Curriculum Vitæ

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French.
Born 26 August 1976 in Roubaix, France.

Education and positions

2014— : 'Directeur de recherche' (Senior Researcher) at the French CNRS, member of the Centre de Mathématiques Laurent Schwartz (CMLS), Ecole Polytechnique.

2010— : Part-time Professor at the Ecole Polytechnique.

2003–2014 : 'Chargé de recherche' (Researcher) at the French CNRS, member of the Institut de Mathématiques de Jussieu (IMJ), Université Pierre et Marie Curie (Paris 6).

2003 : Post-doc at Imperial College, London (advisor Simon Donaldson).

1999–2002 : Ph. D. thesis (advisor Jean-Pierre Demailly), Institut Fourier, Grenoble.

Awards and grants

2018 : Invited speaker ICM Rio 2018.

2016— : Coordinator of the French ANR research project "GRACK" (Gromov-Hausdorff convergence in Kähler geometry).

2014 : Paul Doistau–Emile Blutet prize of the French Academy of Sciences.

Publications

1. *Le cône kählérien d'une variété hyperkählérienne.* C. R. Acad. Sci., Paris, Série I, Math. **333**, no. 10, 935–938 (2001).

2. *On the volume of a line bundle.* Int. J. Math. **13**, no. 10, 1043–1063 (2002).

3. *Divisorial Zariski decompositions on compact complex manifolds.* Ann. Sci. ENS (4) **37**, no. 1, 45–76 (2004).

4. with Charles Favre and Mattias Jonsson. *Valuations and plurisubharmonic singularities.* Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 449–494.

5. with Charles Favre and Mattias Jonsson. *Degree growth of meromorphic surface maps.* Duke Math. J. **141** (2008), no. 3, 519–538.

6. with Charles Favre and Mattias Jonsson. *Differentiability of volumes of divisors and a problem of Teissier.* J. Algebraic Geom. **18** (2009), no. 2, 279–308.

7. with Philippe Eyssidieux, Vincent Guedj and Ahmed Zeriahi. *Monge-Ampère equations in big cohomology classes.* Acta Math. **205** (2010), no. 2, 199–262

8. with Robert Berman. *Growth of balls of holomorphic sections and energy at equilibrium.* Invent. Math. **181** (2010), no. 2, 337–394.

9. with Huayi Chen. *Okounkov bodies of filtered linear series.* Compositio Math. **147** (2011), no. 4, 1205–1229.

10. with Robert Berman and David Witt-Nyström. *Fekete points and convergence towards equilibrium measures on complex manifolds.* Acta Math. **207** (2011), no. 1, 1–27.

11. with Tommaso de Fernex and Charles Favre. *The volume of an isolated singularity.* Duke Math. J. **161** (2012), no. 8, 1455–1520.

12. Monge-Ampère equations on complex manifolds with boundary. In *Complex Monge-Ampère equations and geodesics in the space of Kähler metrics*, Vincent Guedj Ed. Lecture Notes in Math. **2038**. Springer, Heidelberg, 2012.

13. with Jean-Pierre Demailly, Mihai Păun and Thomas Peternell. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* J. Algebraic Geom. **22** (2013), no. 2, 201–248.

14. with Robert Berman, Vincent Guedj and Ahmed Zeriahi. *A variational approach to complex Monge-Ampère equations.* Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245.

15. with Amaël Broustet and Gianluca Pacienza. *Uniruledness of stable base loci of adjoint linear systems via Mori Theory.* Math. Z. **275** (2013), no. 1-2, 499–507.

16. with Vincent Guedj. Regularizing properties of the Kähler-Ricci flow. In *An introduction to the Kähler-Ricci flow*, Sébastien Boucksom, Philippe Eyssidieux and Vincent Guedj Eds. Lecture Notes in Math. **2086**. Springer, Cham, 2013.

17. *Corps d'Okounkov* (d'après Okounkov, Lazarsfeld-Mustata et Kaveh-Khovanskii). Séminaire Bourbaki. Astérisque **361** (2014), Exp. No. 1059, vii, 1–41.

18. with Charles Favre and Mattias Jonsson. *A refinement of Izumi's theorem.* Valuation theory in interaction, 55–81, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014.

19. with Salvatore Cacciola and Angelo Felice Lopez. *Augmented base loci and restricted volumes on normal varieties.* Math. Z. **278** (2014), no. 3-4, 979–985.

20. with Alex Küronya, Catriona Maclean and Tomasz Szemberg. *Vanishing sequences and Okounkov bodies.* Math. Ann. **361** (2015), no. 3-4, 811–834.

21. Appendix to Gabor Székelyhidi's paper *Filtrations and test-configurations.* Math. Ann. **362** (2015), 451–484.

22. with Tommaso de Fernex, Charles Favre and Stefano Urbinati. *Valuation spaces and multiplier ideals on singular varieties.* Recent advances in algebraic geometry, 29–51, London Math. Soc. Lecture Note Ser. **417**, Cambridge Univ. Press, Cambridge, 2015.

23. with Charles Favre and Mattias Jonsson. *Solution to a non-Archimedean Monge-Ampère equation.* J. Amer. Math. Soc., **28** (2015), 617–667.

24. *Limite thermodynamique et théorie du potentiel.* Gaz. Math. (2015) 146, 16–26.

25. with Charles Favre and Mattias Jonsson. *The Non-Archimedean Monge-Ampère Equation.* Nonarchimedean and tropical geometry, 31–49, Simons Symp., Springer, 2016.

26. with Charles Favre and Mattias Jonsson. *Singular metrics in non-Archimedean geometry.* J. Algebraic Geom. **25** (2016), 77–139.

27. with Tomoyuki Hisamoto and Mattias Jonsson. *Uniform K-stabilty, Duistermaat-Heckman measures and singularities of pairs.* Ann. Inst. Fourier **67** (2017), 743–841

28. with Mattias Jonsson. *Tropical and non-Archimedean limits of degenerating families of volume forms.* J. Ec. Polytechnique **4** (2017), 87–139.

29. with Robert Berman, Philippe Eyssidieux, Vincent Guedj and Ahmed Zeriahi. *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties.* J. reine angew. Math. **751** (2019), 27–89.

30. Appendix to David Witt Nyström's paper *Duality between the pseudoeffective and the movable cone on a projective manifold.* J. Amer. Math. Soc. **32** (2019), 675-689.

31. with Tomoyuki Hisamoto and Mattias Jonsson. *Uniform K-stabilty and asymptotics of energy functionals in Kähler geometry.* J. Eur. Math. Soc. **21** (2019), 2905–2944.

32. *Variational and non-Archimedean aspects of the Yau–Tian–Donaldson conjecture.* Proc. ICM 2018 (2019), 591–617

33. with Robert Berman and Mattias Jonsson. *A variational approach to the Yau-Tian-Donaldson conjecture.* Preprint (2015) arXiv :1509.04561. To appear in J. Amer. Math. Soc.

34. with Mattias Jonsson. *Singular semipositive metrics on line bundles on varieties over trivially valued fields.* Preprint (2018) arXiv :1801.08229.

35. with Dennis Eriksson. *Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry.* Preprint (2018) arXiv :1805.01016. to appear in Advances Math.

36. with Mattias Jonsson. *A non-Archimedean approach to K-stability.* Preprint (2018) arXiv :1805.11160.

37. with Simone Diverio. *A note on Lang's conjecture for quotients of bounded domains.* Preprint (2018) arXiv :1809.02398. To appear in Epiga.

38. with Walter Gubler and Florent Martin. *Differentiability of relative volumes over an arbitrary non-Archimedean field.* Preprint (2020) arXiv :2004.03847. To appear in IMRN.

39. with Walter Gubler and Florent Martin. *Non-Archimedean volumes of metrized nef line bundles.* Preprint (2020) arXiv :2011.06986.

Research statement

My research lies at the meeting point of Kähler geometry, algebraic geometry and non-Archimedean geometry, the main directions over the last few years being

— the Yau-Tian-Donaldson conjecture and K-stability ;

— degenerations of Calabi-Yau manifolds.

My most significant contribution over this period has been a new variational proof of the Yau-Tian-Donaldson conjecture for Fano manifolds, jointly with Robert Berman and Mattias Jonsson, substantially simpler than the previous approach by Chen-Donaldson-Sun.

The Yau-Tian-Donaldson conjecture

Let (X, L) be a polarized manifold, i.e. a smooth complex projective variety X, together with an ample line bundle L. The Yau-Tian-Donaldson conjecture relates the existence of a Kähler metric $\omega \in c_1(L)$ with constant scalar curvature (cscK metric for short) to the algebro-geometric notion of K-stability of (X, L).

When $c_1(X)$ is a multiple of $c_1(L)$, a metric $\omega \in c_1(L)$ is cscK if and only it is Kähler-Einstein, i.e. with constant Ricci curvature. Conversely, the latter condition implies that $c_1(X) = \lambda c_1(L)$ with $\lambda \in \mathbf{Q}$. The canonical bundle K_X is then either ample ($\lambda < 0$, canonically polarized case), torsion ($\lambda = 0$, Calabi-Yau case), or negative ($\lambda > 0$, Fano case). In the first two cases, it was showed by Aubin and Yau in the 70's that $c_1(L)$ contains a unique Kähler-Einstein metric. In the Fano case, there are various known obstructions to the existence of a Kähler-Einstein metric, and the Yau-Tian-Donaldson conjecture states in particular that they are of purely algebro-geometric nature.

More precisely, let $(\mathcal{X},\mathcal{L})$ be a *test configuration* for (X,L), i.e. a $\mathbf{C}^*$-equivariant degeneration induced by a $\mathbf{C}^*$-action on a projective space in which X is embedded via the sections of a power of L. One associates to $(\mathcal{X},\mathcal{L})$ a *Donaldson-Futaki invariant* $\mathrm{DF}(\mathcal{X},\mathcal{L})$, which can either be described as a limit of Hilbert-Mumford weights from geometric invariant theory (GIT), or in terms of intersection numbers computed on (a compactification of) $\mathcal{X}$. One then says that (X,L) is K-stable if $\mathrm{DF}(\mathcal{X},\mathcal{L}) > 0$ for every nontrivial test configuration.

On the other hand, the existence of a cscK metric is a variational problem, the solutions being precisely the minimizers of a functional on the space of Kähler metric in $c_1(L)$, namely the Mabuchi K-energy functional M. This functional is further convex in a nontrivial sense, and the intuition underlying the Yau-Tian-Donaldson conjecture is that one can test the growth properties of M by considering its behavior along certain rays of Kähler metrics. Each test configuration $(\mathcal{X},\mathcal{L})$ produces such a ray, consisting of Fubini-Study type metrics, and the slope at infinity of M along this ray is essentially equal to the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X},\mathcal{L})$. The conjecture thus predicts the existence of critical points for M as soon as M is proper along certain algebro-geometric rays of metrics.

This point of view was developped in joint work with Tomoyuki Hisamoto and Mattias Jonsson. This is done using non-Archimedean geometry, the starting point being that a test configuration defines in a natural way a metric on the Berkovich space attached to (X,L). We reinterpret various invariants attached to a test configuration as the non-Archimedean analogue F^{NA} of well-known functionals F in Kähler geometry. In particular, the non-Archimedean version M^{NA} of the K-energy M coincides with the Donaldson-Futaki invariant DF, up to an explicit error term. This analogy is given more substance by the fact that for each functional F we are dealing with, $F^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ coincides with the slope at infinity $\lim_{t\to+\infty} F(\phi_t)/t$ along a ray (ϕ_t) compatible with $(\mathcal{X},\mathcal{L})$.

These results were exploited in a joint work with Robert Berman and Mattias Jonsson, in which we give a new proof of the Yau-Tian-Donaldson conjecture in the Fano case. After the groundbreaking work of Chen-Donaldson-Sun, a variant of their approach, yielding a more general result, was obtained by Datar-Szekelyhidi. In contrast to these proofs, which rely on an in-depth and very difficult study of Gromov-Hausdorff limits of Kähler-Einstein manifolds, our approach is based on the natural interpretation of the Yau-Tian-Donaldson conjecture provided above. Arguing by contradiction, a compactness argument produces a ray of metrics, possibly very singular, along which M remains bounded above. Using the Demailly-Siu theory of multiplier ideals, we then approximate this ray by algebro-geometric ones, and eventually contradict the growth of M along such rays.

Our variational approach to the YTD conjecture opens several important perspectives, which are at the heart of my current research activity. On the one hand, it is very reasonable to hope to extend this approach, which relies purely on pluripotential theory, to the case of singular Fano varieties, that has come to play a key role in recent works by Odaka-Spotti-Sun and Li-Wang-Xu on modulo spaces of K-stable Fano varieties. On the other hand, our strategy opens the way to the general case of the YTD conjecture, dealing with cscK metrics. Combined with the important recent work of Chen-Cheng, it reduces the general case of the conjecture to the study of the monotonicity properties of the slopes at infinity of the K-energy under algebraic approximation of geodesic rays.

Degenerations of Calabi-Yau manifolds

The Kontsevich-Soibelman conjecture deals with the limiting behavior, as metric spaces, of a family of Calabi-Yau manifolds approaching a "cusp" in the boundary of the moduli space. With Mattias Jonsson, we established a weak version of this conjecture, dealing with the associated volume forms.

More precisely, let $(X, L) \to \mathbf{D}^*$ be a degeneration of Calabi-Yau manifolds, parametrized by the punctured unit disc $\mathbf{D}^* \subset \mathbf{C}$, and polarized by an ample line bundle. Each fiber (X_t, L_t) comes with a unique Ricci-flat metric $\omega_t \in c_1(L_t)$, and it is a natural problem to study the limit behavior in the Gromov-Hausdorff sense of (X_t, ω_t) as $t \to 0$. This is further motivated by the Ströminger-Yau-Zaslow conjecture in mirror symmetry, which tries to produce Lagrangian fibrations on X_t for t close to 0, as a step to obtain a mirror family.

Following Kontsevich and Soibelman, one introduces the *essential skeleton* of the degeneration, a simplicial complex $\mathrm{Sk}(X)$ endowed with a $\mathbf{Z}$-PL structure, which can be interpreted as the dual complex of a minimal model of the degenerations, as follows from work of Mustata, Nicaise and Xu. The dimension $\dim \mathrm{Sk}(X) \le \dim X_t$ measures the degree of degeneration. Work of Donaldson-Sun, Tosatti and Takayama shows that the minimally degenerate case $\dim \mathrm{Sk}(X) = 0$ occurs precisely when the diameter of (X_t, ω_t) remains bounded. In that case, (X_t, ω_t) converges to the central fiber X_0 of the minimal model of the degeneration, endowed with its unique singular Ricci-flat metric.

At the other end of the spectrum, the maximally degenerate case $\dim \mathrm{Sk}(X) = \dim X_t$ corresponds to the case where (X_t, L_t) approaches a cusp. Kontsevich and Soibelman then conjecture that X_t, endowed with the metric $\tilde{\omega}_t$ rescaled to diameter 1, converges in the Gromov-Hausdorff sense to the essential skeleton $\mathrm{Sk}(X)$ endowed with a Monge-Ampère metric, i.e. the analogue in convex geometry of a Calabi-Yau manifold.

In joint work with Mattias Jonsson, we proved that the volume forms ω_t^n converge in a natural sense to the Lebesgue measure of $\mathrm{Sk}(X)$ determined by its $\mathbf{Z}$-PL structure. We deal more generally with the convergence to a $\mathbf{Z}$-PL measure of a family of volume forms on compact complex manifolds with appropriate singularities.

In order to make sense of this convergence, we rely on a general analytification procedure due to Berkovich, which completes in a natural way $X \to \mathbf{D}^*$ to a topological fibration over the unit disc $\mathbf{D}$, with central fiber the Berkovich non-Archimedean space X^{an} attached to the degeneration. The essential skeleton $\mathrm{Sk}(X)$ as initially described by Kontsevich and Soibelman lies indeed as a subspace of X^{an}, their idea being to then solve a non-Archimedean Monge-Ampère equation to produce the desired metric on $\mathrm{Sk}(X)$.