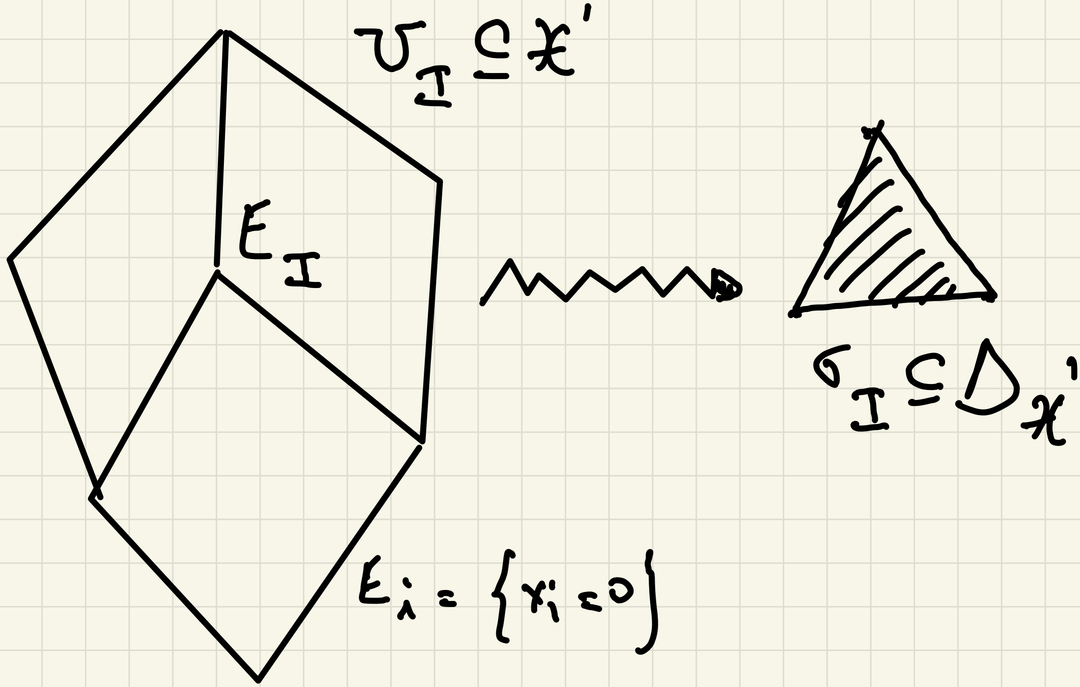
and $\mathcal{X}'_* \xrightarrow[\mu]{\sim} \mathcal{X}_*$

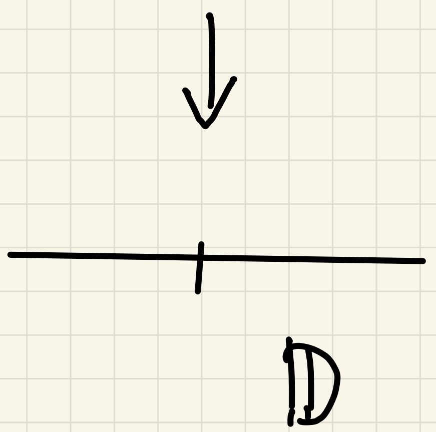
recall the retractions :

$$\begin{tikzcd} X^{an} \arrow[rr, "r_{\mathcal{X}'}"] \arrow[dr, "r_{\mathcal{X}}"'] & & \Delta_{\mathcal{X}'} \arrow[dl, "r_{\mathcal{X}'\mathcal{X}}"] \\ & \Delta_{\mathcal{X}} & \end{tikzcd}$$

$\Delta_{\mathcal{X}'}$ = affine structure

$$\boxed{\varphi \in \mathrm{Aff}(\Delta_{\mathcal{X}'}) \iff \exists\, Z \in \mathrm{Div}(X'_0) \quad \varphi = \varphi_Z}$$





$$Z = \sum \mathrm{ord}_{E_i}(Z)\, E_i$$

$$[X'_0] = \sum b_i\, E_i$$

$$\sigma_I \longleftrightarrow \{ s \in \mathbb{R}^I \quad \textstyle\sum b_i s_i = +1 \}$$

$$\big\downarrow$$

$X^{an} \ni x(s, I)$ semi-norms on $G(\mathcal{U}_I)$

$$- \operatorname{Log} |f(x(s, I))| = \sum \operatorname{ord}_{E_i}(f)\, s_i$$

if $Z \in \operatorname{Div}(X_0')$ set

$$\varphi_Z(x(s, I)) = \sum \operatorname{ord}_{E_i}(Z)\, s_i$$

$$\boxed{\begin{array}{l} \underline{\text{Def}} \quad \varphi \text{ model function if } \exists \text{ model } \\ \mathcal{X}' \text{ such that } \begin{cases} \varphi|_{\Delta_{\mathcal{X}'}} \in \operatorname{Aff}(\Delta_{\mathcal{X}'}) \\ \varphi = \varphi \circ r_{\mathcal{X}'} \end{cases} \end{array}}$$

. $\mathfrak{a}$ ideal sheaf supported on X_0

$$\operatorname{Log}|\mathfrak{a}|(x(\rho, I)) = \min - \operatorname{Log}|f_\alpha(x(\rho, I))|$$

$\mathfrak{a} = \langle f_\alpha \rangle$ in a nghd of E_I

$\boxed{\text{Thm}}$ $\mathcal{D}(X) = \{$model functions$\}$ generated by $\operatorname{Log}|\mathfrak{a}|$, stable by max, by sum and C^0 dense in $C^0(X^{an})$

□ model metrics on L

$L \to X \ / \ \mathbb{C}((t)) \qquad X \hookrightarrow \mathbb{P}^N_{\mathbb{C}((t))}$

$L = \mathcal{O}_X(D)$

metric $\sigma \in H^0(U, L) \rightsquigarrow |\sigma|_U \in C^0(U)$

$$\begin{cases} |\sigma|_V|_V = |\sigma||_V \\ |f\sigma|_U = |f|\,|\sigma|_U \\ |\sigma|_U \neq 0 \text{ if } \sigma \text{ is non-vanishing} \end{cases}$$

$\mathcal{L} \to \mathcal{X}$ relatively ample

$\mathcal{D}$ = closure of D in $\mathcal{X}$

$\sigma \in H^0(X, L) \rightsquigarrow$ mero function

on $\mathcal{X}^*$ $\qquad$ $\mathrm{div}(\sigma) + D \geq 0$

extends to $\mathcal{X}$ as mero function

$$\mathrm{div}(\sigma) + \mathcal{D} = Z$$

$$|\sigma|_{\mathcal{L}} = e^{-\varphi_Z}$$

scheme theoretic interpretation

if $\sigma \in H^0(\mathcal{U}, \mathcal{L})$ local frame

$|\sigma|_{v} \equiv 1$ $\qquad$ $\mathcal{L} \to \mathcal{X} \to \mathrm{Spec}(\Gamma[1/f])$

□ Monge-Ampère measure of a model metric

$$\mathcal{L} \to \mathcal{X} \rightsquigarrow |\cdot|_{\mathcal{L}} \text{ on } L \to X/_{\mathbb{C}((t))}$$

$MA(|\cdot|_{\mathcal{L}})$ signed measure on X^{an}

$$\int \varphi_Z \, MA(|\cdot|_{\mathcal{L}}) := Z \cdot \mu^* c_1(\mathcal{L})^n$$

$$\begin{tikzcd} & \mathcal{X}' \arrow[dl, "\mu"'] \arrow[ddl] \\ \mathcal{X} \arrow[d] \arrow[r, phantom, "\circ"] & \\ \mathbb{D} & \end{tikzcd}$$

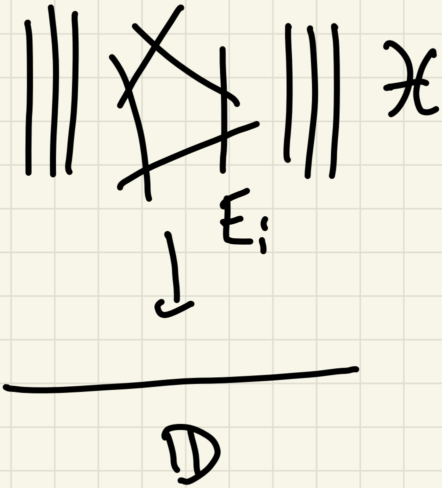
$Z \in \mathrm{Div}(\mathcal{X}'_0)$

$Z = \sum a_i E_i$

$$Z \cdot \mu^* c_1(\mathcal{L})^n = \sum a_i \, \mu^* c_1(\mathcal{L})|_{E_i}^n$$

notation $\quad MA(|\cdot|_{\mathcal{L}}) = MA_{\mathcal{L}}(0)$

② The case $n = 1$ (linear situation)



$$\mu = \sum a_i \, \delta_{x_{E_i}}$$

$$\mathcal{Z} \to \mathcal{X} \quad \mathcal{L} = \mathcal{O}_{\mathcal{X}}(\mathcal{D})$$

$$\mu(X) = \deg(L)$$

Look for $\varphi = \varphi_Z$ s.t.

$$MA_{\mathcal{L}}(\varphi) = \mu$$

$$\iff \forall\, W \in \mathrm{Div}(\mathcal{X}_0)$$

$$\int \varphi_W \, MA_{\mathcal{L}}(\varphi) = \int \varphi_W \, d\mu$$

$$MA_{\mathcal{Z}}(\varphi) = \mu$$

$\Leftrightarrow \forall W \in \mathrm{Div}(X_0)$

$$\int \varphi_W \, MA_{\mathcal{Z}}(\varphi_{\mathcal{Z}}) = \int \varphi_W \, d\mu$$

$$\Leftrightarrow W.(\mathcal{D} + \mathcal{Z}) = \sum a_i \varphi_W(x_{E_i})$$

$$\begin{aligned} \mathrm{Div}(X_0) &\xrightarrow{\Phi} \mathbb{R} \\ W &\longmapsto \sum a_i \varphi_W(x_{E_i}) - W.\mathcal{D} \end{aligned}$$

linear form

ker $= \mathbb{R}[X_0]$ since

$$\int \varphi_{[X_0]} \, d\mu = \mu(X^{an}) = \deg(L)$$

$$= [X_0].\mathcal{Z}$$

$$= [X_E].\mathcal{Z}$$

$\Phi : \mathrm{Div}(X_0)/\mathbb{R}[X_0] \to \mathbb{R}$

$\langle\, , \rangle$ is negative definite on

$\mathrm{Div}(X_0)/\mathbb{R}[X_0]$ (Hodge index)

$\Rightarrow \exists! Z \quad \Phi(W) = (Z \cdot W)$

observation : when $\mu \geq 0$

$(\mathcal{D} + Z) . W \geq 0 \quad \forall W \in \mathrm{Div}(X_0)$

i.e $\quad \mathcal{L}_Z = \mathcal{O}_{\mathcal{X}}(\mathcal{D}+Z)$ is

relatively nef

③ Semi positive metrics

$\mathscr{Z} \to \mathscr{X} \to \mathbb{D}$ reference model $|\cdot|_{\mathscr{Z}}$

Want to define the notion of $\mathscr{Z}$-psh

$\varphi : X^{an} \to [-\infty, +\infty]$

□ Model functions

$\varphi = \varphi_Z$

φ $\mathscr{Z}$-psh (or $|\cdot| = |\cdot|_{\mathscr{Z}} e^{-\varphi_Z}$ is semi-positive) if $Z + \mu^* \mathscr{Z}$ is relatively nef

$$\begin{tikzcd} \mathscr{X} \arrow[d] & \mathscr{X}' \arrow[l, "\mu"'] \arrow[dl] \\ \mathbb{D} & \end{tikzcd}$$

$Z \in \mathrm{Div}(X'_0)$

ex $\mathscr{Z} \otimes \mathfrak{a}$ π-globally generated

$\Rightarrow \operatorname{Log}|\mathfrak{a}|$ is $\mathscr{Z}$-psh

General $\mathcal{X}$-psh functions

φ $\mathcal{X}$-psh if

$$\begin{cases} - \ \varphi \text{ usc} \\ - \ \varphi \leq \varphi \circ r_{\mathcal{X}'} \quad \forall \mathcal{X}' \\ - \ \forall \mathcal{X}' \ \exists \varphi_n \text{ model } \mathcal{X}\text{-psh} \\ \quad \sup_{\Delta_{\mathcal{X}'}} |\varphi_n - \varphi| \to 0 \end{cases}$$

observation: $\mathbb{C}$ analog would say

φ usc + mean-value inequality on balls

+ $\exists \varphi_n \in \omega\text{-psh} \cap \mathcal{C}^\infty$

$$\|\varphi_n - \varphi\|_{L^1_{loc}} \to 0$$

(Demailly approximation)

Comments on the definition

- $\varphi_Z \in \mathcal{D}(X)$

  $\mathcal{D} + Z$ relatively nef $\Rightarrow$ $\varphi_Z \leq \varphi_Z \circ r_{\mathcal{X}}$

[negativity lemma]

Thm (approximation)
φ $\mathcal{Z}$-psh. Then there exist $\varphi_m \in \mathcal{Q}(X)$
φ_m $\mathcal{Z}$-psh $\searrow \varphi$

Idea : $\varphi \circ r_{\mathcal{X}}' \searrow \varphi$

Take $\mathcal{Z}$-psh envelope
$P(\varphi \circ r_{\mathcal{X}}') = \sup\{\psi \leq \varphi \circ r_{\mathcal{X}}'\} \searrow \varphi$

$\rightsquigarrow$ reduce the theorem to $P(\varphi_Z)$

Thm $P(\varphi_Z)$ is a uniform limit of model $\mathcal{Z}$-psh functions (hence $\mathcal{C}^0$)

Tools = multiplier ideals + Nadel vanishing

φ $\mathcal{Z}$-psh and continuous

Dini $\Rightarrow$ $|\varphi - \varphi_n|_{\mathcal{C}^0} \to 0$

Proposition definition

$$MA_{\mathcal{Z}}(\varphi) = \lim_n MA_{\mathcal{Z}}(\varphi_n)$$

sketch of proof (existence of the limit)

$\varphi_j = \varphi_{Z_j}$ $\quad \varepsilon = |\varphi_j - \varphi_\ell|_{\mathcal{C}^0} \ll 1$

$$\left| \int \varphi_W \, MA_{\mathcal{Z}}(\varphi_j) - MA_{\mathcal{Z}}(\varphi_\ell) \right|$$

$$= \left| W.(\mathcal{Z} + Z_j)^n - W.(\mathcal{Z} + Z_\ell)^n \right|$$

$$\leq \varepsilon \sup |\varphi_W| \, C$$

///

Brief comments on the proof of the resolution of the MA operator

μ supported on $\Delta_{\mathcal{X}}$ $\qquad \mu(X) = c_1(L)^n$

$\exists !\ \varphi$ $\mathcal{Z}$-psh $\quad MA_{\mathcal{Z}}(\varphi) = \mu$

$$E(\varphi) = \frac{1}{n+1} \int_X \sum_{j=0}^{n} \varphi\, \omega_\varphi^j \wedge \omega^{n-j}$$

to define $\omega_\varphi^j \wedge \omega^{n-j}$ use polarization

$$MA_{\mathcal{Z}}(t\varphi) = \sum \binom{n}{j} t^j (1-t)^{n-j}\, \omega_\varphi^j \wedge \omega^{n-j}$$

or directly $MA_{\mathcal{Z}}(\varphi_1, \dots, \varphi_n)$

$$\mathcal{Z} = G_{\mathcal{H}}(\mathcal{D}) \qquad \varphi_i = \varphi_{Z_i}$$

$$\int \varphi_W \, MA_{\mathcal{Z}}(\varphi_1, \dots, \varphi_n) = W \cdot (\mathcal{D}+Z_1) \dots (\mathcal{D}+Z_n)$$

and follow BBEGZ + Kolodjiez

$\to$ extend $MA_{\mathcal{Z}}$ to $\mathcal{Z}$-psh of finite energy

$\to$ maximize $E(\varphi) - \int \varphi \, d\mu \rightsquigarrow \varphi_*$

topology on $\mathcal{Z}$-psh = pointwise convergence on $\bigcup_{\mathcal{H}'} \Delta_{\mathcal{H}'}$ $\rightsquigarrow$ E is usc

$\mathrm{Supp}(\mu) \Rightarrow \varphi \mapsto \int \varphi \, d\mu$ is $\mathcal{C}^0$

$\rightarrow \quad MA_{\mathcal{L}}(\varphi_x) = \mu$

use differentiability of volume :

Boucksom - Gubler - Martin

$\rightarrow$ continuity uses $\mathrm{Supp}(\mu) \subseteq \Delta_{\mathcal{X}}$,

(Kolodziej' capacity estimates).

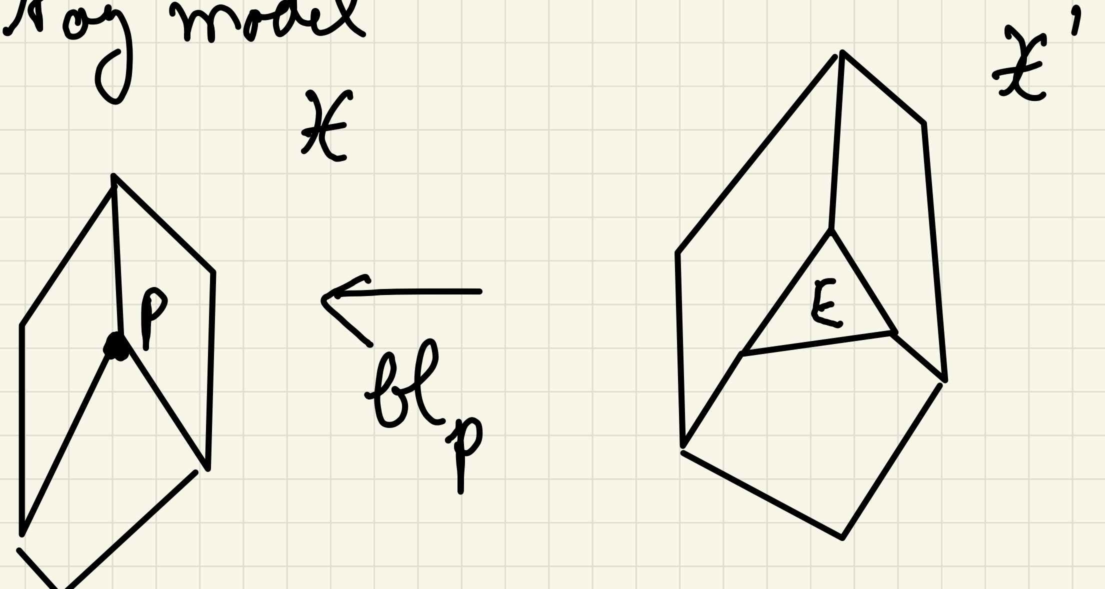
④ Connection with $\mathbb{R}$ Monge-Ampère

$\mathcal{X}'$ snc model

φ model $\mathcal{L}$-psh function

Fact : $\varphi|_\sigma$ is convex for any face σ of $\Delta_{\mathcal{X}}$

Toy model



$E \simeq \mathbb{P}^{n-1}$

$\varphi = \varphi_Z \qquad \varphi \in \mathrm{Dir}(X_0')$

Arol $\quad Z \cdot \mathcal{O}_E(1)^l \geq 0$

$\Delta_{\mathcal{X}}$ $\qquad\qquad\qquad\qquad$ $\Delta_{\mathcal{X}'}$

$\Rightarrow \varphi_Z$ convex on $\Delta_{\mathcal{X}'}$

φ = uniform limit of $\operatorname{Log}|\theta|$

$$\operatorname{Log}|\theta| = \max \operatorname{Log}|f(x(\theta, I))|$$

$$\operatorname{Log}|f(x(\theta, I))| = \max_{a_i \geq 0} \sum a_i \theta_i$$

$\Rightarrow$ $\operatorname{Log}|\theta|$ is convex

Toroidal technics

+ ref divisors correspond to cvx fcts

+ more play with intersection numbers

$\boxed{\text{Thm}}$ For any model $\mathcal{X}'$, for any φ $\mathbb{Z}$-psh then $\varphi|_\sigma$ is convex for all faces of $\Delta\mathcal{X}'$ and

$$\mathrm{Lip}(\varphi)|_{\Delta\mathcal{X}'} \leq C$$

Real Monge-Ampère

$U \subseteq \mathbb{R}^N$ $f : U \to \mathbb{R}$ convex

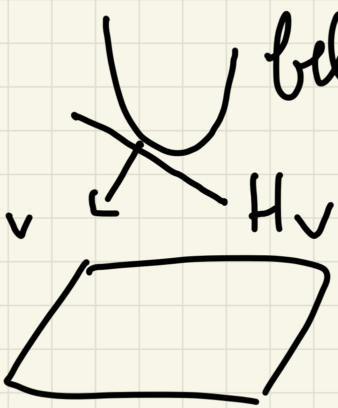
$MA_{\mathbb{R}}(f)$ positive measure in U

$$MA_{\mathbb{R}}(f)\,(E) = leb\left(\nabla f(E)\right)$$

where

$\nabla f(x_0)$ = set of vectors $v \in \mathbb{R}^N$ s.t.

$H_v = \left(y - f(x_0)\right) - v \cdot (x - x_0)$ lies below the epigraph of f



$\mathbb{R}^n$

$\to$ exist by a theorem of Alexandrov

$$\{v,\ \exists\, x_1 \neq x_2 \quad v \in \partial f(x_1) \cap \partial f(x_2)\}$$

is a null-set

$\to$ if f is C^2 $MA_{\mathbb{R}} = \det(\operatorname{Hess} f)$

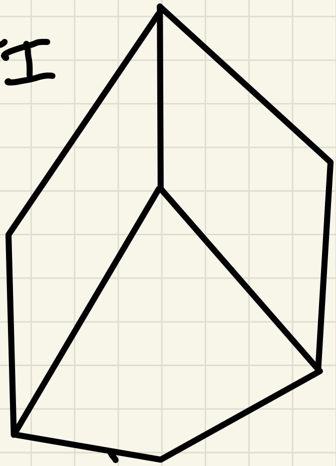
$\to$ $MA_{\mathbb{R}}(f \circ A) = |\det A|\, MA_{\mathbb{R}}(f)$

only depends on the choice of a volume form on $\mathbb{R}^n$ (not the euclidean structure)

σ face $\dim(\sigma) = n$

recall $\sigma = \left\{ \sum_{i \in I} b_i a_i = +1 \right\}$

E_I



locally $[X_0] = \sum b_i [E_i]$

vol. form on σ = normalization by

$\mathrm{vol}(\sigma) = \prod b_i^{-1}$

$\rightsquigarrow$ get $MA_{\mathbb{R}}$ on convex function

on σ

$\boxed{\text{Thm (Vilsmeier)}}$

$\varphi_0 \in \mathcal{C}^0(\Delta_{\mathcal{X}})$ such that

$\varphi = \varphi_0 \circ r_{\mathcal{X}}$ is $\mathcal{Z}$-psh.

σ any face of $\Delta_{\mathcal{X}}$ of dim. n

Then

$$MA_{\mathbb{R}}\left(\varphi|_{\mathring{\sigma}}\right) = MA_{\mathcal{Z}}(\varphi)\Big|_{\mathring{\sigma}}$$

Explanation

purely toric computation

avatar of the formula $L \to X(\Delta)$

$$Z \cdot c_1(L)^n = \int h_Z \, \mu_L$$

μ_L: surface area measure of the polytope associated to L

(precise proof Vilomein p. 23)

Last comments

- φ Model $\mathcal{L}$-psh

$$\Rightarrow \mathrm{MA}_{\mathcal{L}}(\varphi) \text{ atomic supported}$$

on divisorial points

- Converse is unlikely except in special situations

- More generally $\mathrm{Supp}(\mu) \subseteq \Delta_{\mathcal{X}}$ should not imply $\varphi = \varphi \circ r_{\mathcal{X}}$!